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MATEMATISK INSTITUT, ODENSE UNIVERSITET
DK-5230 Odense M., Denmark
and
DEPARTMENT OF MATHEMATICS
OHIO STATE UNIVERSITY
Columbus, OH 43210 U.S.A.

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On weak restricted estimates and endpoints problems for convolutions with oscillating kernels (II)

by

W. B. JURKAT* and G. SAMPSON (Syracuse, N.Y.)

Abstract. Throughout we consider $K(t) = e^{i|t|^a}/|t|^b$, $a > 0$, $a \neq 1$, $b < 1$ and $t \in \boldsymbol{R}$. Here we consider for fixed λ, μ the function

$$M(\lambda, \mu;\ K) = M(\lambda, \mu) = \sup_{\chi_\mu} \left|\{x\colon\ |K * \chi_\mu(x)| > \lambda\}\right|$$

over all "characteristic" functions χ_μ with complex signs (i.e. χ_μ is a measurable function for which $|\chi_\mu| = 1$ on E, $|\chi_\mu| = 0$ off E and $|E| \leqslant \mu$ $(\mu > 0)$). We first note that

$$|K * \chi_\mu| \leqslant \int_{-\mu}^{\mu} \frac{dt}{|t|^b} = c\mu^{1-b},$$

and so if $c\mu^{1-b} < \lambda$ then $M(\lambda, \mu) = 0$. And so we assume throughout that $\lambda \leqslant c_1\mu^{1-b}$ for some constant c_1, independent of λ and μ (but may depend on K). Under these conditions ($\lambda \leqslant c_1\mu^{1-b}$) we estimate $M(\lambda, \mu)$ within constant factors from above and below.

This paper is a continuation of part I [3] where we estimated the function $B(\lambda, \mu)$ within constant factors from above and below. For fixed $\lambda, \mu > 0$ we set

$$B(\lambda, \mu) = \sup_{\chi_\lambda, \chi_\mu} \int \chi_\lambda(x) K * \chi_\mu(x)\, dx,$$

where the sup is taken over all "characteristic" functions χ_λ, χ_μ with complex signs.

§ 1. An interpolation theorem with respect to the kernel. Here we prove the analogue for M of the corresponding theorem for B given in part I. Again we consider a decomposition of the kernel

$$K = K_1 + K_2$$

and make use of the decreasing rearrangement K^* of K (if it exists), so that

$$\sup_{\chi_\mu} \left|\int K(x)\chi_\mu(x)\, dx\right| = \int_0^\mu K^*(t)\, dt \qquad (x \in \boldsymbol{R}^n,\ t \in \boldsymbol{R}).$$

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We also make use of the (distributional) Fourier transform

$$\hat{K}(x) = \int e^{2\pi i x\cdot y} K(y)\,dy.$$

Now let $m_1(\lambda, \mu) = \sup m$, where $m > 0$ and

$$K_1^{**}(m) = \frac{1}{m}\int_0^m K_1^*(t)\,dt > \lambda/2\mu;$$

if no such m exists set $m_1 = 0$. In our applications m_1 can easily be determined explicitly. See also the remark given later.

THEOREM 1. *If K_1^* exists and $\hat{K}_2$ is bounded then we have the following estimates*:

$$M(\lambda, \mu) \leqslant 4\mu\lambda^{-2}\|\hat{K}_2\|_\infty^2 \quad \textit{if} \quad \int_0^\mu K_1^*(t)\,dt \leqslant \lambda/2,$$

and

$$M(\lambda, \mu) \leqslant 4\mu\lambda^{-2}\|\hat{K}_2\|_\infty^2 + m_1(\lambda, \mu)$$

generally.

Furthermore,

$$M(\lambda, \mu) = 0 \quad \textit{if} \quad \int_0^\mu K^*(t)\,dt \leqslant \lambda.$$

Proof. Consider,

$$E = \{x\colon |K*\chi_\mu(x)| > \lambda\}$$

and note that $|K*\chi_\mu| \leqslant \int_0^\mu K^*(t)\,dt$ and if $\int_0^\mu K^*(t)\,dt \leqslant \lambda$ then $E = \emptyset$ and $|E| = 0$. Also note that

$$\begin{aligned} E &= \{|K*\chi_\mu| > \lambda\} \subseteq \{|K_1*\chi_\mu| > \lambda/2\} \cup \{|K_2*\chi_\mu| > \lambda/2\} \\ &= E_1 \cup E_2, \end{aligned}$$

and we estimate $|E_2|$ noting that

$$\|K_2*\chi_\mu\|_2^2 \leqslant \|\hat{K}_2\|_\infty^2 \mu.$$

And since

$$\|K_1*\chi_\mu\|_\infty \leqslant \int_0^\mu K_1^*(t)\,dt$$

so that E_1 must be empty if $\int_0^\mu K_1^*(t)\,dt \leqslant \lambda/2$. Now in case $|E_1| > 0$, say $|E_1| = m$, we see that

$$\begin{aligned} m(\lambda/2) &< \int_{E_1} |K_1*\chi_\mu| = \left|\int \chi_{E_1}(K_1*\chi_\mu)\right| \\ &\leqslant \int dy\,|\chi_\mu(y)| \int_0^m dt\,K_1^*(t) \end{aligned}$$

hence $\lambda/2\mu < K_1^{**}(m)$ and $m \leqslant m_1(\lambda, \mu)$. This completes the proof.

To clarify the meaning of m_1 and also to interrelate M with B we make the following observations.

Remark 1. *We have the following estimates for $1 < q < \infty$, $1/q + 1/q' = 1$:*

$$m_1 \leqslant 2\frac{\mu}{\lambda}\|K_1\|_1, \qquad m_1^{1/q} \leqslant 2q'\frac{\mu}{\lambda}\|K_1\|_q^{\#},$$

where

$$\|K_1\|_q^{\#} = \sup_{\lambda>0} \lambda |\{x\colon |K_1(x)| > \lambda\}|^{1/q} = \sup_{t>0} t^{1/q} K_1^*(t).$$

Proof. Use the upper estimates

$$\int_0^m K_1^*(t)\,dt \leqslant \|K_1\|_1, \qquad \int_0^m K_1^*(t)\,dt \leqslant \Big(\int_0^m t^{-1/q}\,dt\Big) \sup_{t>0} t^{1/q} K_1^*(t)$$

and compare with the lower estimate $m\lambda/2\mu$.

Remark 2. *We have $B(m, \mu) \geqslant m\lambda$ whenever $0 < m < M(\lambda, \mu)$.*

Proof. Since there is a set $E \subseteq \{|K*\chi_\mu| > \lambda\}$ with $|E| = m$, the result follows by integrating.

Our examples will show that the "right" decomposition of K in Theorem 1 gives sharp estimates for all λ, μ. On the other hand, the same estimates also follow from those for B using Remark 2.

§ 2. Upper estimates for M. From now on (as in part I) we consider the kernels

$$K(t) = \frac{e^{i|t|^a}}{|t|^b} \quad (0 \neq t \in R) \quad \text{with} \quad a > 0,\ a \neq 1,\ b < 1.$$

By $c_1, c_2, \ldots$ we denote suitable positive constants which depend only on a and b and we use c generically. We again consider (as in part I) $K_w(t) = K(t)$ for $|t| \leqslant w$, $K^w = K$ for $|t| \geqslant w$, $K_{u,v} = K$ for $u \leqslant |t| \leqslant v$ and $K_w = 0$, $K^w = 0$, $K_{u,v} = 0$ elsewhere. Thus K_1 and K_2 are made up by choosing one to be K_w and the other one being K^w. Accordingly, we distinguish between the four cases

(I) $b \geq 1-a/2,\ b \geq 0$,

(II) $b \geq 1-a/2,\ b < 0$ (implies $a > 2$),

(III) $b < 1-a/2,\ b > 0$ (implies $a < 2$),

(IV) $b < 1-a/2,\ b \leq 0$.

Let me begin by defining the function $\tilde{M}(\lambda, \mu)$. In case (I) for $M(\lambda, \mu) > 0$ we set

(1) $$\tilde{M} = \min\left(\mu(1/\lambda)^{a/(1-b)},\ \mu(1/\lambda)^{a/(a+b-1)},\ (\mu/\lambda)^{1/b}\right), \quad b > 0$$

$$\text{and} \quad \tilde{M} = \mu\lambda^{-2},\ b = 0 \text{ and } a = 2.$$

In case (II) for $M(\lambda, \mu) > 0$ we set

(2) $$\tilde{M} = \min\left(\mu(1/\lambda)^{a/(1-b)},\ \mu(1/\lambda)^{a/(a+b-1)}\right).$$

In case (III) for $M(\lambda, \mu) > 0$ we set

(3) $$\tilde{M} = \min\left(\mu^{(2-a-b)/b}\lambda^{(a-2)/b},\ (\mu/\lambda)^{1/b}\right).$$

And in case (IV) for $M(\lambda, \mu) > 0$ we set $\tilde{M} = \infty$.

THEOREM 2. *If $M(\lambda, \mu) > 0$ then $M(\lambda, \mu) \leq c\tilde{M}(\lambda, \mu)$ for some positive constant c.*

Proof. The most direct way to do Theorem 2 is to use Remark 2 and our estimates in part I for $B(\lambda, \mu)$. In [3], Theorem 2, we proved that $B(\lambda, \mu) \leq c\tilde{B}(\lambda, \mu)$ where $\tilde{B}$ was defined as follows.

In case (I):

$$\tilde{B}(\lambda, \mu) = \min(\lambda\mu^{1-b},\ \mu\lambda^{1-b},\ \lambda^{(a+b-1)/a}\mu^{(1-b)/a},\ \mu^{(a+b-1)a}\ \lambda^{(1-b)/a}).$$

In case (II):

$$\tilde{B}(\lambda, \mu) = \min\left((\lambda\mu)^{(a+b-2)/(a-2)},\ \lambda^{(a+b-1)/a}\mu^{(1-b)/a},\ \mu^{(a+b-1)/a}\lambda^{(1-b)/a}\right).$$

In case (III):

$$\tilde{B}(\lambda, \mu) = \min\left(\lambda\mu^{1-b},\ \mu\lambda^{1-b},\ (\lambda\mu)^{(2-a-b)/(2-a)}\right).$$

To do case (I) we have for $0 < m < M(\lambda, \mu)$ (using Remark 2) that

$$m \leq \frac{1}{\lambda} B(m, \mu) \leq \frac{c}{\lambda}\tilde{B}(m, \mu)$$

$$\leq \frac{c}{\lambda}\min(m\mu^{1-b},\ \mu m^{1-b},\ m^{(a+b-1)/a}\mu^{(1-b)/a}\ \mu^{(a+b-1)/a}\ m^{(1-b)/a});$$

hence $\lambda \leq c\mu^{1-b}$, $m^b \leq (c/\lambda)\mu$, $m^{(1-b)/a} \leq (c/\lambda)\mu^{(1-b)/a}$, $m^{(a+b-1)/a} \leq (c/\lambda)\mu^{(a+b-1)/a}$. Thus, for $b > 0$ we get $M(\lambda, \mu) \leq c\tilde{M}(\lambda, \mu)$. While for $b = 0$ $(a = 2)$ we get our result and hence the proof of case (I) is complete.

To do case (II) we note that for $0 < m < M(\lambda, \mu)$ we get

$$m \leq \frac{c}{\lambda}\min\left((m\mu)^{(a+b-2)/(a-2)},\ m^{(a+b-1)/a}\mu^{(1-b)/a},\ \mu^{(a+b-1)/a}\ m^{(1-b)/a}\right)$$

and so it follows that

$$M(\lambda, \mu) \leq c\lambda^{(a-2)/b}\mu^{(2-a-b)/b}, \quad M(\lambda, \mu) \leq c\lambda^{a/(b-1)}\mu, \quad M(\lambda, \mu) \leq c\lambda^{a/(1-a-b)}\mu.$$

But note that $\lambda^{(a-2)/b}\mu^{(2-a-b)/b} \leq c^{-1}\mu\lambda^{a/(b-1)}$ implies that $\mu^{1-b} \leq c^{-1}\lambda$, but in this case

$$\int_0^{\mu} K^*(t)\,dt \leq c\mu^{1-b} \leq \lambda$$

and hence $M(\lambda, \mu) = 0$. And hence $M(\lambda, \mu) \leq c\tilde{M}(\lambda, \mu)$ and now case (II) is complete.

Now for case (III), we note for $0 < m < M(\lambda, \mu)$ we get

$$m \leq \frac{c}{\lambda}\min\left(m\mu^{1-b},\ \mu m^{1-b},\ (m\mu)^{(2-a-b)/(2-a)}\right)$$

and the proof of case (III) is complete.

§ 3. Lower estimates for M. From the Proposition of part I one gets directly the following:

COROLLARY. *Assume $0 < \delta \leq \mu$, $T \geq \delta$, $\lambda \leq c_1\delta T^{-b}$. Then*

$$M(\lambda, \mu) \geq c_2\min(T,\ T^{2-a}\delta^{-1});$$

furthermore, in case $a+b > 1$, even the stronger estimate

$$M(\lambda, \mu) \geq c_2\mu\delta^{-1}\min(T,\ T^{2-a}\delta^{-1})$$

holds.

THEOREM 3. *If $\mu^{1-b} \geq c_1\lambda$, then $M(\lambda, \mu) \geq c_2\tilde{M}(\lambda, \mu)$, for some constant c_2 independent of λ and μ.*

Proof. Let us begin with case (I). Now from (1) it follows (we assume $\lambda \leq \mu^{1-b}$)

$$\tilde{M}(\lambda, \mu) = \mu(1/\lambda)^{a/(1-b)} \quad \text{if} \quad \lambda^{(a-1)/(a+b-1)} \leq \mu \text{ and } \lambda \geq 1,$$

$$\tilde{M}(\lambda, \mu) = \mu(1/\lambda)^{a/(a+b-1)} \quad \text{if} \quad \lambda^{(a-1)/(a+b-1)} \leq \mu \text{ and } \lambda \leq 1,$$

and

$$\tilde{M}(\lambda, \mu) = (\mu/\lambda)^{1/b} \quad \text{if} \quad \lambda^{(a-1)/(a+b-1)} \geq \mu \text{ and } \lambda \leq 1 \ (b > 0).$$

Let me add that the case where $\lambda^{(a-1)/(a+b-1)} \geq \mu$ and $\lambda \geq 1$ never comes up. Since

$$\lambda^{(a-1)/(a+b-1)} \geq \mu$$

implies

$$(\mu/\lambda)^{1/b} \leqslant (1/\lambda)^{1/(a+b-1)}$$

but for $\lambda \geqslant 1$ that implies

$$(\mu/\lambda)^{1/b} \leqslant (1/\lambda)^{1/(a+b-1)} \leqslant \lambda^{1/(1-b)}$$

and that contradicts the assumption that $\lambda \leqslant \mu^{1-b}$.

In this case $a+b > 1$ and hence from the Corollary we get $c_3\lambda T \leqslant \delta \leqslant \min(T, \mu)$ and

$$M(\lambda, \mu) \geqslant c_2\mu\delta^{-1}\min(T, T^{2-a}\delta^{-1})$$

for some constants c_2, c_3. We wish to take δ as small as possible (in order to maximize $M(\lambda, \mu)$) and so we choose $\delta = c^{-1}\lambda T^b$ with c to be determined. Thus,

$$M(\lambda, \mu) \geqslant cc_2(\mu/\lambda)T^{-b}\min(T, c\lambda^{-1}T^{2-a-b}). \tag{4}$$

In order for such a δ to exist we need that

$$c_3\lambda T^b \leqslant \min(T, \mu) \quad \text{or} \quad \lambda \leqslant c_2 T^{1-b} \quad \text{and} \quad T^b \leqslant c_2(\mu/\lambda).$$

Hence for $b > 0$ we get

$$c_3\lambda^{1/(1-b)} \leqslant T \leqslant c_2(\mu/\lambda)^{1/b}, \tag{5}$$

thus in order for such a T to exist we need $\lambda \leqslant c_4\mu^{1-b}$ for some constant c_4.

In estimating $M(\lambda, \mu)$ from (4) we have two cases to consider $T \geqslant c\lambda^{-1}T^{2-a-b}$ or $T < c\lambda^{-1}T^{2-a-b}$. Let us first assume $T \geqslant c\lambda^{-1}T^{2-a-b}$ and that implies

$$T \geqslant c(1/\lambda)^{1/(a+b-1)} \tag{6}$$

(note $a/2+b-1 \geqslant 0$ here). So in this case we have that

$$c(1/\lambda)^{1/(a+b-1)} \leqslant c_2(\mu/\lambda)^{1/b}$$

and that implies that

$$\mu \geqslant cc_3\lambda^{(a-1)/(a+b-1)} \quad (b > 0).$$

Hence from (4) we get that

$$M(\lambda, \mu) \geqslant \frac{cc_2\mu\lambda^{-2}}{T^{2b+a-2}} \quad (2b+a-2 \geqslant 0). \tag{7}$$

Now in the cases where $2b+a-2 = 0$ and $b > 0$, then from (7) and since

$$\mu \geqslant cc_3\lambda^{(a-1)/(a+b-1)}$$

these cases are completed.

Now we are left with the cases where $2b+a-2 > 0$. And we wish to select the smallest admissible T. So it follows from (5) and (6) that

$$c(1/\lambda)^{1/(a+b-1)} \leqslant c_2(\mu/\lambda)^{1/b}.$$

Now if $\lambda \leqslant 1$ we choose

$$T = c(1/\lambda)^{1/(a+b-1)}$$

and hence from (7) we get

$$M(\lambda, \mu) \geqslant c\frac{\mu\lambda^{-2}}{(1/\lambda)^{(2b+a-2)/(a+b-1)}} = c\mu(1/\lambda)^{a/(a+b-1)}$$

for $\lambda \leqslant c_4\mu^{1-b}$, $\lambda^{(a-1)/(a+b-1)} \leqslant \mu$ and $\lambda \leqslant 1$. Now if $\lambda \geqslant 1$ we choose $T = c\lambda^{1/(1-b)}$ and hence we get

$$M(\lambda, \mu) \geqslant \frac{c\mu\lambda^{-2}}{\lambda^{(2b+a-2)/(1-b)}} = c\mu(1/\lambda)^{a/(1-b)},$$

for $\lambda \leqslant c_4\mu^{1-b}$, $\lambda^{(a-1)/(a+b-1)} \leqslant \mu$ and $\lambda \geqslant 1$.

On the other hand we could have that $T \leqslant c\lambda^{-1}T^{2-a-b}$ and hence

$$T \leqslant (c/\lambda)^{1/(a+b-1)}.$$

But by (5) in order for a T to exist we need $\lambda \leqslant 1$. And since we already did the case where

$$\lambda^{(a-1)/(a+b-1)} \leqslant \mu,$$

we can assume here that

$$\lambda^{(a-1)/(a+b-1)} \geqslant \mu$$

that implies

$$(1/\lambda)^{1/(a+b-1)} \geqslant (\mu/\lambda)^{1/b}.$$

And we can select the largest admissible T, i.e. $T = c(\mu/\lambda)^{1/b}$, then from (4)

$$M(\lambda, \mu) \geqslant c(\mu/\lambda)(\mu/\lambda)^{(1-b)/b} = c(\mu/\lambda)^{1/b},$$

for $\lambda \leqslant c_4\mu^{1-b}$, $\lambda^{(a-1)/(a+b-1)} \geqslant \mu$ and $\lambda \leqslant 1$.

Now to do the case where $b = 0$ and $a = 2$. Now we need that $c_4\mu \geqslant \lambda$ and $T \geqslant c_3\lambda$. And we get from (4) that

$$M(\lambda, \mu) \geqslant c(\mu/\lambda)\min(T, c\lambda^{-1})$$

and we choose $T = c\lambda^{-1}$ and hence

$$M(\lambda, \mu) \geqslant c\mu\lambda^{-2}$$

for $\lambda \leqslant c_4\mu$ and $\lambda \leqslant 1$.

Now we do case (II). Hence from (2) we get

$$\tilde{M}(\lambda, \mu) = \begin{cases} \mu(1/\lambda)^{a/(1-b)} & \text{if} \quad \lambda \geqslant 1, \\ \mu(1/\lambda)^{a/(a+b-1)} & \text{if} \quad \lambda \leqslant 1. \end{cases}$$

And from (4) we get that

$$M(\lambda,\mu)\geqslant cc_2(\mu/\lambda)T^{-b}\min(T,\,c\lambda^{-1}T^{2-a-b})$$

as long as $\lambda\leqslant c_2T^{1-b}$ and $T^b\leqslant c_3(\mu/\lambda)$. Thus, $T\geqslant(\lambda/c_2)^{1/(1-b)}$ and $T\geqslant(\lambda/c_3\mu)^{-1/b}$. And since $\lambda\leqslant c_4\mu^{1-b}$, that implies

$$(\lambda/c_2)^{1/(1-b)}\geqslant c_5(\lambda/c_3)^{-1/b}$$

for some constant c_5. If we first assume that $\lambda\geqslant 1$ and choose $T=c\lambda^{1/(1-b)}$ then

$$\begin{aligned}M(\lambda,\mu)&\geqslant c(\mu/\lambda)\lambda^{-b/(1-b)}\min(c\lambda^{1/(1-b)},\,c\lambda^{-1}\lambda^{(2-a-b)/(1-b)})\\&=c(\mu/\lambda)\lambda^{-b/(1-b)}\lambda^{-1}\lambda^{(2-a-b)/(1-b)}=c\mu\lambda^{-a/(1-b)}.\end{aligned}$$

Now if $\lambda<1$ then we can choose $T=c(1/\lambda)^{1/(a+b-1)}$ and hence

$$M(\lambda,\mu)\geqslant c(\mu/\lambda)(1/\lambda)^{-b/(a+b-1)}\min(\lambda^{-1/(a+b-1)},\,\lambda^{-1/a+b-1)})=c\mu\lambda^{-a/(a+b-1)}.$$

Now this case is finished.

Now to do case (III). Hence from (3) it follows (we assume $\lambda\leqslant\mu^{1-b}$), when $a+b>1$:

$$\tilde M(\lambda,\mu)=\begin{cases}\mu^{(2-a-b)/b}\lambda^{(a-2)/b} & \text{if}\quad \lambda^{(a-1)/(a+b-1)}\leqslant\mu,\\ (\mu/\lambda)^{1/b} & \text{if}\quad \mu\leqslant\lambda^{(a-1)/(a+b-1)},\ \lambda\leqslant 1,\end{cases}$$

when $a+b<1$:

$$\tilde M(\lambda,\mu)=\begin{cases}(\mu/\lambda)^{1/b} & \text{if}\quad \lambda^{(a-1)/(a+b-1)}\leqslant\mu,\\ \mu^{(2-a-b)/b}\lambda^{(a-2)/b} & \text{if}\quad \mu\leqslant\lambda^{(a-1)/(a+b-1)},\ \lambda\leqslant 1,\end{cases}$$

when $a+b=1$:

$$\tilde M(\lambda,\mu)=\min\big((\mu/\lambda)^{1/b},\,\lambda^{-1}(\mu/\lambda)^{1/b}\big).$$

Note as we argued earlier the case where $\lambda^{(a-1)/(a+b-1)}\geqslant\mu$ and $\lambda\geqslant 1$ never comes up.

First we assume that $a+b>1$. And so both (4) and (5) hold in this case. Now suppose we first assume that $T\geqslant c\lambda^{-1}T^{2-a-b}$ and hence that implies (6) (i.e. $T\geqslant c(1/\lambda)^{1/(a+b-1)}$). And so in order for a T to exist we need from (5) and (6) that $c(1/\lambda)^{1/(a+b-1)}\leqslant c_2(\mu/\lambda)^{1/b}$. Thus the largest T admissible in this case would be $T=c_2(\mu/\lambda)^{1/b}$ and hence

$$M(\lambda,\mu)\geqslant c\mu\lambda^{-2}(\mu/\lambda)^{(2-a-2b)/b}=c\mu^{(2-a-b)/b}\lambda^{(a-2)/b}$$

if $\lambda^{(a-1)/(a+b-1)}\leqslant\mu$ and $\lambda\leqslant c_4\mu^{1-b}$. Otherwise, $T\leqslant c\lambda^{-1}T^{2-a-b}$ and hence this implies that $T\leqslant c(1/\lambda)^{1/(a+b-1)}$ and so from (5) in order for a T to exist we need that $c(1/\lambda)^{1/(a+b-1)}\geqslant c_2(\mu/\lambda)^{1/b}$. And so here again we choose $T=c_2(\mu/\lambda)^{1/b}$ and we get that

$$M(\lambda,\mu)\geqslant c(\mu/\lambda)T^{1-b}=c(\mu/\lambda)(\mu/\lambda)^{(1-b)/b}=c(\mu/\lambda)^{1/b}$$

if $\lambda^{(a-1)/(a+b-1)}\geqslant\mu$, $\lambda\leqslant c_4\mu^{1-b}$ and $\lambda\leqslant 1$.

Now to complete case (III) with $a+b\leqslant 1$. From the Corollary it follows that

$$M(\lambda,\mu)\geqslant c\min(T,\,c\lambda^{-1}T^{2-a-b})\tag{8}$$

and once again (5) holds. First we assume $T\leqslant cT^{2-a-b}\lambda^{-1}$. Hence $T\geqslant c\lambda^{1/(1-a-b)}$ $(a+b<1)$ and also $M(\lambda,\mu)\geqslant cT$. And now by (5) in order for a T to exist we need that $c\lambda^{1/(1-a-b)}\leqslant c_2(\mu/\lambda)^{1/b}$ and so we choose $T=c_3(\mu/\lambda)^{1/b}$ the largest admissible T and hence

$$M(\lambda,\mu)\geqslant c(\mu/\lambda)^{1/b}$$

for $\lambda^{(a-1)/(a+b-1)}\leqslant\mu$ and $\lambda\leqslant c_4\mu^{1-b}$. Next we assume that $T\geqslant cT^{2-a-b}\lambda^{-1}$ and hence $T\leqslant c\lambda^{1/(1-a-b)}(a+b<1)$. And in order for a T to exist here we need from (5) that $c_3\lambda^{1/(1-b)}\leqslant c\lambda^{1/(1-a-b)}$ and hence $\lambda\leqslant 1$. Now we are left with the case where $\mu\leqslant\lambda^{(a-1)/(a+b-1)}$ and so that implies $\lambda^{1/(1-a-b)}\geqslant(\mu/\lambda)^{1/b}$ and so we select $T=c(\mu/\lambda)^{1/b}$ and hence from (8) we get,

$$M(\lambda,\mu)\geqslant c\lambda^{-1}(\mu/\lambda)^{(2-a-b)/b}=c\mu^{(2-a-b)/b}\lambda^{(a-2)/b}$$

for $\mu\leqslant\lambda^{(a-1)/(a+b-1)}$, $\lambda\leqslant c_4\mu^{1-b}$ and $\lambda\leqslant 1$.

Now to do the case where $a+b=1$. And so from (5) and (8) we get with $T=c(\mu/\lambda)^{1/b}$ that

$$M(\lambda,\mu)\geqslant c\begin{cases}(\mu/\lambda)^{1/b} & \text{if}\quad \lambda\leqslant 1,\\ \lambda^{-1}(\mu/\lambda)^{1/b} & \text{if}\quad \lambda\geqslant 1.\end{cases}$$

And now the proof for case (III) is complete.

Now to do case (IV). We begin with the case where $a+b\leqslant 1$. Now once again (8) holds but since $b<0$ this forces $T\geqslant c(\lambda/\mu)^{-1/b}$ and $T\geqslant c\lambda^{1/(1-b)}$. And so, since $M(\lambda,\mu)\geqslant c_2T\min(1,\,\lambda^{-1}T^{1-a-b})$ on letting $T\to\infty$, which is admissible here, we get that $M(\lambda,\mu)=\infty$.

Now for the case where $a+b>1$ we get that

$$M(\lambda,\mu)\geqslant c_2\mu\lambda^{-1}T^{-b}\min(T,\,c\lambda^{-1}T^{2-a-b})$$

and once again $T\geqslant c\lambda^{1/(1-b)}$ and $T\geqslant c(\lambda/\mu)^{-1/b}$. And since $1-b>0$ and $2-a-2b>0$ we again let $T\to\infty$ and get our result that $M(\lambda,\mu)=\infty$.

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Filtres: mesurabilité, rapidité, propriété de Baire forte

par

MICHEL TALAGRAND (Paris)

Résumé. On montre que contrairement à ce qui se passe pour la propriété de Baire il y a peu d'espoir de caractériser la mesurabilité des filtres par des conditions de rapidité. On étudie ensuite divers aspects de la propriété de Baire. En particilier, sous l'hypothèse du continu, il existe un filtre ayant la propriété de Baire forte qui n'est pas mesurable. On étudie enfin quels sont les filtres tels qu'il existe une suite d'ultrafiltres convergent selon ce filtre. De façon surprenante, ils peuvent être relativement réguliers.

Le but de ce travail est de complèter et de préciser un travail précédent [4], dont on conserve la terminologie et les notations. Pour simplifier on appelle filtre sur $\boldsymbol{N}$ un filtre propre, c'est-à-dire contenant les complémentaires de parties finies de $\boldsymbol{N}$. On appelle m la mesure canonique de $K = \{0,1\}^{\boldsymbol{N}} = \boldsymbol{P}(\boldsymbol{N})$. Un filtre, étant un ensemble de parties de $\boldsymbol{N}$, est donc un sous-ensemble de K, on peut ainsi parler de filtres mesurables, où définir diverses propriétés topologiques pour les filtres.

I. Mesurabilité et rapidité. Le Théorème 17 de [4] montre qu'un filtre non-mesurable $\mathcal{F}$ possède des parties ayant "peu" d'éléments, au sens suivant: $\mathcal{F}$ satisfait

(1) Si (I_n) est une partition de $\boldsymbol{N}$ en ensembles finis telle qu'il existe un réel $a > 0$ avec $\sum a^{\operatorname{card} I_n} < +\infty$, alors, pour tout α il existe $x \in \mathcal{F}$ avec $\operatorname{card}(I_n \cap x) \leqslant \alpha \operatorname{card} I_n$.

Le problème est posé de savoir si cette propriété entraine la non-mesurabilité de $\mathcal{F}$. Nous allons montrer que sous l'hypothèse du continu (HC) il n'en est rien. Disons qu'un filtre $\mathcal{F}$ est *dichotomique* si pour toute partition $(J_n)_n$ de $\boldsymbol{N}$ en ensembles à deux éléments, il existe $x \in \mathcal{F}$ avec $\operatorname{card}(J_n \cap x) \leqslant 1$ $\forall n$. Il est facile de voir qu'un filtre dichotomique satisfait (1).

THÉORÈME 1 (HC). *Il existe un filtre dichotomique mesurable.*

L'idée est de montrer qu'un filtre non measurable satisfait des conditions analogues à (1), mais assez différentes pour qu'on puisse les nier sans nier la dichotomie de $\mathcal{F}$.

PROPOSITION 1. *Soit* p_n, k_n *deux suites d'entiers,* $(I_n^l)_{l \leqslant p_n}$ *des sous-ensembles deux à deux disjoints de* $\boldsymbol{N}$, *avec pour* $l \leqslant p_n$, $\operatorname{card} I_n^l = k_n$. *Supposons qu'il existe un* $a > 0$ *avec*

$$\sum (1-(1-a)^{k_n})^{p_n} < +\infty.$$