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Approximation by Abel means and Tauberian Theorems in sequence spaces*

by

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Abstract. Let E be a topological sequence space. We say the Abel means of a sequence $x \in E$ exist if for $0 < r < 1$ the sequences $A_n^r x = (x_0, rx_1, r^2x_2, \ldots \ldots, r^n x_n, 0, 0, \ldots)$ converge to the sequence $A^r x = (x_0, rx_1, r^2 x_2, \ldots, r^n x_n, \ldots)$ in the topology of E. We say that x can be approximated by Abel means if $A^r x$ converges to x (as $r \to 1^-$) in the topology of E. Approximation by Abel means in topological sequence spaces is investigated. It is a concept more general than Abel summability since in the FK-space

$$\mathscr{A} = \{x\colon \lim_{r\to 1^-} \sum_{k=0}^{\infty} x_k r^k \text{ exists}\}$$

of all summable sequences, every sequence can be approximated by Abel means. Also a sequence $x \in \mathscr{A}$ has the property of sectional convergence (AK) if and only if it is in the summability field $cs = \{x\colon \sum_{k=0}^{\infty} x_k \text{ exists}\}$. Yet the properties of approximation by Abel means and AK apply also to spaces which are not summability fields. Thus statements which give conditions under which approximation by Abel means implies AK are generalizations of Tauberian Theorems for the Abel method. Several such approximation statements are obtained which extend classical Tauberian Theorems. Approximation by Abel means is useful in spaces of Fourier coefficients as well as being of interest in the general theory of sequence spaces. As an application, a generalization of results of Tietz and Goes is obtained about the equivalence of absolute Tauberian conditions.

1. Introduction. Section 2 contains definitions. In Section 3 we investigate approximation by Abel means in topological sequence spaces. It is shown that in the FK-space

$$\mathscr{A} = \left\{x\colon \lim_{r\to 1^-} \sum_{k=0}^{\infty} x_k r^k \text{ exists}\right\}$$

of Abel summable sequences, every sequence can be approximated by Abel

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means and a sequence x in $\mathscr{A}$ has sectional convergence if and only if its series $\sum_{k=0}^{\infty} x_k$ converges. We also give some criteria for approximation by Abel means in general topological sequence spaces. In Section 4 it is shown that many of the Tauberian Theorems for Abel summability such as Tauber's First and Second Theorem [16], the O-Tauberian Theorem of Littlewood [10] and the gap Tauberian Theorem of Hardy and Littlewood [7] can be generalized to approximation statements. As a further application of Abel means, in Section 5 we generalize results of Tietz [17] and Goes [5] on the equivalence of absolute Tauberian conditions.

2. Definitions. A K-*space* is a Hausdorff locally convex space of sequences $x = (x_k)$ of real or complex coordinates with continuous coordinate functionals f_k: $x \to x_k$ $(k = 0, 1, 2, \ldots)$. An FK-*space* (respectively BK-*space*) is a K-space with a Fréchet (respectively Banach) space topology ([6], [18], [20]). For each $k = 0, 1, 2, \ldots$, let δ^k be the sequence with 1 in the kth position and zero elsewhere and let φ be the space of all finite linear combinations of the δ^k's. All sequence spaces E and F considered will be assumed to be K-spaces which contain φ.

Let $x \in E$. We say that the *Abel means of x exist* if, for all $0 < r < 1$, the series $A^r x = \sum_{k=0}^{\infty} x_k r^k \delta^k$ converges with respect to the topology of E. We say *x can be approximated by its Abel means* if the Abel means of x exist and $\lim_{r \to 1-} A^r x = x$, the limit being convergent with respect to the topology of E.

Let T be an infinite matrix with rows in φ and with columns converging to 1. A sequence x in E has T-*sectional convergence* (TK) if $\lim_{n\to\infty} T^n x = x$, where $T^n x = \sum_k t_{nk} x_k \delta^k$, convergence being with respect to the topology of E. $t^n x$ is called the *n-th* T-*section of x*. For $\alpha \geqslant 0$, C_α-sectional convergence is TK with respect to the triangular matrix given by $t_{nk} = \binom{n-k+\alpha}{n-k} / \binom{n+\alpha}{n}$, $k \leqslant n$. Sectional convergence (AK) is the same as C_0-sectional convergence; the nth sections of x being $S^n x = \sum_{k=0}^{n} x_k \delta^k$. The C_1-sections are denoted

$$\sigma^n x = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) x_k \delta^k = \frac{1}{n+1} \sum_{k=0}^{n} S^k x.$$

A sequence x in E has bounded sections if $\{S^n x\}_{n=0}^{\infty}$ is a bounded subset of E.

Let $\mathscr{A} = \{x = (x_k)_{k=0}^{\infty} : \lim_{r \to 1-} \sum_{k=0}^{\infty} x_k r^k \text{ exists}\}$ be the series-sequence Abel summability field. $\mathscr{A}$ is an FK-space with seminorms given by:

$$p_0(x) = \sup_{0<r<1} \left| \sum_{k=0}^{\infty} x_k r^k \right|,$$

$$p_j(x) = \sup_{m=0,1,2,\ldots} |x_m| \left(\frac{j}{j+1}\right)^m, \quad j = 1, 2, \ldots$$

This was proved in [19] (Theorem IV) (and stated earlier without proof in [23]). The seminorms p_j, $j = 1, 2, 3, \ldots$ may be replaced by

$$p'_j(x) = \sum_{m=0}^{\infty} |x_m| \left(\frac{j}{j+1}\right)^m, \quad j = 1, 2, 3, \ldots$$

3. Approximation by Abel means. We start with a proposition about the existence of Abel means.

PROPOSITION 1. *Let E be a sequentially complete* K-*space and $x \in E$. The following statements are equivalent.*

(a) *The Abel means of x exist;*

(b) *for every $0 < r < 1$, $x_k r^k \delta^k \to 0$ as $n \to \infty$;*

(c) *for every $0 < r < 1$, $\{x_k r^k \delta^k\}$ is a bounded subset of E;*

(d) *for every $0 < r < 1$, $r^n S^n x = r^n \sum_{k=0}^{n} x_k \delta^k \to 0$ as $n \to \infty$;*

(e) *for every $0 < r < 1$, $A^r x = (1-r) \sum_{k=0}^{\infty} r^n S^n x$.*

Proof. Since $x_k r^k \delta^k = \sum_{k=0}^{n} x_k r^k \delta^k - \sum_{k=0}^{n-1} x_k r^k \delta^k$, (a) $\Rightarrow$ (b) follows. (b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (d): Let $0 < r < t < 1$ and let p be a continuous seminorm on E. Then

$$p(r^n S^n x) \leqslant \left(\frac{r}{t}\right)^n \sup_k p(x_k t^k \delta^k) \sum_{k=0}^{n} t^{n-k} = \left(\frac{r}{t}\right)^n \sup_k p(x_k t^k \delta^k) \left(\frac{1 - t^{n+1}}{1-t}\right)$$

which tends to zero as $n \to \infty$.

(d) $\Rightarrow$ (e): Let $0 < r < t < 1$ and let p be a continuous seminorm on E. Then

$$p\left(\sum_{k=M}^{N} x_k r^k \delta^k\right) \leqslant \sup_k p(t^k S^k x) \sum_{k=M}^{N} \left(\frac{r}{t}\right)^k.$$

Since E is sequentially complete, $\sum_{k=0}^{\infty} r^k S^k x$ exists. The rest follows from the observation that

$$\sum_{k=0}^{n} x_k r^k \delta^k = (1-r)\sum_{k=0}^{n} r^k S^k x + r^{n+1} S^n x.$$

Finally, (e)$\Rightarrow$(a) is trivial. ∎

A sequence space E is *tempered* if for each $x \in E$, there exists n such that $x_k = O(k^n)$. All spaces of Fourier coefficients are tempered. If an FK space E contains the space l^1 of absolutely convergent series, then $\{\delta^k\}_{k=0}^{\infty}$ is a bounded subset of E. It follows from (c) above that in a tempered FK-space E containing l^1 the Abel means of every sequence exists.

The following shows that approximation by Abel means is more general than C_α-sectional convergence.

Proposition 2. *Suppose E is sequentially complete and $\alpha \geqslant 0$. If x has C_α-sectional convergence, then x can be approximated by its Abel means. In particular,* AK *implies approximation by Abel means.*

Proof. Since C_α-sectional convergence implies C_β-sectional convergence for $0 \leqslant \alpha \leqslant \beta$, it is sufficient to assume that α is a non-negative integer. Let p be a continuous semi-norm on E and suppose x in E has C_α-sectional convergence. Let

$$T^n x = \sum_{k=0}^{n} t_{nk} x_k \delta^k, \quad \text{where} \quad t_{nk} = \binom{n-k+\alpha}{n-k}\Big/\binom{n+\alpha}{n}.$$

Clearly $p(x_k\delta^k) = O(k^\alpha)$. By Proposition 1 (c), the Abel means of x exist. Further,

$$A^r x = (1-r)^{\alpha+1}\sum_{k=0}^{\infty}\binom{k+\alpha}{k} r^k T^k x.$$

If $p(T^k x - x) < \varepsilon$ whenever $k > N$, then

$$p(A^r x - x) \leqslant (1-r)^{\alpha+1}\sum_{k=0}^{N}\binom{k+\alpha}{k} r^k \sup_n p(T^n x - x) + \varepsilon,$$

which tends to ε as r tends to 1. ∎

The following two results show that for a general class of summability fields, approximation by Abel means (respectively, C_α-sectional convergence) of a sequence is the same as Abel summability (resp. C_α-summability). The following result is stated in [23] without proof. For completeness we give a proof.

Theorem 1. *Every element of $\mathscr{A}$ can be approximated by its Abel means.*

Proof. Let $x \in \mathscr{A}$. We show (a): $p_j(x - A^r x) \to 0$ (as $r \to 1^-$) for $j = 1, 2, 3, \ldots$ and (b): $p_0(A^{r_1}x - A^{r_2}x) \to 0$ (as $r_1, r_2 \to 1$).

(a): Let $\varepsilon > 0$, $j \neq 0$. Choose N such that

$$|x_m|\left(\frac{j}{j+1}\right)^m < \varepsilon \text{ for } m > N.$$

For r sufficiently large,

$$|x_m|\left(\frac{j}{j+1}\right)^m (1-r^m) < \varepsilon \quad \text{for} \quad m = 1, 2, \ldots, N.$$

Then

$$p_j(x - A^r x) = \sup_m |x_m|\left(\frac{j}{j+1}\right)^m (1-r^m) < \varepsilon.$$

(b): Let $\varepsilon > 0$. Since $x \in \mathscr{A}$, there exists j such that $\left|\sum_k x_k r_1^k - x_k r_2^k\right| < \varepsilon$ whenever $\left(\frac{j}{j+1}\right)^2 < r_1 < r_2 < 1$. By the equivalence of the seminorms $\{p_j\}$ and $\{p_j'\}$ we have $p_j'(A^{r_1}x - A^{r_2}x) < \varepsilon$ for $0 < r_1, r_2 < 1$ sufficiently close to 1. Let, further, $\frac{j}{j+1} < r_1 < r_2 < 1$. For $t > \frac{j}{j+1}$ we have

$$\left|\sum_k (x_k r_1^k - x_k r_2^k) t^k\right| = \left|\sum_k x_k (r_1 t)^k - x_k (r_2 t)^k\right| < \varepsilon$$

since $\left(\frac{j}{j+1}\right)^2 < r_1 t < r_2 t < 1$. For $t < \frac{j}{j+1}$ we have

$$\left|\sum_k (x_k r_1^k - x_k r_2^k) t^k\right| \leqslant p_j'(A^{r_1}x - A^{r_2}x) < \varepsilon.$$

Thus

$$p_0(A^{r_1}x - A^{r_2}x) = \sup_{0<t<1}\left|\sum_k (x_k r_1^k - x_k r_2^k) t^k\right| < \varepsilon. \ ∎$$

Let A be a summability method given by $A - \sum x_k = \lim_{r\to r_\infty} \sum_{k=0}^{\infty} a_k(r) x_k$ satisfying:

(1) $a_k(r)$ is continuous for $0 < r < r_\infty \leqslant \infty$, $k = 0, 1, 2, \ldots$ and some fixed r_∞;

(2) for any x, $0 < r_1 < r_\infty$, convergence of $\sum_{k=0}^{\infty} a_k(r_1) x_k$ implies uniform convergence of $\sum a_k(r) x_k$ in the interval $0 \leqslant r \leqslant r_1$;

(3) for all $n = 0, 1, 2, \ldots$, if $x = \delta^n$, then $A - \sum x_k = 1$.

The summability field $c_A = \{x\colon A - \sum x_k \text{ exists}\}$ of such a method is an FK-space [19]. All summability methods defined by a matrix $\left(A - \sum x_k = \lim_{n\to\infty} \sum_k a_{nk} x_k\right)$ satisfy (1) and (2) when $r_\infty = \infty$, $a_k(n) = a_{nk}$ and $a_k(r)$ is linear between $a_k(n)$ and $a_k(n+1)$. Condition (3) states that the method A sums finite sequences to their usual sum.

THEOREM 2. *Let c_A be the summability field of a method satisfying* (1)–(3).

(a) *If $x \in c_A$ can be approximated by its Abel means, then $x \in \mathscr{A}$.*

(b) *If $x \in c_A$ has C_a-sectional convergence, then x is C_a-summable.*

Proof. By standard equicontinuity arguments on FK-spaces, one can show that $f(x) = A - \sum x_k$ is a continuous linear functional on c_A. If $x = \lim_{r\to 1-} A^r x$, then

$$f(x) = \lim_{r\to 1-} f(A^r x) = \lim_{r\to 1-} \sum_k x_k r^k f(\delta^k) = \lim_{r\to 1-} \sum_k x_k r^k .$$

Thus $x \in \mathscr{A}$. Similarly, if x has C_a-sectional convergence, then $f(x) = C_a - \sum x_k$. ∎

Remark. Suppose c_A satisfies (1)–(3). If $\mathscr{A} \subset c_A$, the inclusion map $i\colon \mathscr{A} \to c_A$ is continuous. Then a sequence x in c_A can be approximated by Abel means if and only if $x \in \mathscr{A}$. Such a statement can also be made about C_a-sections ($\alpha \geqslant 0$) since the summability fields of the C_a methods have C_a-sectional convergence [22]. In particular, a sequence x in $\mathscr{A}$ has the property AK if and only if $\sum_{k=0}^{\infty}$ exists.

A sequence λ is a *multiplier from E to F* if $x\cdot\lambda = (x_k\lambda_k) \in F$ for all x in E. The space of all multipliers from E to F is denoted by $(E\to F)$. For example $(l^p \to l^1) = l^q$ where $1/p+1/q = 1$ and $1 \leqslant p, q \leqslant \infty$. Multiplier maps $x \to x\cdot\lambda$ between FK-spaces are continuous [20]. If E is an FK-space and λ is a multiplier from E to $\mathscr{A}$, it follows that the linear functional $f(x) = \lim_{r\to 1-} \sum_k x_k\lambda_k r^k$ is continuous on E. We define $E^{\mathscr{S}}$ as the space of all sequences $\lambda = (\lambda_k)$, $\lambda_k = f(\delta^k)$, $k = 0, 1, 2, \ldots$, where f ranges over all continuous linear functionals on E.

PROPOSITION 3. *Let E be an FK-space containing φ. $(E\to\mathscr{A}) = E^{\mathscr{S}}$ if and only if $\lim_{r\to 1-} \sum_k r^k x_k f(\delta^k)$ exists for every continuous linear functional f on E and every $x \in E$.*

Proof. ($\Rightarrow$): Suppose $(E\to\mathscr{A}) = E^{\mathscr{S}}$. Let f be a continuous linear functional on E and $\lambda_j = f(\delta^j)$. Then $\lambda\cdot x \in \mathscr{A}$ for every $x \in E$. That is, $\lim_{r\to 1-} \sum_k \lambda_k r^k = \lim_{r\to 1-} \sum_k x_k r^k f(\delta^k)$ exists for every $x \in E$.

($\Leftarrow$): Let $\lambda \in (E\to\mathscr{A})$. Then the linear functional $f(x) = \lim_{r\to 1-} \sum_k x_k\lambda_k r^k$ is continuous. Since $f(\delta^j) = \lambda_j$, we have $\lambda \in E^{\mathscr{S}}$. Hence $(E\to\mathscr{A}) \subset E^{\mathscr{S}}$. Conversely let f be a continuous linear functional on E and let $\lambda_j = f(\delta^j)$. Then for every $x \in E$, $\lim_{r\to 1-} \sum_k x_k r^k f(\delta^k) = \lim_{r\to 1-} \sum_k x_k r^k \lambda_k$ exists. Thus $\lambda\cdot x \in \mathscr{A}$ for every $x \in E$ and $\lambda \in E^{\mathscr{S}}$. ∎

THEOREM 3. *Let E be a K-space containing φ. If every element of E can be approximated by Abel means, then every continuous functional f on E is of the form*

$$f(x) = \lim_{r\to 1-} \sum_k x_k \lambda_k r^k$$

for some multiplier λ from E to $\mathscr{A}$. If E is an FK-space, the converse is also true.

The proof if omitted. The first statement can be proved easily for $\lambda_k = f(\delta^k)$. The converse uses an equicontinuity argument as in [21], Satz 3.4, and [4], Proposition 1.

In particular, $\mathscr{A}^{\mathscr{S}} = (\mathscr{A}\to\mathscr{A})$ by Theorem 1. A multiplier λ from $\mathscr{A}$ to $\mathscr{A}$ is of the form $\lambda_k = \int_0^1 t^k dg(t) + O(r^k)$, where $0 < r < 1$ and g is of bounded variation ([23], [19]).

In the theory of Fourier series, approximation by Abel means is a natural concept. Some examples of spaces of sequences of Fourier coefficients in which all sequences can be approximated by Abel means (but not by Cesàro sections) are given in [3]. Although T-sections are finite sequences whereas Abel means are not, many of the results for T-sectional convergence carry over to approximation by Abel means without major changes in proofs. Below we given three such results.

The following can be proved using equicontinuity arguments similar to [21], Satz 3.3, and [4], Proposition 1.

THEOREM 4. *Suppose E is a barreled K-space containing φ. Every element of E can be approximated by Abel means if and only if φ is a dense subset of E and for each $x \in E$ the Abel sections $\{A^r x\}_{0<r<1}$ form a bounded subset of E.*

The proof of the following is similar to [15], Theorem 3.4.

THEOREM 5. *Let E be an FK-space containing φ. Every element of E can be approximated by Abel means if and only if φ is a dense subset of E and every multiplier from $\mathscr{A}$ to $\mathscr{A}$ is a multiplier from E to E.*

The proof of the following is similar to [1], Theorem 4 and [1], Proposition 2.

THEOREM 6. *Let E be a BK-space containing φ. If every element of E can be approximated by Abel means, then the space $(E\to\mathscr{A})$ is a BK-space in which for each multiplier $\lambda \in (E\to\mathscr{A})$ the set of Abel means $\{A^r\lambda\}_{0<r<1}$ is bounded.*

4. Tauberian theorems as approximation statements. Here we consider statements giving conditions under which approximation by Abel means or C_1-sectional convergence implies sectional convergence. When applied to the FK-space $\mathscr{A}$ or

$$C_1 = \left\{x:\ \lim_{n\to\infty} \sum_k \left(1 - \frac{k}{n+1}\right) x_k \text{ exists}\right\},$$

such statements become Tauberian Theorems in the classical sense because every element of $\mathscr{A}$ can be approximated by Abel means (Theorem 1), every C_1 summable sequence has C_1-sectional convergence ([22], Satz 5) and a sequence in these spaces has sectional convergence if and only if it has a convergent series (Theorem 2). Some Tauberian Theorems for Cesàro sections are given in [2].

Sectional boundedness can often be deduced from Tauberian conditions using the equivalence of boundedness and weak boundedness along with a classical Tauberian Theorem on $\mathscr{A}$. This technique is used in Theorems 8 and 10. The following theorem uses sectional boundedness to reduce a Tauberian Theorem for Abel means to one for C_1-sections. The proof is a modification of a Tauberian Theorem of Karamata [9]. It has been adapted to an approximation statement instead of a summability statement and also differs from the original by using the uniform approximation by polynomials of a continuous function g instead of almost everywhere approximation by polynomials of an integrable function. For Banach spaces, the result was also obtained in [11] using another modification of Karamata's argument.

THEOREM 7. *Let E be a sequentially complete* K*-space containing φ. If a sequence x in E can be approximated by Abel means and has bounded sections, then it has C_1-sectional convergence.*

Proof. Let $r = e^{-t}$. Since $\lim_{t\to\infty}(1-e^{-t})/t = 1$ and

$$A^r x = (1-r)\sum_{k=0}^{\infty} r^k S^k x \to x \quad (r\to 1^-),$$

we have

$$\lim_{t\to 0^+} t\sum_{k=0}^{\infty} e^{-tk} S^k x = x.$$

For $m = 0, 1, 2, \ldots$ we have

$$\lim_{t\to 0^+} t\sum_{k=0}^{\infty} e^{-tk}(e^{-tk})^m S^k x = \frac{1}{m+1}\lim_{t\to 0^+}(m+1)t\sum_{k=0}^{\infty} e^{-(m+1)tk} S^k x$$
$$= x/m+1 = x\int_0^1 y^m\,dy$$

and hence for each polynomial P, we have

$$\lim_{t\to 0^+} t\sum_{k=0}^{\infty} e^{-tk} P(e^{-tk}) S^k x = x\int_0^1 P(y)\,dy.$$

Let $\varepsilon > 0$, $N > 1/\varepsilon$ and

$$g(x) = \begin{cases} e^{N+1}x^N & \text{if } 0 \leqslant x \leqslant e^{-1}, \\ x^{-1} & \text{if } e^{-1} \leqslant x \leqslant 1. \end{cases}$$

Since g is continuous, we can find a polynomial P such that $|g(x) - P(x)| < \varepsilon$ for $0 \leqslant x \leqslant 1$. Let p be a continuous seminorm on E. We have

$$p(\sigma^n x - x) \leqslant p\left(\frac{1}{n+1}\sum_{k=0}^{n} S^k x - t\sum_{k=0}^{n} e^{-tk}P(e^{-tk})S^k x\right) +$$
$$+ p\left(t\sum_{k=n+1}^{\infty} e^{-tk}P(e^{-tk})S^k x\right) + p\left(t\sum_{k=0}^{\infty} e^{-tk}P(e^{-tk})S^k x -\right.$$
$$\left. - x\int_0^1 P(y)\,dy\right) + p\left(x\int_0^1 P(y)\,dy - x\right).$$

There exists $\delta > 0$ such that

$$p\left(t\sum_{k=0}^{\infty} e^{-tk}P(e^{-tk})S^k x - x\int_0^1 P(y)\,dy\right) < \varepsilon$$

whenever $0 < t < \delta$. Let $t = \dfrac{1}{n+1} < \delta$ and $M = \sup_k p(S^k x)$. Then

$$p(\sigma^n x - x) \leqslant Mt\sum_{k=0}^{n} |1 - e^{-tk}P(e^{-tk})| + Mt\sum_{k=n+1}^{\infty} e^{-tk}|P(e^{-tk})| + \varepsilon +$$
$$+ \left|\int_0^1 P(y)\,dy - 1\right| p(x) < Mt\sum_{k=0}^{n}\varepsilon + Mt\sum_{k=n+1}^{\infty} e^{-tk}(\varepsilon + e^{N+1}e^{-tNk}) + \varepsilon +$$
$$+ \left(\varepsilon + \frac{1}{N+1}\right)p(x) \leqslant M\varepsilon + Mt\varepsilon\sum_{k=0}^{\infty} e^{-tk} + Mt\sum_{k=n+1}^{\infty} e^{N+1}e^{-tNk} + \varepsilon + 2\varepsilon p(x)$$
$$= M\varepsilon + Mt\varepsilon\left(\frac{1}{1-e^{-t}}\right) + Mt\left(\frac{1}{1-e^{-Nt}}\right) + \varepsilon + 2\varepsilon p(x).$$

This can be made arbitrarily small since $Mt\varepsilon\left(\dfrac{1}{1-e^{-t}}\right) \to M\varepsilon$ $(n\to\infty)$ and $Mt\left(\dfrac{1}{1-e^{-Nt}}\right) \to M/N+1 < M\varepsilon$ $(n\to\infty)$. ∎

THEOREM 8 (Tauber's second Theorem for Abel means). *Let E be a sequentially complete* K*-space containing φ. Suppose that a sequence x in E can be approximated by its Abel means. Let $d^n x = \frac{1}{n+1}\sum_{k=0}^{n} kx_k\delta^k$. Then x has sectional convergence if and only if $d^n x\to 0$ $(n\to\infty)$.*

Proof. Suppose that x has sectional convergence. Then

$$d^n x = \frac{1}{n+1}\Big(nS^n x - \sum_{k=0}^{n-1} S^k x\Big) = \frac{1}{n+1}\Big(n(S^n x - x) - \sum_{k=0}^{n-1}(S^k x - x)\Big)$$
$$= \frac{n}{n+1}(S^n x - x) - \frac{1}{n+1}\sum_{k=0}^{n-1}(S^k x - x)$$

each term of which tends to zero. Conversely let f be a continuous linear functional on E, and let $\lambda_j = f(\delta^j)$. Since x can be approximated by Abel means it follows that $\lambda\cdot x$ is an element of $\mathscr{A}$. Clearly $f(d^n x) = \frac{1}{n+1}\sum_{k=0}^{n} k\lambda_k x_k$ tends to zero. By the second Theorem of Tauber, $\sum_{k=0}^{\infty}\lambda_k x_k$ converges. In particular $f(S^n x) = \sum_{k=0}^{n}\lambda_k x_k$ is bounded for every continuous linear functional and hence $\{S^n x\}$ is a bounded subset of E. By Theorem 7, x has C_1-sectional convergence. But $S^n x - \sigma^n x = d^n x \to 0$ as $n\to 0$. Since $\sigma^n x$ converges to x, so does $S^n x$. ∎

LEMMA 1 (Hardy's Tauberian Theorem for Cesàro sections). *Let E be a* K*-space containing φ. Suppose that a sequence x in E has C_1-sectional convergence. If $\{nx_n\delta^n\}_{n=0}^{\infty}$ is a bounded subset of E, then x has the property* AK.

Proof. It is sufficient to show that $S^n x - \sigma^n x\to 0$ $(n\to\infty)$ under the conditions of the hypothesis. We have

$$S^n x - \sigma^n x = \frac{1}{n+1}\sum_{k=0}^{n} kx_k\delta^k$$

and for $m\geqslant n$ we have

$$\sigma^m x - \sigma^n x = \frac{m-n}{(m+1)(n+1)}\sum_{k=0}^{n} kx_k\delta^k + \frac{1}{m+1}\sum_{k=n+1}^{m}(m+1-k)x_k\delta^k.$$

Substituting we get

$$S^n x - \sigma^n x = \frac{m+1}{m-n}(\sigma^m x - \sigma^n x) + \frac{1}{m-n}\sum_{k=n+1}^{m}(m+1-k)x_k\delta^k.$$

For each continuous seminorm p and for $m\leqslant n$, we have

$$p(S^n x - \sigma^n x)\leqslant\Big(1+\frac{n+1}{m-n}\Big)p(\sigma^m x - \sigma^n x) + \Big(\frac{m-n}{n+1}\Big)\sup_k p(kx_k\delta^k).$$

Let j be any positive integer and choose N so that $N>2^j$ and $p(\sigma^m x - \sigma^n x) < 4^{-j}$ whenever $m\geqslant n > N$. Let m be the smallest positive integer for which $\Big(\frac{n+1}{m-n}\Big) < 2^j$. Then $m>n$ and $\Big(\frac{n+1}{m-n}\Big)\leqslant 2^j < \Big(\frac{n+1}{m-n-1}\Big)$ and hence

$$p(S^n x - \sigma^n x)\leqslant(1+2^j)4^{-j} + 2^{-j+1}\sup_k p(kx_k\delta^k) < 2^{-j+1}\big(1+\sup_k p(kx_k\delta^k)\big).\ ∎$$

THEOREM 9 (Littlewood's Tauberian Theorem for Abel means). *Let E be a sequentially complete* K*-space containing φ. Suppose a sequence x in E can be approximated by its Abel means and $\{nx_n\delta^n\}_{n=0}^{\infty}$ is a bounded subset of E. Then x has sectional convergence.*

Proof. By Theorem 7 and Lemma 1 it is sufficient to show that x has bounded sections. Let p be a continuous seminorm on E, let $r = \Big(\frac{n}{n+1}\Big)$ and let $M = \sup p(kx_k\delta^k)$. Then

$$p(S^n x)\leqslant p(S^n x - S^n A^r x) + p(S^n A^r x)$$
$$\leqslant\sum_{k=0}^{n}(1-r^k)p(x_k\delta^k) + \sum_{k=0}^{n} r^k p(x_k\delta^k)$$
$$\leqslant\frac{1}{n+1}\sum_{k=0}^{n}\Big(\frac{1-r^k}{1-r}\Big)p(x_k\delta^k) + \frac{1}{n+1}\sum_{k=0}^{n} r^k M$$
$$= \frac{1}{n+1}\sum_{k=0}^{n}(1+r+r^2+\ldots+r^{k-1})p(x_k\delta^k) + M(1-r^{n+1})$$
$$\leqslant\frac{1}{n+1}\sum_{k=0}^{n} kp(x_k\delta^k) + M\leqslant 2M.\ ∎$$

COROLLARY. *Let E be a sequentially complete* K*-space containing φ. Suppose a sequence x in E can be approximated by its Abel means and $x_k\lambda_k = O(k^{-1})$ for every $\lambda\in(E\to\mathscr{A})$. Then x has sectional convergence.*

The proof follows from Theorem 9 using the equivalence of boundedness and weak boundedness.

LEMMA 2 (Lacunary Tauberian Theorem for Cesàro sections). *Let E be a* K*-space containing φ. Suppose that $x\in E$ has C_1-sectional convergence.*

If $x_n = 0$ except perhaps for $n = n_k$, where $(n_{k+1}/n_k) \geqslant r > 1$ for $k = 0, 1, 2, \ldots$, then x has the property AK.

Proof (cf. [24], vol. I, p. 79). Since $(n_{k+1}-n_k)S^{n_k}x = n_{k+1}\sigma^{n_{k+1}-1}x - n_k\sigma^{n_k-1}x$, we have for each continuous seminorm p,

$$p(S^{n_k}x-x) \leqslant \frac{n_{k+1}}{n_{k+1}-n_k}p(\sigma^{n_{k+1}}x-x)+\frac{n_k}{n_{k+1}-n_k}p(\sigma^{n_k-1}x-x)$$
$$\leqslant \frac{1}{r-1}p(\sigma^{n_k-1}x-x)+\frac{1}{r-1}p(\sigma^{n_k-1}x-x)$$

which can be made arbitrarily small for sufficiently large n_k. ■

THEOREM 10 (Lacunary Tauberian Theorem for Abel means). *Let E be a sequentially complete* K*-space containing φ. Suppose that $x \in E$ can be approximated by Abel means and $x_n = 0$ except perhaps for $n = n_k$, where $(n_{k+1}/n_k) \geqslant r > 1$ for $k = 0, 1, 2, \ldots$ Then x has sectional convergence.*

Proof. By Theorem 7 and Lemma 2 it is sufficient to show that x has bounded sections. Let f be a continuous linear functional on E and let $\lambda_j = f(\delta^j)$. Clearly, $\lambda\cdot x$ is a lacunary sequence in $\mathscr{A}$. By the gap Tauberian Theorem of Hardy and Littlewood [7], $f(S^n x) = \sum_{k=0}^{n}\lambda_k x_k$ converges as $n\to\infty$. In particular the set $\{S^n x\}$ is weakly bounded and therefore bounded. ■

5. An application of Abel means to Absolute Tauberian Conditions. For each sequence $\lambda = (\lambda_k)$ and $a \geqslant 0$, let

$$\Delta^a\lambda_k = \sum_{j=k}\binom{j-k-a-1}{j-k}\lambda_j.$$

Then $\Delta^0\lambda_k = \lambda_k$ and $\Delta^{n+1}\lambda_k = \Delta^n\lambda_k - \Delta^n\lambda_{k+1}$. The sequence λ is *monotonic of order* a if $\Delta^a\lambda_k \geqslant 0$ for all $k = 0, 1, 2, \ldots$ and it is *of bounded variation of order* a if it is the difference of two bounded monotonic sequences of order a. The set of sequences of bounded variation of order a form a BK-space which we denote by bv^a. We write $\mathrm{bv} = \mathrm{bv}^1$. If $a \leqslant \beta$, then $\mathrm{bv}^\beta \subset \mathrm{bv}^a$ ([12], [13]).

We call a sequence *fully monotonic* if it is monotonic of all orders $n = 1, 2, 3, \ldots$ It is *quasi-fully monotonic* if it is the difference of two bounded fully monotonic sequences. The space of quasi-fully monotonic sequences is denoted by bv^∞. We have

$$\mathrm{bv}^\infty \subsetneqq \bigcap_{n=1}^{\infty}\mathrm{bv}^n.$$

For each sequence $x = (x_k)$, let $d_n = \frac{1}{n+1}\sum_{k=0}^{n}kx_k$ and $d = (d_n)$. For each sequence space E, let $\int E = \left\{\frac{1}{k}x_k : x \in E\right\}$.

Hyslop [8] has shown that the condition $d \in \mathrm{bv}$ is an Absolute Tauberian Condition (ATC) for the Abel method. That is, if the Abel transform of a sequence x is of bounded variation and $d \in \mathrm{bv}$, then $x \in l^1$. Tietz [17] has shown that if V is an absolutely permanent and additive summability method, then "$d \in \mathrm{bv}$" is an ATC for V if and only if "$x \in \int \mathrm{bv}$" is an ATC for V. Goes [5] has further shown the equivalence of "$d \in \mathrm{bv}$", "$x \in \int \mathrm{bv}$" and "$x \in \int \mathrm{bv}^2$" as ATC. We extend these results.

THEOREM 11. *Let V be an absolutely permanent and additive summability method and let a be any number such that $1 \leqslant a \leqslant \infty$. Then "$d \in \mathrm{bv}$" is an* ATC *for V if and only if "$x \in \int \mathrm{bv}^a$" is an* ATC *for V.*

Proof. Since $\mathrm{bv}^\infty \subset \mathrm{bv}^a \subset \mathrm{bv}$, for $1 \leqslant a \leqslant \infty$, and since the result of Tietz is the case $a = 1$, it is sufficient to show that if "$x \in \int \mathrm{bv}^\infty$" is an ATC for V, then "$x \in \int \mathrm{bv}$" is an ATC for V. By the 0-Tauberian Theorem of Littlewood,

$$\mathscr{A} \cap \int l^\infty \subset \mathrm{cs} = \left\{x\colon \sum_k x_k \text{ exists}\right\}.$$

Thus $(\mathrm{cs}\to\mathscr{A}) \subset \left((\mathscr{A}\cap\int l^\infty)\to\mathscr{A}\right)$. We have $(\mathrm{cs}\to\mathscr{A}) = \mathrm{cs}^{\mathscr{S}} = \mathrm{bv}$ since cs has the property AK [4]. By Proposition 3, it follows that $\left((\mathscr{A}\cap\int l^\infty)\to\mathscr{A}\right) = (\mathscr{A}\cap\int l^\infty)^{\mathscr{S}}$. By [14] (Theorem 3.1), $(\mathscr{A}\cap\int l^\infty)^{\mathscr{S}} = \mathscr{A}^{\mathscr{S}}+(\int l^\infty)^{\mathscr{S}}$. Thus

$$\mathrm{bv} \subset \mathscr{A}^{\mathscr{S}}+\left(\int l^\infty\right)^{\mathscr{S}} = (\mathscr{A}\to\mathscr{A})+\left(\int l^\infty\to\mathscr{A}\right).$$

Every element of $(\mathscr{A}\to\mathscr{A})$ is of the form $x+y$, where $x \in \mathrm{bv}^\infty$ and $y_k = O(r^k)$ for $0<r<1$ [19], [23]. Every element of $\left(\int l^\infty\to\mathscr{A}\right)$ is of the form (kz_k), where $z \in l^1$ since by Theorem 3 and [5], Beispiele 1.19, we have

$$(l^\infty\to\mathscr{A}) = (l^\infty)^{\mathscr{S}} = (l^\infty\to\mathrm{cs}) = l^1.$$

Thus every element λ of $\int \mathrm{bv}$ is of the form $\lambda = x+w$, where $x \in \int \mathrm{bv}^\infty$ and $w \in l^1$. If λ is absolutely V summable, then x is absolutely V summable since V is absolutely permanent. If "$x \in \int \mathrm{bv}^\infty$" is an ATC for V, then $\lambda = x+w \in l^1$. Then "$x \in \int \mathrm{bv}$" is an ATC for V. ■

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Singular integrals supported on submanifolds

by

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Abstract. Three results related to Calderón-Zygmund singular integrals in generalized contexts are established. In the first, Hilbert transforms supported on hypersurfaces of the form $\prod |x_j|^{b_j} = r$ where b_j are non-zero real numbers, are proved bounded on L^p for certain values of p (Nagel and Wainger proved L^2 boundedness). In the second, L^p boundedness is established for convolutions on $\mathbf{R}^2$ with kernels of the form $g(|x_1x_2|^{1/2})\operatorname{sgn} x_1$ by transference from known results about radial kernels. In the third, the "method of rotations" is carried through for the L^2 theory of Knapp–Stein singular integrals on 2-stage nilpotent Lie groups.

§ 1. Introduction. The Calderón–Zygmund theory of singular integrals has been extended in many directions in recent years. One major theme in these extensions is the "method of rotations" introduced in [1]. This leads to the study of various Hilbert transforms supported on curves or more general submanifolds. The reader is urged to consult the excellent exposition of these ideas in Stein and Wainger [10].

In this paper we present three contributions to this study. The first concerns Hilbert transforms supported on hypersurfaces in $\mathbf{R}^n$ defined by the equation $\prod_{j=1}^{n} |x_j|^{b_j} = r$ where $b_1, \ldots, b_n$ are non-zero real numbers. The L^2 boundedness of these operators was established by Nagel and Wainger [7]. We give an independent proof that also establishes L^p boundedness for certain values of p. We do not know if these values of p are best possible.

Our second contribution is a transference result relating convolutions on $\mathbf{R}^2$ with functions of the form $g(|x_1x_2|^{1/2})\operatorname{sgn} x_1$ to convolutions with the radial function $g(\sqrt{x_1^2+x_2^2})$. We show that L^2-boundedness is equivalent for these two operators, and the applicability of the Marcinkiewicz multiplier theorem is also equivalent.

Our third contribution is to the Knapp–Stein [5] theory of singular integrals on nilpotent Lie groups. For 2-stage groups we carry out the method of rotations for L^2-boundedness. We use the Euclidean Plancherel

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