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Translation invariant complemented subspaces of L^p

by

J. BOURGAIN (Brussels)

Abstract. G be a compact Abelian group and Γ the dual group of G. Assume $\Lambda \subset \Gamma$ and L^p_Λ complemented in $L^p(G)$ for some $1 < p < \infty$ $(p \neq 2)$. A necessary and sufficient condition on Λ is given in order that L^p_Λ should be isomorphic to the space $L^p[0, 1]$.

Introduction. Throughout this note G will be a compact Abelian group and Γ the dual group of G. For $\Lambda \subset \Gamma$ and $1 \leqslant p \leqslant \infty$, L^p_Λ will have the usual meaning. The results presented here are a continuation of [2] (see also [3] for more details) in which more particularly the Cantor group was considered.

Our purpose is to show that if $1 < p < \infty$ $(p \neq 2)$ and L^p_Λ complemented in $L^p(G)$, then the analytic condition for Λ considered in Th. 1 of [2] is necessary and sufficient in order that L^p_Λ should be linearly isomorphic to the Banach space L^p. From the descriptive point of view, our approach is satisfactory if the elements of G are of bounded order.

Notations and preliminary results. Let $\gamma_1, \ldots, \gamma_r$ be a finite sequence in Γ. Define by $V_k(\gamma_1, \ldots, \gamma_r)$ for $k = 1, 2, \ldots$ the set of characters γ which can be written in the form

$$\gamma = \gamma_1^{a_1} \ldots \gamma_r^{a_r},$$

where $a_s \in \{0, 1, \ldots, k\}$ for $s = 1, \ldots, r$.

Similarly, let $W(\gamma_1, \ldots, \gamma_r)$ be the set of characters γ of the form

$$\gamma = \gamma_1^{a_1} \ldots \gamma_r^{a_r},$$

where $a_s \in \{0, 1, \ldots, s\}$ for $s = 1, \ldots, r$.

For $k = 1, 2, \ldots$, we agree to say that a subset Λ of Γ *has property* (k) provided there exist a sequence (γ_r) of distinct characters and a sequence of characters (δ_r) in Γ such that for each r

$$V_k(\gamma_1, \ldots, \gamma_r) . \delta_r \subset \Lambda. \tag{1}$$

Analogously, we define property (L) for a subset Λ of Γ replacing (1) by

$$W(\gamma_1, \ldots, \gamma_r).\delta_r \subset \Lambda. \tag{2}$$

Thus property (1) corresponds to the property stated in Th. 1 of [2].

Although the next results are in [2] explicitly stated for the Cantor group, they extend to any compact Abelian group.

PROPOSITION 1. *If $p > 2$ and L^p embeds in L^p_Λ, then Λ has property* (1).

PROPOSITION 2. *Property* (1) *is "primary", i.e. if Λ has* (1) *and $\Lambda = \Lambda_1 \cup \Lambda_2$, then either Λ_1 or Λ_2 has* (1).

In fact, Prop. 2 is purely combinatorial and related to Hindman's theorem (see [1], [7]).

Translation invariant complemented subspaces. In this section we state the new results. Proofs will be outlined in the next section.

PROPOSITION 3. *If $\Lambda \subset \Gamma$ has (L), then L^p embeds in L^p_Λ for all $1 < p < \infty$.*

PROPOSITION 4. *Assume $\Lambda \subset \Gamma$ has* (1) *and L^p_Λ complemented in $L^p(G)$ $(p \neq 2)$. Then Λ has (L).*

THEOREM 5. *Assume $\Lambda \subset \Gamma$, $1 < p \neq 2 < \infty$ and L^p_Λ complemented in $L^p(G)$. Then t.f.a.e.*

(i) *Λ has* (1).

(ii) *Λ has (L).*

(iii) *L^p_Λ is isomorphic to L^p (assuming Λ countable).*

COROLLARY 6. *Assume $\Lambda \subset \Gamma$, $1 < p \neq 2$, $q < \infty$ and L^p_Λ, L^q_Λ complemented. If L^p_Λ is isomorphic to L^p, then L^q_Λ is isomorphic to L^q.*

COROLLARY 7. *Assume the elements of G of bounded order and $\Lambda \subset \Gamma$. Then t.f.a.e.*

(i) *There is a sequence (γ_r) of distinct characters in Γ such that Λ contains a translate of the subgroup $\mathrm{gr}[\gamma_1, \ldots, \gamma_r]$ for each r.*

(ii) *There is a subset Λ' of Λ such that for some $1 < p < \infty$ $(p \neq 2)$ $L^p_{\Lambda'}$ is complemented in $L^p(G)$ and isomorphic to L^p.*

(iii) *There is a subset Λ' of Λ such that for all $1 < p < \infty$ $L^p_{\Lambda'}$ is complemented in $L^p(G)$ and isomorphic to L^p.*

Referring to the classes $\mathscr{S}_a$ introduced in [2].

COROLLARY 8. *Assume $\Lambda \subset \mathbf{Z}$, $1 < p < \infty$ $(p \neq 2)$ and L^p_Λ complemented in $L^p(\Pi)$. If Λ does not belong to any class of finite index, then Λ contains arithmetic progressions.*

Additional proofs. For $k = 1, 2, \ldots$, and (γ_r) a sequence in Γ, we say that (γ_r) is *k-dissociated* provided

$$\gamma_1^{j_1} \ldots \gamma_r^{j_r} = 1 \text{ and } |j_s| \leqslant k \ (1 \leqslant s \leqslant r) \Rightarrow \gamma_s^{j_s} = 1 \ (1 \leqslant s \leqslant r).$$

We omit the proof of following two simple facts.

LEMMA 1. *If Λ is an infinite subset of Γ and k a positive integer, then there is a sequence (γ_r) in Γ and $\delta \in \Gamma$ such that $(\delta . \gamma_r)$ is k-dissociated.*

LEMMA 2. *Let (γ_r) be a sequence of distinct elements of Γ and k a positive integer. Then there exist finite subsets S_a of N satisfying $\max S_a < \min S_{a+1}$, such that (δ_a) is k-dissociated, where $\delta_a = \prod_{r \in S_a} \gamma_r$.*

The next lemma is easily obtained by standard approximation arguments.

LEMMA 3. *Given $1 \leqslant p < \infty$, $\varepsilon > 0$ and positive integers k, r, there exists an integer K such that if $(\gamma_s)_{1 \leqslant s \leqslant r}$ in Γ is K-dissociated and if $f_s \in [\gamma_s^j;\ |j| \leqslant k]$ $(1 \leqslant s \leqslant r)$, then*

$$(1-\varepsilon)\|f_1\|_p \ldots \|f_r\|_p \leqslant \|f_1 \ldots f_r\|_p \leqslant (1+\varepsilon)\|f_1\|_p \ldots \|f_r\|_p.$$

LEMMA 4. *For $1 < p < \infty$ $(p \neq 2)$, there is a constant $c_p > 1$ such that the orthogonal projection from $[1, \gamma, \gamma^2]$ onto $[1, \gamma]$ is of L^p-norm at least c_p, whenever γ in Γ is of order at least 3.*

Proof. By duality, we can take $2 < p < \infty$. Since further

$$\|a + be^{i\theta}\gamma + ce^{2i\theta}\gamma^2\|_p = \|a + be^{i\theta} + ce^{2i\theta}\|_p,$$

we can restrict ourselves to $G = \Pi$ and $\gamma = e^{i\theta}$. Now, one has for $0 < \varepsilon < 1$

$$\|1 + e^{i\theta} - \varepsilon e^{2i\theta}\|_p^p = \|1 + e^{i\theta}\|_p^p - \sigma\varepsilon + o(\varepsilon^2),$$

where

$$\sigma = p\pi^{-1} 2^{2(p-1)} \int_0^{\pi/2} [\cos t]^{p-1} \cos 3t\, dt > 0.$$

The next result is immediate from Lemma 3 and Lemma 4.

LEMMA 5. *For $1 < p < \infty$ $(p \neq 2)$, there is a constant $c_p > 1$ such that for given positive integers k, r, the L^p-norm of the orthogonal projection from $V_{k+1}(\gamma_1, \ldots, \gamma_r)$ onto $V_k(\gamma_1, \ldots, \gamma_r)$ is at least c_p^r, provided $(\gamma_s)_{1 \leqslant s \leqslant r}$ is a sufficiently dissociated sequence in Γ whose elements are at least of order $k+1$.*

LEMMA 6. *Assume L^p_Λ complemented in $L^p(G)$ $(p \neq 2)$ and Λ with property (k). Then for infinitely many characters γ, the set $\bigcap_{j=0}^{k+1} \bar{\gamma}^j \Lambda$ has* (1).

Proof. Remark first that by Lemma 2, the sequence (γ_r) in the definition of property (k) can be chosen arbitrarily dissociated. Since further γ will be obtained as element of $V_1(\gamma_1, \gamma_2, \ldots)$, the existence of infinitely many candidates will be automatic. Denote P the orthogonal projection on L^p_Λ and fix r large enough to ensure $c_p^r > \|P\|$. Take $\gamma_1, \ldots, \gamma_r$ sufficiently dissociated and $\Lambda_1 \subset \Gamma$ with (1) such that

$$V_k(\gamma_1, \ldots, \gamma_r).\delta \subset \Lambda \quad \text{for each } \delta \in \Lambda_1.$$

If some $\gamma \in \{\gamma_1, \ldots, \gamma_r\}$ is of order at most k, then obviously

$$\bigcap_{j=0}^{k} \bar{\gamma}^j \Lambda = \bigcap_{j=0}^{k+1} \bar{\gamma}^j \Lambda$$

has (1), since latter set contains Λ_1.

Otherwise, Lemma 5 asserts that the L^p-norm of the orthogonal projection from $V_{k+1}(\gamma_1, \ldots, \gamma_r)$ onto $V_k(\gamma_1, \ldots, \gamma_r)$ is at least c_p^r. Fixing $\delta \in \Lambda_1$, one has

$$V_k(\gamma_1, \ldots, \gamma_r) \subset \Lambda.\bar{\delta} \cap V_{k+1}(\gamma_1, \ldots, \gamma_r) \subset V_{k+1}(\gamma_1, \ldots, \gamma_r),$$

where in particular the second set is $\|P\|$ complemented in the third. Thus the first and the second set must be different, implying the existence of some $\xi_\delta \in V_{k+1} \setminus V_k$ such that $\delta \in \Lambda \bar{\xi}_\delta$. Applying now Prop. 2, we can fix $\xi \in V_{k+1} \setminus V_k$ for which $\Lambda \bar{\xi} \cap \Lambda_1$ has (1). There is a nonempty subset A of $\{1, \ldots, r\}$ such that

$$\xi = \gamma^{k+1}\eta, \qquad \text{where} \qquad \gamma = \prod_{s \in A} \gamma_s \text{ and } \eta \in V_k(\gamma_s;\ s \notin A).$$

Since

$$\Lambda \bar{\xi} \cap \Lambda_1 \subset \Big(\Lambda \bar{\gamma}^{k+1} \bar{\eta} \cap \bigcap_{j=0}^{k} \Lambda \bar{\gamma}^j \bar{\eta}\Big),$$

the set $\bigcap_{j=0}^{k+1} \bar{\gamma}^j \Lambda$ has property (1), which conclude the proof.

LEMMA 7. *Assume L^p_Λ complemented in $L^p(G)$ $(p \neq 2)$. If Λ has* (1), *then Λ has also* (k) *for any positive integer k.*

Proof. We proceed by induction on k. So assume that under the above hypothesis the implication $(1) \Rightarrow (k)$ holds. Thus in particular Λ has (k) and hence, by Lemma 6, there is a character γ_1 such that the set

$$\Lambda_1 = \bigcap_{j=0}^{k+1} \bar{\gamma}_1^j.\Lambda = \bigcap_{\xi \in V_{k+1}(\gamma_1)} \xi.\Lambda$$

has (1). Observe that $L^p_{\Lambda_1}$ is still complemented in $L^p(G)$ since Λ_1 is finite intersection of L^p-complemented sets. So, by induction hypothesis, Λ_1 has (k).

Apply again Lemma 6 to obtain a character $\gamma_2 \neq \gamma_1$ such that

$$\Lambda_2 = \bigcap_{j=0}^{k+1} \bar{\gamma}_2^j.\Lambda_1 = \bigcap_{\xi \in V_{k+1}(\gamma_1, \gamma_2)} \xi.\Lambda$$

has (1).

Iteration of this procedure leads to a sequence (γ_r) of distinct characters such that for each r

$$\bigcap_{\xi \in V_{k+1}(\gamma_1, \ldots, \gamma_r)} \xi.\Lambda$$

has (1) and hence is nonempty. Thus one can find a sequence (δ_r) in Γ satisfying

$$V_{k+1}(\gamma_1, \ldots, \gamma_r).\delta_r \subset \Lambda \quad \text{for each } r.$$

Consequently Λ has property $(k+1)$.

Proof of Proposition 4. We use the same procedure as in Lemma 7. If $\gamma_1, \ldots, \gamma_r$ are already obtained and

$$\Lambda_r = \bigcap_{\xi \in W(\gamma_1, \ldots, \gamma_r)} \xi.\Lambda$$

has (1), Lemma 7 asserts that Λ_r also has $(r+1)$. In particular, there exists $\gamma_{r+1} \in \Gamma$ so that

$$\Lambda_{r+1} = \bigcap_{j=0}^{r+1} \bar{\gamma}_{r+1}^j.\Lambda_r = \bigcap_{\xi \in W(\gamma_1, \ldots, \gamma_{r+1})} \xi.\Lambda$$

still has (1).

Proof of Proposition 3. We will construct a system of functions on G with spectrum in Λ which is equivalent to the usual Haar system in $L^p[0, 1]$.

Take first sequences (p_m), (q_m) of trigonometric polynomials with positive spectrum on Π such that

(i) $|p_m| + |q_m| \leqslant 1$.

(ii) $\int_\Pi |p_m|^2 \to \frac{1}{2}$, $\int_\Pi |q_m|^2 \to \frac{1}{2}$ "rapidly enough".

Let further p_m, q_m be of degree at most q_m.

Let (γ_r) be a sequence of distinct characters and (δ_r) a sequence of characters s.t.

$$W(\gamma_1, \ldots, \gamma_r).\delta_r \subset \Lambda.$$

We will fix our attention to the case where the γ_r are of unbounded order. We can then replace (γ_r), (δ_r) by sequences (γ_m), (δ_m) satisfying

(iii) $\gamma_1^{j_1} \ldots \gamma_m^{j_m} = 1$ and $|j_s| \leqslant d_s \Rightarrow j_1 = j_2 = \ldots = j_m = 0$.

(iv) $\gamma_1^{j_1} \ldots \gamma_m^{j_m} \delta_m \in \Lambda$ provided $j_s \in \{0, 1, \ldots, d_s\}$ $(1 \leqslant s \leqslant m)$.

Define $\varphi_{m,0} = p_m \circ \gamma_m$ and $\varphi_{m,1} = q_m \circ \gamma_m$. Let further for $n = \sum_{s=0}^{m-1} \varepsilon_s 2^s = 1, \ldots, 2^m$,

$$f_{m,n} = \prod_{s=0}^{m-1} \varphi_{s,\varepsilon_s}.\delta_{m,n},$$

where the $\delta_{m,n}$ in Γ are chosen such that $\operatorname{Spec} f_{m,n} \subset \Lambda$ and the system $(f_{m,n})$ is a martingale difference sequence for the lexicographical order in $L^p(G)$. By (i)

$$1 \geqslant |f_{m,n}| \geqslant |f_{m+1,2n-1}| + |f_{m+1,2n}|$$

and by (iii), (ii)

$$\int_G |\varphi_{m,0}|^2 = \int_H |p_m|^2, \qquad \int_G |\varphi_{m,1}|^2 = \int_H |q_m|^2$$

and

$$\int_G |f_{m,n}|^2 = \prod_{s=0}^{m-1} \int_G |\varphi_{s,\varepsilon_s}|^2 \sim 2^{-m}.$$

By standard techniques, one can then show that $(f_{m,n})$ contains a subsystem equivalent to the Haar system (cf. [8]).

Proof of Theorem 5. (1) ⇒ (2) follows by Prop. 4.

(2) ⇒ (3): By Prop. 3, L^p embeds in L^p_Λ and hence also as complemented subspace (see [8]). The isomorphism follows from Pełczyński's decomposition method (see [9]).

(3) ⇒ (1) follows by Prop. 1 and duality in case $1 < p < 2$.

Corollary 6 is now straightforward.

Proof of Corollary 7. (iii) ⇒ (ii) is obvious and (ii ⇒ (i) is a consequence of Th. 5.

Notice that the orthogonal projection on $\operatorname{gr}[\gamma_1, \ldots, \gamma_r]$ is given by a conditional expectation. The implication (i) ⇒ (iii) follows from standard Burkhölder–Gundi square-function techniques for martingale difference sequences and Stein's inequality (cf. [6] and [10]).

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DEPARTMENT OF MATHEMATICS
VRIJE UNIVERSITEIT BRUSSEL
Pleinlaan 2, F7 1050 Brussels

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