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The Selberg trace formula for the Picard group $\mathrm{SL}(2, Z[i])$

by

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1. Introduction. The paper contains a derivation of the Selberg trace formula for the complex group $\mathrm{SL}(2, Z[i])$ with the entries being Gaussian integers. We apply Zagier's method [24] which gives the better understanding of the trace formula in this case and exhibits its connections with number theory.

The original Selberg's paper [18] of 1956 became inspiring for further investigations of the eigenvalue problem for the Laplacian on Riemann surfaces and also for invariant operators acting on homogeneous spaces of a more general type. The case of non-compact spaces with finite volume is more difficult than the compact one, the reason being the presence of continuous spectrum of differential operators in the non-compact case. For the unimodular group $\Gamma = \mathrm{SL}(2, Z)$ with integer entries the spectral decomposition of the space $L^2(\Gamma\backslash H)$, H being the upper half plane, and the explicit trace formula describing the discrete spectrum of the Laplacian have been given by Selberg [18]. This paper contains also foundations of the theory concerning more general groups. Don Zagier in his paper [24] has presented a new proof of the Selberg trace formula for $\mathrm{SL}(2, Z)$ which is based on the spectral decomposition of the space $L^2(\Gamma\backslash H)$ but the trace of an integral operator with the kernel K_0 is calculated as the residue of the integral

$$I(s) = \int_{\Gamma\backslash H} K_0(z, z) E(z; s)\, dz.$$

The following property of Eisenstein series is then used: $\operatorname{res}_{s=1} E(z; s)$ does not depend of the first argument; hence $\operatorname{res}_{s=1} I(s)$ is proportional to the trace of the integral operator. The investigation of the integral $I(s)$ is based on the simple principle, due to Rankin [16] and Selberg [17], asserting that the integral of a Γ-invariant function against the Eisenstein series equals the Mellin transform of the constant term in the Fourier expansion of this function. This principle applied to the kernel K_0 gives rise to a representation of $I(s)$ as an infinite sum each of whose terms represents the contribution of either a conjugacy class of elements of Γ or of the continuous spectrum. The Dedekind zeta functions of quadratic extensions of the rationals Q appear here in a natural way. The terms on the right-hand side of the trace formula are either proportional to the class numbers of quadratic fields or they are constants in the Laurent expansion of the Riemann zeta function (e.g. the Euler constant). The contribution from the continuous spectrum leads to integrals of the logarithmic derivative of Riemann zeta function or of gamma functions.

The above method of deriving the trace formula provides a direct connection with the arithmetic theory of binary quadratic forms. Compared with the original Selberg's proof (e.g. Appendix to Kubota's book [12]) we make use of more properties of special functions than of the geometry of fundamental domains; the latter for $\mathrm{SL}(2, Z[i])$ is rather complicated. The convergence questions in a neighbourhood of a cusp of a discrete subgroup, existing for non-compact fundamental domains, reduce here to a study of some properties of the involved special functions. For these reasons the proposed method is (besides its theoretical significance giving a deeper understanding of the origin of various terms appearing in the trace formula) more satisfactory from the aesthetic point of view due to its invariant character.

The explicit form of the trace formula for $\mathrm{SL}(2, Z[i])$, considered as a discontinuous group of transformations of the three-dimensional hyperbolic space, has been established by Venkov [21], Tanigawa [19] and de la Torre [20]; each one applying basically Selberg's approach. The general case of real R-rank one semisimple Lie groups and non-compact fundamental domains has been treated by Warner [22] and Gangoli, Warner [5].

This paper is an attempt to apply Zagier's method to the complex case. The individual contributions to the trace formula are now related to quadratic extensions of the field $k = Q(i)$; the rôle of the Riemann zeta function for $\mathrm{SL}(2, Z)$ is played by the Dedekind zeta function of this field. There appear bilinear quadratic forms with coefficients being complex integers together with the corresponding Dedekind zeta functions of the quadratic extensions of k. This fact is consistent with the similar phenomenon appearing in the paper [9] of Jacquet and Zagier concerning GL_2 over the adèle group of a number field. The integral transforms which appear in our paper are in many cases similar to those from [24], but also there are some new ones related to the conjugacy classes which are not presented for the group $\mathrm{SL}(2, Z)$. Perhaps it is worthly of mentioning that elliptic elements of trace zero give the contribution to continuous spectrum and we need here the generalized hypergeometric series of the type ${}_3F_2$; this place required subtle considerations of fundamental domains

in the papers cited above. The method used here is also connected with the utilization of the Fourier expansion of Eisenstein series; it corresponds to the application of the Maass–Selberg relations in the original proof.

The second section of the paper has an introductory character and contains some known material. It seems that Proposition 2 contains a new information about location of the first eigenvalue of the Laplace–Beltrami operator, $\lambda_1 \geqslant \pi^2$; the important thing here is the estimation $\lambda_1 > 1$ which implies that there are not purely imaginary r_j's ($\lambda_j = 1 + r_j^2$, $j = 1, 2, \ldots$) in the trace formula. The situation here is similar to that of $\mathrm{SL}(2, Z)$ where the exceptional eigenvalues do not exist either; i.e. such that $\lambda_j < \frac{1}{4}$. Section 3 constitutes the main part of this work, in which the integral $I(s)$ is calculated for $\mathrm{Re}\, s > 2$ (the Eisenstein series for $\mathrm{SL}(2, Z[i])$ has a pole at $s = 2$) and the analytic continuation of individual terms of $I(s)$ together with their principal parts is investigated. The value of $\operatorname*{res}_{s=2} I(s)$ gives us the trace formula which we find in Section 4.

The title of the paper is related to the fact that a systematic investigation of discrete subgroups of $\mathrm{SL}(2, C)$ was begun by Picard [15] who gave a description of the fundamental domain for the group $\mathrm{SL}(2, Z[i])$. This subject embraces a part of the monograph [3] of Fricke and Klein where we find connections with the theory of complex quadratic forms due to Dirichlet [2] (see also references added in proof).

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2. Preliminaries.

2.1. Eisenstein series. The symmetric space corresponding to the group $\tilde{G} = \mathrm{SL}(2, C)$ can be parametrized as follows

$$H = \{u = (z, v);\ z \in C,\ v > 0\}.$$

An element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ operates on H by the linear fractional transformations

$$g\cdot(z, v) = \left(\frac{(az+b)\overline{(cz+d)} + a\bar{c}v^2}{|cz+d|^2 + |c|^2 v^2},\ \frac{v}{|cz+d|^2 + |c|^2 v^2}\right). \tag{2.1}$$

This action is transitive, the isotropy group of the point $u_0 = (0, 1)$ is the maximal compact subgroup $\tilde{K} = \mathrm{SU}(2)$ of $\tilde{G}$, and one can identify the space H with the homogeneous space

$$\tilde{G}/\tilde{K} \ni gK \mapsto g\cdot u_0 \in H.$$

We give now, following Kubota [11], a description of the Eisenstein series related to the discrete subgroup $\Gamma = \mathrm{PSL}(2, Z[i]) = \mathrm{SL}(2, Z[i])/\{\pm I\}$ of the group $G = \tilde{G}/\{\pm I\}$, I being the identity matrix. Let $\Gamma_\infty = \left\{\pm\begin{pmatrix} a & b \\ & d\end{pmatrix} \in \Gamma\right\}$ be the stabilizer of the unique cusp ∞ for Γ. For a complex variable s the Eisenstein series is defined as follows

$$E(u;s) = \sum_{\sigma\in\Gamma_\infty\backslash\Gamma} v(\sigma\cdot u)^s, \quad u\in H, \tag{2.2}$$

where $v(\cdot)$ stands for the corresponding v-part in (2.1). The Eisenstein series converges absolutely in the region $\mathrm{Re}\, s > 2$ and defines there a holomorphic function of s. It is a real analytic function in the variables (x, y, v) and can be written more explicitly in the form

$$E(z, v; s) = \frac{1}{4}\sum_{\substack{c,d\in Z[i]\\(c,d)=1}} \frac{v^s}{(|cz+d|^2+|c|^2v^2)^s}, \quad z = x+iy.$$

The summation is taken over all pairs of relatively prime Gaussian integers. For later needs let us write the Fourier expansion of Eisenstein series. Letting $\mathfrak{a} = Z[i]$, $\mathfrak{a}^*$ the subset of all non-zero elements, $X = \{a\in\mathfrak{a};\ \mathrm{Re}\, a > 0,\ \mathrm{Im}\, a \geqslant 0\}$ the sector in $\mathfrak{a}$, and $\zeta_k(s)$ the Dedekind zeta function of the field $k = Q(i)$, i.e.

$$\zeta_k(s) = \sum_{n\in X} |n|^{-2s} \quad \text{with } |n|^2 = n\bar{n}.$$

We have then

$$E(z, v; s) = v^s + \pi\frac{\Gamma(s-1)\zeta_k(s-1)}{\Gamma(s)\zeta_k(s)}v^{2-s} + \\ + \frac{2\pi^s v}{\Gamma(s)\zeta_k(s)}\sum_{n\in\mathfrak{a}^*}\tau_{s-1}(n)K_{s-1}(2\pi|n|v)\exp\{\pi i(nz+\overline{nz})\}. \tag{2.3}$$

Here $\tau_\nu(n)$ is the divisor function which for any $\nu\in C$ and any $n\in\mathfrak{a}^*$ is defined by

$$\tau_\nu(n) = \sum_{\substack{ad=1\\ a\in\mathfrak{a}^*,\, d\in X}} \left|\frac{a}{d}\right|^\nu = |n|^\nu \sum_{\substack{d|n\\ d\in X}} |d|^{-2\nu}.$$

The Bessel functions are given, for example, by the integral formula

$$K_\nu(z) = \int_0^\infty \exp(-z\cosh t)\cosh(\nu t)\,dt, \quad \nu, z\in C,\ \mathrm{Re}\, z > 0.$$

If one introduces the normalized zeta function $\zeta_k^*(s) = 4\pi^{-s}\Gamma(s)\zeta_k(s)$, then $\zeta_k^*(s) = \zeta_k^*(1-s)$, and the normalized Eisenstein series $E^*(u;s) = \zeta_k^*(s)E(u;s)$, then (2.3) takes the form

$$E^*(z, v; s) = \zeta_k^*(s)v^s + \zeta_k^*(s-1)v^{2-s} + \\ + 8v\sum_{n\in\mathfrak{a}^*}\tau_{s-1}(n)K_{s-1}(2\pi|n|v)\exp\{\pi i(nz+\overline{nz})\}. \tag{2.4}$$

As an immediate corollary from the Fourier expansion (2.4) one deduces the meromorphic continuation of $E^*(u;s)$ to the whole complex plane, the only poles are at $s = 0$, $s = 2$, and they are simple ones. Since

$$\tau_\nu(n) = \tau_{-\nu}(n) \quad \text{and} \quad K_\nu(z) = K_{-\nu}(z),$$

another corollary from (2.4) is the functional equation:

$$E^*(u;s) = E^*(u;2-s).$$

Furthermore (2.3) and (2.4) yield:

$$\operatorname*{res}_{s=2} E^*(u;s) = \operatorname*{res}_{s=1}\zeta_k^*(s) = 1, \quad \operatorname*{res}_{s=2} E(u;s) = \frac{1}{\zeta_k^*(2)} = \frac{\pi^2}{4\zeta_k(2)},$$

$$E(z, v; s) = O(v^{\max(\mathrm{Re}\, s,\, 2-\mathrm{Re}\, s)}) \tag{2.5}$$

for fixed z as $v\mapsto\infty$, because the sum of Bessel functions tends to zero exponentially.

2.2. The Rankin–Selberg method. The symmetric space H has the G-invariant measure

$$d\mu(u) = \frac{dx\,dy\,dv}{v^3}, \quad u = (x+iy, v).$$

By the Rankin–Selberg method ([16], [17], [24]) we mean the principle that a scalar product of a function $f\colon \Gamma\backslash H\mapsto C$ with Eisenstein series equals the Mellin transform of the constant term in the Fourier expansion of f.

PROPOSITION 1. *Let f be a Γ-invariant function on H of a rapid decay, i.e. $f(z, v) = O(v^{-\varepsilon})$ as $v\mapsto\infty$, for some $\varepsilon > 0$. Then the scalar product*

$$\Psi(s) = (f, E(\cdot;\bar{s})) = \int_{\Gamma\backslash H} f(u)E(u;s)\,d\mu(u)$$

converges absolutely in the infinite strip: $-\varepsilon < \mathrm{Re}\, s < 2+\varepsilon$, and it is given by

$$\Psi(s) = \frac{1}{2}\int_0^\infty\int_0^1\int_0^1 f(x+iy, v)v^{s-3}\,dx\,dy\,dv.$$

The proof goes by inserting the series (2.2) into the scalar product and applying termwise integration.

2.3. The spectral decomposition. The Laplace–Beltrami operator on the symmetric space H is given in the standard coordinates by

$$D = v^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial v^2}\right)-v\frac{\partial}{\partial v}.$$

It is G-invariant and hermitian operator acting on a dense subspace of $L^2(\Gamma\diagdown H)$. The space $L^2(\Gamma\diagdown H)$ of square integrable functions has the spectral decomposition with respect to D:

$$L^2(\Gamma\diagdown H) = C\oplus L_0^2(\Gamma\diagdown H)\oplus L^2_{\mathrm{cont}}(\Gamma\diagdown H),$$

C stands for the one-dimensional subspace of constant functions, $L_0^2(\Gamma\diagdown H)$ for the subspace of functions with constant term zero (the space of cusp forms). The operator D has only discrete spectrum in $L_0^2(\Gamma\diagdown H)$, $f_0\equiv 1$ corresponds to the eigenvalue $\lambda_0=0$, and we take $\{f_j\}_{j=0}^{\infty}$ to be an orthogonal basis of $C\oplus L_0^2(\Gamma\diagdown H)$ consisting of eigenfunctions of $-D$:

$$-Df_j=\lambda_j f_j,\quad \lambda_0<\lambda_1\leqslant\lambda_2\leqslant\ldots,\quad \lim_{j\to\infty}\lambda_j=\infty.$$

We write the eigen-equation in the form

$$-Df_j=(1+r_j^2)f_j,\quad j=0,1,2,\ldots,\quad r_0=\sqrt{-1},$$

the numbers r_j are real or purely imaginary since $-D$ is positive definite and there is a priori only a finite number of imaginary ones; necessarily with $|r_j|<1$. The operator D has continuous spectrum in the subspace $L^2_{\mathrm{cont}}(\Gamma\diagdown H)$.

Now any bounded function $f\in L^2(\Gamma\diagdown H)$ has an expansion

$$f(u)=\sum_{j=0}^{\infty}\frac{(f,f_j)}{(f_j,f_j)}f_j(u)+\frac{1}{2\pi}\int_{-\infty}^{\infty}\big(f,E(u;1+ir)\big)E(u;1+ir)\,dr. \tag{2.6}$$

The spectral decomposition is a consequence of the general theory of Eisenstein series (Langlands [14]) or can be proved in this case along the lines presented in Zagier [24].

The fundamental domain $\mathscr{D}$ of the group $\Gamma=\mathrm{PSL}(2,Z[i])$ acting on H can be chosen as follows (Picard [15]):

$$\mathscr{D}=\{(x+iy,v)\in H;\ 0\leqslant x+y,\ x,y\leqslant 1/2,\ x^2+y^2+v^2\geqslant 1\}.$$

The picture of the fundamental domain can be found in Fricke and Klein ([3], p. 82); it is a tetrahedron in R^3. Its volume is related to the residue of the Eisenstein series; namely one has

$$\mathrm{vol}(\Gamma\diagdown H)^{-1}=2\operatorname*{res}_{s=2}E(u;s)=\frac{\pi^2}{2\zeta_k(2)}.$$

In the case of the modular group $\mathrm{SL}(2,Z)$ the first eigenvalue of the Laplacian satisfies $\lambda_1>1/4$, this fact is due to Roecke (Sitzung. d. Heidelberger Akad. d. Wiss., 1956). A simple variant of the proof, elaborated by M. F. Vigneras and published in Deshouillers, Iwaniec ([1], Th. 3), gives in fact the sharper estimation $\lambda_1\geqslant 3\pi^2/2$. We adopt this method to the group $\mathrm{PSL}(2,Z[i])$ to prove the following

PROPOSITION 2. *The first eigenvalue of the operator $-D$ satisfies: $\lambda_1\geqslant\pi^2$ (for the final shape of the trace formula it suffices a weaker bound $\lambda_1>1$).*

Proof. Let us take the two mappings of the space H:

$$\tau=\begin{pmatrix} & -1\\ 1 & \end{pmatrix}\quad\text{and}\quad \omega=\begin{pmatrix} i & \\ & -i\end{pmatrix};$$

then

$$\tau\cdot(z,v)=\left(\frac{-\bar z}{|z|^2+v^2},\frac{v}{|z|^2+v^2}\right),\quad \omega\cdot(z,v)=(-z,v),$$

where τ sends the cusp ∞ to the point 0 and ω is a rotation by the angle π. From the particular domain $\mathscr{D}$ of Γ we build up the union of four copies $\mathscr{B}=\mathscr{D}\cup\omega\mathscr{D}\cup\tau\mathscr{D}\cup\omega\tau\mathscr{D}$. It can be checked that $\mathscr{B}$ contains the solid figure

$$I=\{(x,y,v);\ -\tfrac12\leqslant x\leqslant\tfrac12,\ -\tfrac12\leqslant y\leqslant\tfrac12,\ v\geqslant v_0\}$$

with $v_0=\sqrt2/2$; the point $(\frac12,\frac12,\sqrt2/2)$ lies on the sphere $x^2+y^2+v^2=1$ and we have $\mathscr{D}\subset I\subset\mathscr{B}$.

Let f be a cusp form such that

$$\int_{\mathscr{D}}|f|^2d\mu(u)=1,\quad -Df=\lambda f\quad\text{with }\lambda>0$$

and we can take f to be real valued. Then using Green's formula (the integrals over the boundary of $\mathscr{B}$ disappear because of the periodicity of automorphic functions and the boundary integral over the top of the truncated fundamental domain vanishes in the limit) we deduce that

$$4\lambda=\int_{\mathscr{B}}fDfd\mu(u)=-\int_{\mathscr{B}}f(f_{xx}+f_{yy}+f_{vv})\frac{dxdydv}{v}+\int_{\mathscr{B}}ff_v\frac{dxdydv}{v^2}$$

$$=\int_{\mathscr{B}}\left(f_x^2+f_y^2+f_v^2-\frac{f}{v}f_v\right)\frac{dxdydv}{v}+\int_{\mathscr{B}}\frac{f}{v}f_v\frac{dxdydv}{v}$$

$$\geqslant\int_I(f_x^2+f_y^2+f_v^2)\frac{dxdydv}{v}\geqslant\int_I(f_x^2+f_y^2)\frac{dxdydv}{v}.$$

From the Fourier expansion

$$f(x, y, v) = \sum_{n\in\mathfrak{a}^*} a_n(v)\exp\{\pi i(nz+\overline{nz})\}, \quad z = x+iy,$$

it follows that ($n = \mathrm{Re}\,n+i\,\mathrm{Im}\,n$)

$$\int_I (f_x^2+f_y^2)\frac{dxdydv}{v} = \int_{v_0}^{\infty}\sum_{n\in\mathfrak{a}^*}\{|2\pi(\mathrm{Re}\,n)|a_n(v)||^2+|2\pi(\mathrm{Im}\,n)|a_n(v)||^2\}\frac{dv}{v}$$

$$\geqslant 2(2\pi v_0)^2\int_{v_0}^{\infty}\sum_n |a_n(v)|^2\frac{dv}{v^3} = 2(2\pi v_0)^2\int_I f^2 d\mu(u)$$

$$\geqslant 2(2\pi v_0)^2\int_{\mathscr{D}} f^2 d\mu(u) = 4\pi^2.$$

Hence $\lambda \geqslant \pi^2$.

2.4. The spherical transform. Let φ be a smooth function on $G = \mathrm{PSL}(2, \boldsymbol{C})$ of sufficiently rapid decay which is K-biinvariant ($K = \mathrm{PSU}(2)$). Then using the map (the Cartan decomposition)

$$t\colon K\begin{pmatrix} a & b\\ c & d\end{pmatrix}K \mapsto |a|^2+|b|^2+|c|^2+|d|^2-2,$$

we can view φ as a function of a real variable $t \in [0, \infty)$. Following the Selberg notation [18] we introduce the new functions Q, g and h by

$$Q(w) = \pi\int_w^{\infty}\varphi(t)\,dt \quad \text{if} \quad w > 0, \tag{2.7}$$

$$g(u) = Q(w) \quad \text{where} \quad w = e^u+e^{-u}-2,\ u\in R,$$

and

$$h(r) = \int_{-\infty}^{\infty} g(u)e^{iru}du, \quad r\in \boldsymbol{C}.$$

The inverse formulae are

$$g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)e^{-iru}du$$

and

$$\varphi(t) = -\frac{1}{\pi}Q'(t) = -\frac{g'\left(\operatorname{arc\,cosh}\frac{t+2}{2}\right)}{\pi\sqrt{t(t+4)}}$$

$$= \frac{1}{2\pi^2\sqrt{t(t+4)}}\int_{-\infty}^{\infty} rh(r)\sin\left(r\operatorname{arc\,cosh}\frac{t+2}{2}\right)dr.$$

The integral representation of the Legendre functions $P_\nu^\mu(\cosh t)$, ([EH], 3.7(7)), gives in the particular case

$$P_{ir-1/2}^{-1/2}(\cosh u) = \sqrt{\frac{2}{\pi\sinh u}}\,\frac{\sin ru}{r},$$

and now the expression of φ in terms of its spherical transform h becomes

$$\varphi(t) = \frac{1}{4\pi^{3/2}}\big(t(t+4)\big)^{-1/4}\int_{-\infty}^{\infty} r^2 P_{ir-1/2}^{-1/2}\left(1+\frac{t}{2}\right)h(r)\,dr. \tag{2.8}$$

We impose on φ the decay condition

$$\varphi(t) = O(t^{-(1+\Delta)/2}) \quad \text{as} \quad t\mapsto\infty,$$

for some constant $\Delta > 2$, this is equivalent to the holomorphy of the function $h(r)$ in the infinite strip

$$\{r\in \boldsymbol{C};\ |\mathrm{Im}\,r| < \Delta/2\}, \quad h(r) = h(-r).$$

The smoothness of φ is equivalent to the statement that h is of the rapid decay; precisely that

$$h(r) = O(|r|^{-n}) \quad \text{for all natural } n.$$

2.5. The kernel function. Let φ be a function on the group G with the properties described above. The function of two variables

$$k(g, g') = \varphi(g'^{-1}g), \quad g, g'\in G,$$

defines an operator

$$(L_\varphi f)(g) = \int_G k(g, g')f(g')dg', \quad f\in L^2(\Gamma\backslash H),$$

which can be viewed as an integral one

$$(L_\varphi f)(u) = \int_{\Gamma\backslash H} K(u, u')f(u')\,d\mu(u')$$

with the left Γ-invariant kernel

$$K(g, g') = \sum_{\gamma\in\Gamma}\varphi(g'^{-1}\gamma g).$$

We infer from the rapid decay of φ that $K(u, u')$ is in $L^2(\Gamma\backslash H)$ with respect to each variable separately. Now we appeal to the celebrated Theorem of Selberg saying that every eigenfunction of an invariant differential operator is also an eigenfunction of L_φ with the following correspondence:

$$\big(Df = -(1+r^2)f\big) \Rightarrow \big(L_\varphi f = h(r)f\big)$$

for the Laplace–Beltrami operator D. From this follows that

$$\big(K(\cdot, u'), f_j\big) = h(r_j)\overline{f_j(u')}, \quad j = 0, 1, 2, \ldots,$$

and

$$\big(K(\cdot, u'), E(\cdot; 1+ir)\big) = h(r)\, E(u'; 1-ir).$$

The Eisenstein series are eigenfunctions of D which are not square-integrable. The spectral decomposition (2.6) applied to $K(\cdot, \cdot)$ gives

$$K(u, u') = \sum_{j=0}^{\infty} \frac{h(r_j)}{(f_j, f_j)} f_j(u)\overline{f_j(u')} + \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r)\, E(u; 1+ir)\, E(u'; 1-ir)\, dr.$$

The last integral, denoted by $H(u, u')$, describes the continuous spectrum of the operator L_φ. For suitable functions φ an integral operator $L_{\varphi,0}$ with the kernel

$$K_0(u, u') = K(u, u') - \mathrm{vol}(\Gamma\backslash H)^{-1} h(i) - H(u, u') \tag{2.9}$$

is of the trace class (it has only discrete spectrum on $L_0^2(\Gamma\backslash H)$) and

$$\mathrm{trace}\big(L_{\varphi,0} \text{ on } L_0^2(\Gamma\backslash H)\big) = \sum_{j=1}^{\infty} h(r_j) = \int_{\Gamma\backslash H} K_0(u, u)\, d\mu(u). \tag{2.10}$$

This is "a general formulation" of the Selberg trace formula. The problem is to compute explicitly the integral on the right-hand side.

2.6. The conjugacy classes. We shall write down the classification of the conjugacy classes of elements $g \in \mathrm{SL}(2, C)$ and their Jordan canonical form (e.g. Ford [4], Chap. 1).

The unit element: $\pm\begin{pmatrix}1 & \\ & 1\end{pmatrix}$.

Elliptic elements: $\mathrm{tr}\, g$ real and $|\mathrm{tr}\, g| < 2$, their canonical form is $\pm\begin{pmatrix}\varepsilon & \\ & \bar\varepsilon\end{pmatrix}$ where $|\varepsilon| = 1$ and $\varepsilon \neq \pm 1$.

All elliptic elements have finite orders, specifically:

$$\begin{array}{lll} \text{order } 4 & \text{if} & \mathrm{tr}\, g = 0, \\ \text{order } 3 & \text{if} & \mathrm{tr}\, g = -1, \\ \text{order } 6 & \text{if} & \mathrm{tr}\, g = 1. \end{array}$$

Parabolic elements: $\mathrm{tr}\, g = 2$ or $\mathrm{tr}\, g = -2$, with canonical form $\pm\begin{pmatrix}1 & z\\ & 1\end{pmatrix}$ where $z \in C$ and $z \neq 0$.

Hyperbolic elements: $\mathrm{tr}\, g$ real, $|\mathrm{tr}\, g| > 2$, with canonical form $\pm\begin{pmatrix}\lambda & \\ & \lambda^{-1}\end{pmatrix}$ where $\lambda > 1$.

Loxodromic elements: $\mathrm{tr}\, g$ non-real, with canonical form $\pm\begin{pmatrix}\varepsilon\lambda & \\ & \bar\varepsilon\lambda^{-1}\end{pmatrix}$ where $\lambda > 1$ and $|\varepsilon| = 1$, $\varepsilon \neq \pm 1$.

Every loxodromic element is a composition of an elliptic and a hyperbolic one.

3. The computation of the integral $I(s)$ for $\mathrm{Re}\, s > 2$ and its analytic continuation.

3.1. The decomposition of $I(s)$. Following the general idea announced in the introduction we consider the integral

$$I(s) = \int_{\Gamma\backslash H} K_0(u, u)\, E(u; s)\, d\mu(u)$$

which converges for all $s \in C$ different from all the poles of $E(u; s)$. Applying the Rankin–Selberg method we obtain

$$I(s) = \tfrac{1}{2}\int_0^{\infty} \mathscr{K}(v)\, v^{s-3}\, dv \quad \text{for} \quad \mathrm{Re}\, s > 2,$$

where $\mathscr{K}(v) = \int_0^1\int_0^1 K_0(u, u)\, dx\, dy$, $u = (x+iy, v)$ is the constant term of $K_0(\cdot, \cdot)$. To evaluate $\mathscr{K}(v)$ we make use of (2.9):

$$K_0(u, u) = \sum_{\gamma\in\Gamma} k(u, \gamma\cdot u) - \mathrm{vol}(\Gamma\backslash H)^{-1} h(i) - \frac{1}{2\pi}\int_{-\infty}^{\infty} E(u; 1+ir)\, E(u; 1-ir)\, h(r)\, dr.$$

From the Fourier expansion of the Eisenstein series we find that the constant term of the function

$$H \ni u \mapsto E(u; 1+ir)\, E(u; 1-ir)$$

equals

$$2v^2 + \frac{\zeta_k^*(ir)}{\zeta_k^*(1+ir)} v^{2-2ir} + \frac{\zeta_k^*(-ir)}{\zeta_k^*(1-ir)} v^{2+2ir} + \frac{4^3 v^2}{\zeta_k^*(1-ir)\,\zeta_k^*(1+ir)} \sum_{n\in\mathfrak{a}^*} \tau_{ir}(n)^2 K_{ir}(2\pi|n|v)^2.$$

The integration of the second and the third term against the even function $h(r)$ gives the same contribution to the integral $I(s)$. We decompose the constant term $\mathscr{K}(v)$ in several parts, the idea of this decomposition is taken from Zagier [24].

$$\mathscr{K}(v) = \sum_{n=1}^{4} \mathscr{K}_n(v)$$

where

$$\mathscr{K}_1(v) = \int_0^1\int_0^1 \sum_{\gamma\in\Gamma, c\neq 0} k(u, \gamma\cdot u)\,dxdy, \qquad \gamma = \pm\begin{pmatrix} a & b\\ c & d\end{pmatrix}, \qquad \Gamma = \mathrm{PSL}(2, Z[i]),$$

$$\mathscr{K}_2(v) = \int_0^1\int_0^1 \sum_{\gamma\in\Gamma, c=0} k(u, \gamma\cdot u)\,dxdy - \frac{v^2}{\pi^2}\int_{-\infty}^{\infty} h(r)\,dr,$$

$$\mathscr{K}_3(v) = -\frac{v^2}{\pi}\int_{-\infty}^{\infty} v^{2ir}\frac{\zeta_k^*(-ir)}{\zeta_k^*(1-ir)}\,h(r)\,dr - \frac{\pi^2}{2\zeta_k(2)}\,h(i),$$

$$\mathscr{K}_4(v) = -\frac{4^3v^2}{2\pi}\int_{-\infty}^{\infty}\frac{h(r)}{\zeta_k^*(1+ir)\,\zeta_k^*(1-ir)}\,dr\sum_{n\in\mathfrak{a}^*}\tau_{ir}(n)^2 K_{ir}(2\pi|n|v)^2.$$

According to the classification of conjugacy classes we decompose $\mathscr{K}_1$ and $\mathscr{K}_2$ further:

$$\mathscr{K}_1 = \mathscr{K}_{1,\mathrm{ell}(3)} + \mathscr{K}_{1,\mathrm{ell}(6)} + \mathscr{K}_{1,\mathrm{ell}(4)} + \mathscr{K}_{1,\mathrm{hyp}} + \mathscr{K}_{1,\mathrm{lox}} + \mathscr{K}_{1,\mathrm{par}}$$

(we have distinquished elliptic elements of given order) and $\mathscr{K}_2 = \mathscr{K}_{2,\mathrm{par}} + \mathscr{K}_{2,\mathrm{ell}}$ where

$$\mathscr{K}_{2,\mathrm{par}}(v) = \int_0^1\int_0^1 \sum_{\substack{\gamma = \pm\left(\begin{smallmatrix}1 & b\\ & 1\end{smallmatrix}\right)\\ b\in\mathfrak{a}}} k(u, \gamma\cdot u)\,dxdy - \pi v^2\int_0^{\infty}\varphi(t)\,dt,$$

$$\mathscr{K}_{2,\mathrm{ell}}(v) = \int_0^1\int_0^1 \sum_{\substack{\gamma = \pm\left(\begin{smallmatrix}i & b\\ & -i\end{smallmatrix}\right)\\ b\in\mathfrak{a}}} k(u, \gamma\cdot u)\,dxdy - \pi v^2\int_0^{\infty}\varphi(t)\,dt.$$

Notice that in $\mathscr{K}_{2,\mathrm{ell}}$ we have only elliptic elements of trace zero (i.e. those of order 4 and $c = 0$). Thus we may write

$$I(s) = \sum_{n=1}^{4} I_n(s) \quad \text{where} \quad I_n(s) = \frac{1}{2}\int_0^{\infty}\mathscr{K}_n(v)\,v^{s-3}\,dv.$$

3.2. The integral $I_4(s)$. At the first instance we calculate

$$I_4(s) = -\frac{16}{\pi}\int_{-\infty}^{\infty}\Big(\sum_{n\in\mathfrak{a}^*}\frac{\tau_{ir}(n)^2}{|n|^s}\Big)\Big(\int_{-\infty}^{\infty} K_{ir}(2\pi v)^2 v^{s-1}\,dv\Big)\times$$
$$\times\frac{h(r)\,dr}{\zeta_k^*(1+ir)\,\zeta_k^*(1-ir)}.$$

The inner integral equals (see [ET], 6.8 (45))

$$\frac{1}{8\pi^s}\frac{\Gamma(s/2)^2}{\Gamma(s)}\Gamma\Big(\frac{s}{2}+ir\Big)\Gamma\Big(\frac{s}{2}-ir\Big) \quad \text{for} \quad \mathrm{Re}\,s > 1,$$

while the series can be transformed into

$$\sum_{n\in\mathfrak{a}}\frac{\tau_{ir}(n)^2}{|n|^s} = 4\sum_{n\in X}\frac{\sigma_{-ir}(n)}{|n|^{s-2ir}}.$$

Next there holds the identity

$$\sum_{n\in X}\frac{\sigma_a(n)}{|n|^{2s}} = \frac{\zeta_k(s)\zeta_k(s-a)^2\zeta_k(s-2a)}{\zeta_k(2s-2a)}$$

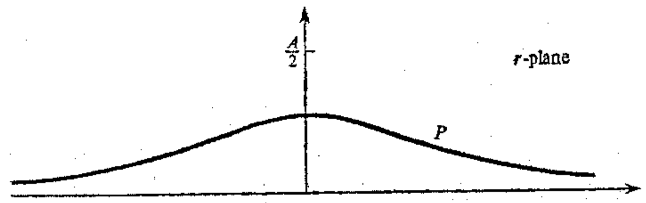
valid in the region of convergence of the zeta functions on the right-hand side; it can be proved comparing the local factors of the Euler products for example.

Putting all this together we obtain for $\mathrm{Re}\,s > 2$

$$I_4(s) = -\frac{1}{8\pi}\frac{\zeta_k^*(s/2)^2}{\zeta_k^*(s)}\int_{-\infty}^{\infty}\frac{\zeta_k^*(s/2+ir)\,\zeta_k^*(s/2-ir)}{\zeta_k^*(1+ir)\,\zeta_k^*(1-ir)}\,h(r)\,dr. \tag{3.1}$$

The *analytic continuation* of $I_4(s)$ follows the same lines as in Zagier's paper [24]. We denote the integral in (3.1) by $J(s)$ and introduce the new one

$$J_P(s) = \int_P\frac{\zeta_k^*(s/2+ir)\,\zeta_k^*(s/2-ir)}{\zeta_k^*(1+ir)\,\zeta_k^*(1-ir)}\,h(r)\,dr$$



where the path of integration is such that all zeros of the Dedekind zeta function lie on the left of $1+iP$. For the normalized integral $I_4^*(s) = \zeta_k^*(s)I_4(s)$ we can prove the following

PROPOSITION 3. *We have*

$$I_4^*(s) = \begin{cases} -\dfrac{1}{8\pi}\zeta_k^*\left(\dfrac{s}{2}\right)^2 J(s) & \text{for } \operatorname{Re} s > 2, \\ -\dfrac{1}{8\pi}\zeta_k^*\left(\dfrac{s}{2}\right)^2 J_P(s) - \dfrac{1}{4}\dfrac{\zeta_k^*(s/2)\zeta_k^*(s-1)}{\zeta_k^*(s/2-1)} h\left(i\dfrac{s-2}{2}\right) & \text{for } s \in U \end{cases}$$

where U is a neighbourhood of the point $s = 2$.

Let us calculate the residue of $I_4(s)$ at $s = 2$. The Riemann zeta function has the Laurent expansion

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1), \quad \gamma \text{ the Euler constant},$$

and

$$\zeta^*(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{s-1} + \frac{1}{2}(\gamma - \log 4\pi) + O(s-1).$$

The corresponding expansion for the Dedekind zeta function of the field $k = Q(i)$ is

$$\zeta_k(s) = \frac{\pi}{4(s-1)} + \frac{1}{4}\gamma_{Q(i)} + O(s-1)$$

where $\gamma_{Q(i)}$ is the generalized Euler constant equal to

$$\lim_{n\to\infty}\left(\sum_{N\mathfrak{a}<n}\frac{1}{N\mathfrak{a}} - \pi\log n\right).$$

The summation is taken over all integral ideals of $Q(i)$ with the norm less than the natural number n. Then

$$\zeta_k^*(s) = 4\pi^{-s}\Gamma(s)\zeta_k(s) = \frac{1}{s-1} + \frac{1}{\pi}\gamma_{Q(i)} - \gamma - \log\pi + O(s-1).$$

The expansion of $J_P(s)$ around $s = 2$ together with the last equality lead to

(3.2) $$\operatorname*{res}_{s=2} I_4(s) = \frac{1}{4}h(0) - \left(\frac{1}{\pi}\gamma_{Q(i)} - \gamma - \log\pi\right)g(0) - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\zeta_k^{*\prime}}{\zeta_k^*}(1+ir)h(r)\,dr.$$

3.3. The integral $I_3(s)$. The integrand in $\mathscr{K}_3(v)$ is holomorphic for $0 < \operatorname{Im} r < A/2$ except for a simple pole at $r = i$. This enables to transform $\mathscr{K}_3(v)$ into

$$\mathscr{K}_3(v) = \frac{iv^2}{2\pi}\int_{B-i\infty}^{B+i\infty} v^{-s}\frac{\zeta_k^*(s/2)}{\zeta_k^*(1+s/2)} h\left(i\frac{s}{2}\right)ds, \quad 2 < B < A.$$

In order to calculate $I_3(s)$ we use the Mellin inversion formula giving

$$I_3(s) = -\frac{1}{2}\frac{\zeta_k^*(s/2)}{\zeta_k^*(1+s/2)} h\left(i\frac{s}{2}\right) \quad \text{for} \quad 2 < \operatorname{Re} s < A.$$

Now we can state

PROPOSITION 4. *The function*

$$I_3^*(s) = \zeta_k^*(s)\,I_3(s) = -\frac{\zeta_k^*(s)\zeta_k^*(s/2)}{2\zeta_k^*(1+s/2)} h\left(i\frac{s}{2}\right)$$

has a meromorphic continuation, since ζ_k^* does, the poles come from those of the zeta functions involved, and

(3.3) $$\operatorname*{res}_{s=2} I_3^*(s) = -h(i).$$

3.4. The integral $I_2(s)$. To calculate the contribution to the integral $I(s)$ coming from parabolic elements with $c = 0$ we use the method of [24], but now we employ the Fourier transform in R^2 and the corresponding Poisson summation formula.

PROPOSITION 5. *We have*

$$I_2(s) = \frac{\zeta_k^*(s/2)}{8\cdot 2^s\pi^{\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)}\int_{-\infty}^{\infty}\frac{\Gamma(s/2+ir)\Gamma(s/2-ir)}{\Gamma(ir)\Gamma(-ir)}h(r)\,dr$$

in the region $\operatorname{Re} s > 1$, *and for $I_2^*(s) = \zeta_k^*(s)I_2(s)$ we have*

(3.4) $$\operatorname*{res}_{s=2} I_2^*(s) = \frac{\zeta_k(2)}{2\pi^4}\int_{-\infty}^{\infty} r^2 h(r)\,dr.$$

3.5. The zeta functions related to binary quadratic forms. To investigate the integral $I_1(s)$ we have to consider some zeta functions related to the Dedekind zeta functions for quadratic extensions of the field $k = Q(i)$. Let us consider a binary quadratic form

$$\Phi(m, n) = am^2 + bmn + cn^2, \quad m, n \in \mathfrak{a} = Z[i],$$

with coefficients a, b, c from the ring $\mathfrak{a}$. The group SL$(2, \mathfrak{a})$ operates on the set of such forms by the unimodular transformations of variables

$$(\gamma\cdot\Phi)(m,n) = \Phi(am+cn,\, bm+dn), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $\Delta = b^2-4ac$ be the discriminant of the form Φ. The collection of such forms with fixed discriminant $\Delta \neq 0$ breaks into a finite number of equivalence classes. Define the following zeta functions

$$(3.5) \qquad \zeta(s,\Delta) = \sum_{\substack{[\Phi]\\ \operatorname{discr}\Phi=\Delta}} \sum_{\substack{(m,n)\in(\mathfrak{a}\times\mathfrak{a})/\operatorname{Aut}\Phi\\ (m,n)\in X}} |\Phi(m,n)|^{-2s}.$$

The first sum is taken over all $SL(2,\mathfrak{a})$-equivalence classes of quadratic forms with discriminant Δ,

$$\operatorname{Aut}\Phi = \{\gamma\in SL(2,\mathfrak{a});\ \gamma\cdot\Phi = \Phi\}$$

is the stabilizer of the form and the second sum is taken over inequivalent pairs of Gaussian integers with respect to the group $\operatorname{Aut}\Phi$.

If Δ is the discriminant of a field K being a quadratic extension of $k = Q(i)$, then $\zeta(s,\Delta)$ coincides with the Dedekind zeta function $\zeta_K(s)$ of this extension. This assertions is based on the fact that we have here (analogously to the case of quadratic extensions of the rationals Q) the one-to-one correspondence between ideal classes of K and the $SL(2,\mathfrak{a})$-equivalence classes of binary quadratic forms with coefficients from $\mathfrak{a}$ (see Kaplansky [10] for the proof of such correspondence formulated there in a more general seeting). The first sum in (3.5) corresponds to the ideal classes of K, the second sum to the ideals in a fixed class and the value $|\Phi(m,n)|^2$ is the norm of the ideal generated by (m,n).

The investigation of biquadratic extensions of Q was started by Dirichlet [2] and continued by Hilbert [7].

In the case when $\Delta = Df^2$, D being the discriminant of the corresponding extension of $Q(i)$, $f\in X$, we have the situation analogous to that of Zagier's paper ([23], Prop. 3).

PROPOSITION 6. *Recall that $\zeta(s,\Delta)$ is defined by (3.5), where Δ is as above, $s\in C$, $\operatorname{Re}s>1$. Then*

(i) $\zeta(s,\Delta) = \zeta_k(2s)\sum_{n\in X} n(a)|n|^{-2s}$ *with $n(a)$ being the number of solutions $b\,(\mathrm{mod}\,2a)$ of the congruence $b^2\equiv\Delta\,(\mathrm{mod}\,4a)$ in the ring of Gaussian integers.*

(ii)

$$\zeta(s,\Delta) = \begin{cases} 0 & \text{if } \Delta\equiv 2 \text{ or } 3\,(\mathrm{mod}\,4),\\ \zeta_k(s)\zeta_k(2s-1) & \text{if } \Delta = 0,\\ \zeta_k(s)L_D(s)\displaystyle\sum_{d|f}\mu(d)\left[\frac{D}{d}\right]|d|^{-2s}\sigma_{1-2s}\left(\frac{f}{d}\right) & \\ \qquad\qquad \text{if } \Delta\equiv 0 \text{ or } 1\,(\mathrm{mod}\,4),\ \Delta\neq 0. \end{cases}$$

(iii) *$\zeta(s,\Delta)$ has a meromorphic continuation to the whole complex plane, the positions of poles and the values of residues can be read off from* (ii).

We indicate here only the basic ideas which one can use to construct a proof. The identity (i) is equivalent to the assertion of theorem on representation of Gaussian integers by binary quadratic forms over $\mathfrak{a}$; $n(a)$ is the number of $SL(2,\mathfrak{a})$-inequivalent primitive representations of the integer a by forms with discriminant Δ. In the case of rational integers it is the main theorem of the theory of binary quadratic forms, see Landau [13], Th. 203. Hence we must write down the analog of this theorem in the case of $\mathfrak{a} = Z[i]$ to obtain (i). It can be also proved directly as in [23]. Concerning (ii) we have written $\Delta = Df^2$ with the above meaning of D and f. $\left[\frac{D}{\cdot}\right]$ is the corresponding Kronecker symbol for the quadratic extension of $Q(i)$ with discriminant D. If

$$L_D(s) = \sum_{n\in X}\left[\frac{D}{n}\right]|n|^{-2s}$$

is the associated L-series, then $\zeta_k(s)L_D(s)$ is the Dedekind zeta function for this quadratic extension.

$$\sigma_\nu(n) = \sum_{d|n,\, d\in X}|d|^{2\nu}, \qquad \nu\in C,\ n\in\mathfrak{a}^*,$$

where $\mu(\cdot)$ is the Möbius function in $\mathfrak{a}$ defined using prime ideals factorization, analogously as for Z. The formula for $\zeta(s,\Delta)$, $\Delta\neq 0$, can be proved using (i) and the technique presented in Hirzebruch, Zagier ([8], Prop. 2).

(iii) is a consequence of analytic properties of the Dedekind zeta functions.

In the further section (3.10) we will need the case of (ii) when $\Delta = -4$ (i.e. $D = 1$, $f = 2i = (1+i)^2$; $1+i$ is a prime number in $\mathfrak{a}$). Then

$$\zeta(s,-4) = \zeta_k(s)^2\sum_{\substack{d|(1+i)^2\\ d\in X}}\mu(d)|d|^{-2s}\sigma_{1-2s}\left(\frac{f}{d}\right);$$

we have here $\mu(1) = 1$, $\mu(1+i) = -1$, $\mu\big((1+i)^2\big) = 0$. Hence

$$(3.6) \qquad \zeta(s,-4) = \zeta_k(s)^2 l(s)$$

with

$$l(s) = 1+2^{1-2s}+2^{2-4s}-2^{-s}-2^{1-3s}.$$

3.6. The integral $I_1(s)$. There are employed here the ideas from Zagier's paper [24]. It is an important place of the work, since there appear the zeta functions related to number fields. Inserting the formula for $\mathscr{K}_1(v)$ into $I_1(s)$ we obtain

$$I_1(s) = \frac{1}{2}\int_0^\infty\int_0^1\int_0^1 \sum_{\gamma\in\Gamma,\gamma\notin\Gamma_\infty} k(u,\gamma\cdot u)v^s d\mu(u)$$

$$= \frac{1}{2}\int_0^\infty\int_0^1\int_0^1 \sum_{[\gamma]}{}' \sum_{\substack{\sigma\in\Gamma_\gamma\diagdown\Gamma \\ \sigma^{-1}\gamma\sigma\notin\Gamma_\infty}} k\big(u,(\sigma^{-1}\gamma\sigma)\cdot u\big)v^s d\mu(u)$$

where Γ_∞ is the stabilizer of the cusp ∞, Γ_γ the centralizer of $\gamma\in\Gamma$, the brackets $[\cdot]$ denote a conjugacy class in Γ and the first sum is over all non-trivial classes; each conjugacy class contains at least one element which does not belong to Γ_∞. We write further

$$I_1(s) = \frac{1}{2}\int_0^\infty\int_0^1\int_0^1 \sum_{[\gamma]}{}' \sum_{\substack{\sigma\in\Gamma_\gamma\diagdown\Gamma/\Gamma_\infty \\ \sigma^{-1}\gamma\sigma\notin\Gamma_\infty}} \sum_{\beta\in\Gamma_\infty} k\big(\beta\cdot u,(\sigma^{-1}\gamma\sigma)\beta\cdot u\big)v^s d\mu(u)$$

$$= \frac{1}{2}\sum_{[\gamma]}{}' \sum_{\substack{\sigma\in\Gamma_\gamma\diagdown\Gamma/\Gamma_\infty \\ \sigma^{-1}\gamma\sigma\notin\Gamma_\infty}} \int_H k\big(u,(\sigma^{-1}\gamma\sigma)\cdot u\big)v^s d\mu(u).$$

We have met above the integral

$$\int_H k(u,\tau\cdot u)v^s d\mu(u) = \int_H \varphi(g^{-1}\tau g)v^s d\mu(u) \tag{3.7}$$

(φ is right and left $\mathrm{SU}(2)$-invariant, $u = g\,\mathrm{SU}(2)$) with $\tau\in\mathrm{SL}(2,C)$, $\tau = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$, $c\neq 0$.

We consider the contribution to $I_1(s)$ given by non-parabolic elements. Each hyperbolic (loxodromic or elliptic) element τ, when considered as a conformal mapping of the Riemann sphere

$$\tau z = \frac{az+b}{cz+d}, \quad z\in\bar{C},$$

has two different fixed points. We will change in (3.7) the variables by diagonalization of the matrix τ. The fixed points of the conformal mapping τ are

$$z_{1,2} = \frac{a-d\mp\sqrt{\Delta}}{2c}, \quad \Delta = (a+d)^2-4.$$

In the case of non-parabolic elements ($\Delta\neq 0$) the map

$$T = \sqrt{\frac{c}{\sqrt{\Delta}}}\begin{pmatrix}1 & z_1\\ 1 & -z_2\end{pmatrix}$$

sends the points z_1, z_2 to $0, \infty$ and $T\tau T^{-1}$ is a diagonal matrix. The change of variable $g\mapsto T^{-1}g$, in particular

$$v\mapsto \frac{\sqrt{|\Delta|}}{c}\,\frac{v}{|z-1|^2+v^2},$$

carries the integral (3.7) into

$$\frac{|\Delta|^{s/2}}{|c|^s}\int_H \varphi\big(g^{-1}(T\tau T^{-1})g\big)\left(\frac{v}{|z-1|^2+v^2}\right)^s d\mu(u) = \frac{|\Delta|^{s/2}}{|c|^s}V(s,\operatorname{tr}\tau).$$

We can write now

$$I_1(s) = \sum_{t\in\mathfrak{a}}\Bigg(\frac{1}{4}\sum_{\substack{[[\gamma]]\\ \operatorname{tr}\gamma=t}}\sum_{\substack{\sigma\in\Gamma_\gamma\diagdown\Gamma/\Gamma_\infty\\ \sigma^{-1}\gamma\sigma\notin\Gamma_\infty}} |c(\sigma^{-1}\gamma\sigma)|^{-s}\Bigg)|\Delta|^{s/2}V(s,t), \tag{3.8}$$

$$\Delta = t^2-4.$$

We work in $\tilde{\Gamma} = \mathrm{SL}(2,\mathfrak{a})$ rather than $\Gamma = \mathrm{PSL}(2,\mathfrak{a})$ in order to have a well defined trace; $[[\cdot]]$ means a conjugacy class in $\tilde{\Gamma}$ and $c(\sigma^{-1}\gamma\sigma)$ is the element in the lower left-hand corner of $\sigma^{-1}\gamma\sigma$, $\Gamma_\gamma\subset\Gamma$ and $\sigma^{-1}\gamma\sigma\in\mathrm{SL}(2,\mathfrak{a})$ make sense for $\gamma\in\mathrm{SL}(2,\mathfrak{a})$, $\sigma\in\mathrm{PSL}(2,\mathfrak{a})$.

There is a one-to-one correspondence between conjugacy classes $[[\gamma]]$ of trace t and $\mathrm{SL}(2,\mathfrak{a})$-equivalence classes of binary quadratic forms Φ with coefficients from $\mathfrak{a}$ given by

$$\Phi(m,n) = cm^2+(d-a)mn-bn^2,$$

$$\gamma = \begin{pmatrix}a & b\\ c & d\end{pmatrix}, \quad a+d = t, \quad m,n\in\mathfrak{a}.$$

The expression in parantheses in (3.8) equals

$$\sum_{\substack{[[\gamma]]\\ \operatorname{tr}\gamma=t}}\sum_{\substack{\sigma\in\Gamma_\gamma\diagdown\Gamma/\Gamma_\infty\\ c(\sigma^{-1}\gamma\sigma)\in X}} |c(\sigma^{-1}\gamma\sigma)|^{-s}, \tag{3.9}$$

X is the sector in $\mathfrak{a}$. There is a one-to-one correspondence between Γ/Γ_∞ and the set of pairs $(\mathfrak{m},\mathfrak{n})$ of relatively prime ideals in $\mathfrak{a}$ given by mapping an element $\sigma\in\Gamma/\Gamma_\infty$ to its first column. Let (m,n) be a representative of the pair $(\mathfrak{m},\mathfrak{n})$. Under this bijection we have

$$c(\sigma^{-1}\gamma\sigma) = \Phi(m,n), \quad \Gamma_\gamma\cong \mathrm{Aut}\,\Phi/\{\pm 1,\pm i\},$$

and (3.9) becomes

$$\sum_{\substack{[\Phi]\\ \operatorname{discr}\Phi=\Delta}} \sum_{\substack{(m,n)\in(\mathfrak{a}\times\mathfrak{a})/\operatorname{Aut}\Phi\\ \Phi(m,n)\in X\\ (m,n)=1}} |\Phi(m,n)|^{-s} = \frac{\zeta(s/2, t^2-4)}{\zeta_k(s)}.$$

The final formula has the form

$$I_1(s) = \sum_{t\in\mathfrak{a}} \frac{\zeta(s/2, \Delta)}{\zeta_k(s)} |\Delta|^{s/2} V(s,t), \quad \Delta = t^2-4. \tag{3.10}$$

The absolute convergence of this series will be asserted in Section 4. The integral transform $V(s,t)$ of φ can be written, using the correspondence between $K\diagdown G/K$ and $[0,\infty)$, as

$$V(s,t) = \int_H \varphi\left(\frac{|z-\Lambda\cdot z|^2+(v-\Lambda\cdot v)^2}{v(\Lambda\cdot v)}\right)\left(\frac{v}{|z-1|^2+v^2)}\right)^s \frac{dx\,dy\,dv}{v^3}, \tag{3.11}$$

where $\Lambda = T\tau T^{-1}$ is the Jordan canonical form of the matrix τ and $t = \operatorname{tr}\tau$.

3.7. The hyperbolic classes. We carry out the summation (3.10) over t being rational integers with $|t|>2$. After using the expression (2.1) for the action of $\mathrm{SL}(2,C)$ on the symmetric space H and doing some simple manipulations with the argument of φ in (3.11) we obtain that

$$V(s,t) = \int_H \varphi\left(\Delta\,\frac{|z|^2+v^2}{v^2}\right)\left(\frac{v}{|z-1|^2+v^2}\right)^s \frac{dx\,dy\,dv}{v^3}, \quad \Delta>0.$$

The calculations similar to those in [24] lead to

(3.12) $V(s,t)$

$$= \frac{(2\pi)^2\Gamma(s)}{2^{2s}\Gamma\left(\frac{s+1}{2}\right)^2} \int_0^\infty \varphi\big(\Delta(1+r^2)\big)\, F\left(\frac{s}{2},\frac{s}{2},1;\frac{r^2}{1+r^2}\right)(1+r^2)^{-s/2} r\,dr,$$

where F is the hypergeometric function. It can be verified that $V(s,t)$ is a holomorphic function of s in the region $0<\operatorname{Re}s<2+\Delta$. We will need its value at $s=2$;

$$V(2,t) = \frac{1}{2\Delta}Q(\Delta), \quad \Delta = t^2-4,$$

where Q is the function (2.7) used to define the spherical transform of φ.

Proposition 7. *The contribution to the integral $I_1(s)$ of the hyperbolic conjugacy classes has the form*

$$I_{1,\mathrm{hyp}}(s) = \sum_{\substack{t\in Z\\ |t|>2}} \frac{\zeta(s/2,\Delta)}{\zeta_k(s)} |\Delta|^{s/2} V(s,t)$$

with $V(s,t)$ given by (3.12), *and*

$$\operatorname*{res}_{s=2} I_{1,\mathrm{hyp}}(s) = \frac{1}{2\zeta_k(2)} \sum_{\substack{t\in Z\\ |t|>2}} Q(t^2-4) \operatorname*{res}_{s=2}\zeta\left(\frac{s}{2}, t^2-4\right). \tag{3.13}$$

The values of residues of the involved zeta functions can be read off from Proposition 6 (ii); they are expressed by the class numbers of quadratic extensions of the field $k = Q(i)$ having discriminants D which are rational integers (see Hasse [6], Sec. 26, for related formulae).

3.8. The loxodromic classes. The contribution of these classes is similar to that of the hyperbolic ones. The Jordan canonical form of a loxodromic element $\tau\in\mathrm{SL}(2,C)$ ($t=\operatorname{tr}\tau$ has a non-zero imaginary part) is

$$\pm\begin{pmatrix}\varepsilon\lambda & \\ & \bar{\varepsilon}\lambda^{-1}\end{pmatrix} \quad \text{with some } \lambda>0 \text{ and } |\varepsilon|=1,\ \varepsilon\neq\pm1.$$

The argument of φ in the formula for $V(s,t)$ becomes

$$\frac{|z-\tau\cdot z|^2+(v-\tau\cdot v)^2}{v(\tau\cdot v)} = \frac{|1-(\varepsilon\lambda)^2|^2|z|^2+(1-\lambda^2)^2v^2}{v^2} = \frac{|\Delta|\,|z|^2+\Delta_{\mathrm{hyp}}v^2}{v^2},$$

where $\Delta = t^2-4$ and

$$\Delta_{\mathrm{hyp}} = \left(\lambda+\frac{1}{\lambda}\right)^2-4>0. \tag{3.14}$$

Every loxodromic element is a composition of an elliptic and a hyperbolic ones

$$\begin{pmatrix}\varepsilon\lambda & \\ & \bar{\varepsilon}\lambda^{-1}\end{pmatrix} = \begin{pmatrix}\varepsilon & \\ & \bar{\varepsilon}\end{pmatrix}\begin{pmatrix}\lambda & \\ & \lambda^{-1}\end{pmatrix}$$

hence Δ_{hyp} corresponds to "the hyperbolic part" of the given loxodromic element.

The counterpart of the formula (3.12) is now

$$V(s,t) = \frac{(2\pi)^2\Gamma(s)}{2^{2s}\Gamma\left(\frac{s+1}{2}\right)^2}\times \tag{3.15}$$

$$\times\int_0^\infty \varphi(|\Delta|r^2+\Delta_{\mathrm{hyp}})F\left(\frac{s}{2},\frac{s}{2},1;\frac{r^2}{1+r^2}\right)(1+r^2)^{-s/2}r\,dr,$$

where again $V(s,t)$ is holomorphic in the region $0 < \operatorname{Re} s < 2+A$. Its value at $s=2$ is

$$V(2,t) = \frac{1}{2|\Delta|} Q(\Delta_{\text{hyp}}).$$

The final formulae concerning the contribution of loxodromic classes we write down as

PROPOSITION 8. *We have*

$$I_{1,\text{lox}}(s) = \sum_{\substack{t\in\mathfrak{a}\\ \operatorname{Im} t\neq 0}} \frac{\zeta(s/2,\Delta)}{\zeta_k(s)} |\Delta|^{s/2} V(s,t)$$

with $V(s,t)$ *given by* (3.15), *and*

$$\operatorname*{res}_{s=2} I_{1,\text{lox}}(s) = \frac{1}{2\zeta_k(2)} \sum_{\substack{t\in\mathfrak{a}\\ \operatorname{Im} t\neq 0}} Q\big(\Delta_{\text{hyp}}(t)\big) \operatorname*{res}_{s=2} \zeta\left(\frac{s}{2},\Delta\right), \tag{3.16}$$

where Δ_{hyp} *is given by* (3.14); *in fact it is a function of the trace* t.

We are dealing here with the quadratic extensions of the field $k = Q(i)$ having discriminants which are Gaussian integers of non-zero imaginary part.

3.9. The elliptic elements of order 3 and 6. The contribution to the integral $I(s)$ given by these elements (i.e. the ones of trace 1 and -1) is equal to

$$I_{1,\text{ell}(3)}(s) + I_{1,\text{ell}(6)}(s) = \frac{\zeta(s/2,-3)}{\zeta_k(s)} |\operatorname{discr} \boldsymbol{K}|^{s/2} \big(V(s,1) + V(s,-1)\big), \tag{3.17}$$

where $\boldsymbol{K} = Q\left(\sqrt{-1}, \sqrt{-3}\right)$ is a biquadratic extension of the rationals and $\zeta(s,-3) = \zeta_{\boldsymbol{K}}(s)$ the corresponding Dedekind zeta function of this field (discr $\boldsymbol{K} = -3$). The formula for the class number of $\boldsymbol{K}$ (e.g. Hasse [6], Sec. 26) gives

$$\operatorname*{res}_{s=2} \zeta\left(\frac{s}{2},-3\right) = \frac{\pi^2 \log\left(2+\sqrt{3}\right)}{9},$$

where $2+\sqrt{3}$ is the fundamental unit of the real field $Q(\sqrt{3})$. The integral transforms of φ in (3.17) can be calculated following the same way as for the hyperbolic elements. The Jordan canonical form of elements having trace 1 is

$$\tau = \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix} \quad \text{with} \quad \lambda = \frac{1 \mp i\sqrt{3}}{2}$$

and for elements of trace -1 we must put $\lambda = \left(-1 \mp i\sqrt{3}\right)/2$; there are two conjugacy classes in each case. The action looks like

$$\tau\cdot(z,v) = (\lambda^2 z, v).$$

The corresponding formulae for the integral transform have the form

$$V(s,-1) = V(s,1) = \int_H \varphi\left(3\frac{|z|^2}{v^2}\right)\left(\frac{v}{|z-1|^2+v^2}\right)^s \frac{dx\,dy\,dv}{v^3} \tag{3.18}$$

$$= \frac{(2\pi)^2\Gamma(s)}{2^{2s}\Gamma\left(\frac{s+1}{2}\right)^2} \int_0^\infty \varphi(3r^2) F\left(\frac{s}{2},\frac{s}{2},1;\frac{r^2}{1+r^2}\right)(1+r^2)^{-s/2} r\,dr.$$

$V(s,1)$ is a holomorphic function of s in the region $0 < \operatorname{Re} s < 2+A$ and its value at $s=2$ is

$$V(2,1) = \frac{1}{6} g(0) = \frac{1}{12\pi} \int_{-\infty}^{\infty} h(r)\,dr.$$

If we multiply (3.17) by $\zeta_k^*(s)$, we have

PROPOSITION 9. *We have*

$$I^*_{1,\text{ell}(3)}(s) + I^*_{1,\text{ell}(6)}(s) = 8\cdot 3^{s/2}\pi^{-s}\zeta(s/2,-3)\,V(s,1)$$

with $V(s,1)$ *given by* (3.18), *and*

$$\operatorname*{res}_{s=2}\left(I^*_{1,\text{ell}(3)}(s) + I^*_{1,\text{ell}(6)}(s)\right) = \tfrac{4}{9}\log\left(2+\sqrt{3}\right) g(0). \tag{3.19}$$

3.10. The elliptic elements of order 4. Let us consider first the elliptic elements of trace zero and $c=0$, i.e. these

$$\gamma = \pm\begin{pmatrix} i & b \\ & -i \end{pmatrix} \quad \text{with} \quad b\in\mathfrak{a}.$$

The contribution of these elements to the kernel function $K(u,u)$ appears as follows

$$\sum_\gamma k(u,\gamma\cdot u) = \sum_{b\in\mathfrak{a}} \varphi\left(\frac{|2z+b|^2}{v^2}\right)$$

and its constant term equals

$$\int_0^1\int_0^1 \sum_{b\in\mathfrak{a}} \varphi\left(\frac{|2z+b|^2}{v^2}\right) dx\,dy = \pi v^2 \int_0^\infty \varphi(t)\,dt.$$

The decomposition of $\mathscr{K}(v)$, which has been written down in Section 3.1, implies that

$$\mathscr{K}_{2,\mathrm{ell}}(v) \equiv 0.$$

We pay now an attention to the elliptic elements of trace zero and $c = 0$. Their contribution to the integral $I(s)$ is given, according to (3.10), by

$$I_{1,\mathrm{ell}(4)}(s) = \frac{\zeta(s/2, -4)}{\zeta_k(s)} 2^s V(s, 0), \tag{3.20}$$

where from (3.6) we have that

$$\zeta\left(\frac{s}{2}, -4\right) = \zeta_k\left(\frac{s}{2}\right)^2 l\left(\frac{s}{2}\right) = \zeta_k\left(\frac{s}{2}\right)^2 (1+2^{1-s}+2^{2-2s}-2^{-\frac{s}{2}}-2^{1-\frac{3s}{2}}) \tag{3.21}$$

and for the integral transform

$$V(s, 0) = \int_H \varphi\left(\frac{4|z|^2}{v^2}\right)\left(\frac{v}{|z-1|^2+v^2}\right)^s \frac{dx\,dy\,dv}{v^3}.$$

The last formula for $V(s, 0)$ follows from the fact that the Jordan form of elements having trace zero is $\mp\begin{pmatrix} i & \\ & -i\end{pmatrix}$. The calculations similar to those for other non-parabolic classes yield

$$V(s, 0) = \frac{(2\pi)^2\Gamma(s)}{2^{2s}\Gamma\left(\frac{s+1}{2}\right)^2}\int_0^\infty \varphi(4r^2) F\left(\frac{s}{2}, \frac{s}{2}, 1; \frac{r^2}{1+r^2}\right)(1+r^2)^{-s/2} r\,dr$$

$$= \frac{(2\pi)^2\Gamma(s)}{2^{2s}\Gamma\left(\frac{s+1}{2}\right)^2} A(s).$$

We use now the spherical transform (2.7) of φ and make the substitution $r^2/(1+r^2) = u$ to obtain

$$A(s) = 2^{-4}\pi^{-3/2}\int_{-\infty}^{\infty} h(r) r^2 \int_0^1 u^{-1/4}(1-u)^{(s-3)/2} \times$$
$$\times P^{-1/2}_{ir-1/2}\left(\frac{1+u}{1-u}\right) F\left(\frac{s}{2}, \frac{s}{2}, 1; u\right) du\,dr.$$

The representation of Legendre functions ([EH], 3.2(16)),

$$P^{-1/2}_{ir-1/2}\left(\frac{1+u}{1-u}\right) = 2\pi^{-1/2}u^{1/4}(1-u)^{1/2-ir} F(\tfrac{1}{2}-ir, 1-ir, \tfrac{3}{2}; u),$$

implies that

$$A(s) = \frac{1}{8\pi^2}\int_{-\infty}^{\infty} h(r) r^2 \int_0^1 (1-u)^{s/2-ir-1} \times$$
$$\times F\left(\frac{1}{2}-ir, 1-ir, \frac{3}{2}; u\right) F\left(\frac{s}{2}, \frac{s}{2}, 1; u\right) du\,dr. \tag{3.22}$$

To calculate the inner integral above we utilize the formula ([EH], 2.9 (33)),

$$F(a, b, c; z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a, b, a+b+1-c; 1-z) +$$
$$+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b} F(c-a, c-b, c+1-a-b; 1-z) \tag{3.23}$$

in the particular case of the hypergeometric function $F(\frac{1}{2}-ir, 1-ir, \frac{3}{2}; u)$. After putting into $A(s)$, in the place of $F(\frac{1}{2}-ir, 1-ir, \frac{3}{2}; u)$, the first term on the right of (3.23) one has the contribution to the inner integral

$$B_1(s, r) = \frac{\Gamma(\frac{3}{2})\Gamma(2ir)}{\Gamma(1+ir)\Gamma(\frac{1}{2}+ir)}\int_0^1 (1-u)^{s/2-ir-1} \times$$
$$\times F(s/2-ir, 1-ir, 1-2ir; 1-u) F(s/2, s/2, 1; u)\,du.$$

When expanding the first hypergeometric function as an infinite series and using the integral representation ([EH], 2.4 (2)) (which in our case is valid for $0 < \mathrm{Re}\, s < 2$ and the coefficient of u in the hypergeometric function equal to 1) we have to consider the integrals inside the sum over n;

$$\int_0^1 (1-u)^{s/2-ir-1+n} F\left(\frac{s}{2}, \frac{s}{2}, 1; u\right) du$$
$$= \frac{\Gamma(s/2-ir+n)}{\Gamma(s/2-ir+1+n)} F\left(\frac{s}{2}, \frac{s}{2}, \frac{s}{2}-ir+1+n; 1\right)$$
$$= \frac{\Gamma(s/2-ir+n)\Gamma(-s/2-ir+1+n)}{\Gamma(-ir+1+n)^2},$$

see [EH], 2.8 (46).

In this way one can obtain the representation

$$B_1(s,r) = \frac{\Gamma(\frac{3}{2})\Gamma(2ir)}{\Gamma(\frac{1}{2}+ir)\Gamma(1+ir)\Gamma(1-ir)} \times$$

$$\times \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{2}-ir+n)\Gamma(1-2ir)\Gamma(s/2-ir+n)\Gamma(-s/2-ir+1+n)}{\Gamma(\frac{1}{2}-ir)\Gamma(1-2ir+n)\Gamma(1-ir+n)}.$$

We have met here the generalized hypergeometric function of the type

$${}_3F_2(a,b,c;u,v;z), \quad \text{see [EH], Chap. 4.}$$

It is in our case the value at $z=1$ of such series with the parameters:

$$a = \frac{s}{2}-ir, \quad b = -\frac{s}{2}-ir+1, \quad c = \tfrac{1}{2}-ir,$$

$$u = \tfrac{1}{2}(a+b+c) = 1+ir, \quad v = 2c = 1-2ir.$$

This value is given by Watson's theorem ([EH], 4.4 (6))

$${}_3F_2\left(a,b,c;\frac{1}{2}(a+b+1),2c;1\right)$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{a+b+1}{2}\right)\Gamma\left(c+\frac{1}{2}-\frac{a}{2}-\frac{b}{2}\right)}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}-\frac{a}{2}\right)\Gamma\left(c+\frac{1}{2}-\frac{b}{2}\right)}.$$

The application of standard identities for the involved gamma functions reduces the formula for B_1 to

$$B_1(s,r) = \frac{1}{8ir}\cdot\frac{\Gamma\left(\frac{s}{4}-\frac{ir}{2}\right)\Gamma\left(-\frac{s}{4}-\frac{ir}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{s}{4}-\frac{ir}{2}+\frac{1}{2}\right)\Gamma\left(-\frac{s}{4}-\frac{ir}{2}+1\right)}.$$

The second term on the right-hand side of (3.23) gives the contribution to the inner integral in (3.22) as

$$B_2(s,r) = \frac{\Gamma(\frac{3}{2})\Gamma(-2ir)}{\Gamma(\frac{1}{2}-ir)\Gamma(1-ir)}\int_0^1 (1-u)^{s/2+ir-1}\times$$

$$\times F(1+ir,\tfrac{1}{2}+ir,1+2ir;1-u)\,F\left(\frac{s}{2},\frac{s}{2},1;u\right)du.$$

The same way as for B_1 leads to

$$B_2(s,r) = -\frac{1}{8ir}\cdot\frac{\Gamma\left(\frac{s}{4}+\frac{ir}{2}\right)\Gamma\left(-\frac{s}{4}+\frac{ir}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{s}{4}+\frac{ir}{2}+\frac{1}{2}\right)\Gamma\left(-\frac{s}{4}+\frac{ir}{2}+1\right)}.$$

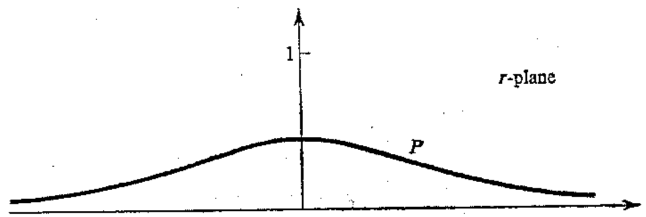
We have calculated above the inner integral in $A(s)$ and after the change $r\mapsto -r$ in $B_1(s,r)$ we have the formula for the integral transform

$$V(s,0) = \frac{i\Gamma(s)}{8\cdot 2^{2s}\Gamma\left(\frac{s+1}{2}\right)^2}\times \tag{3.24}$$

$$\times\int_{-\infty}^{\infty}\frac{\Gamma\left(\frac{s}{4}+\frac{ir}{2}\right)\Gamma\left(-\frac{s}{4}+\frac{ir}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{s}{4}+\frac{ir}{2}+\frac{1}{2}\right)\Gamma\left(-\frac{s}{4}+\frac{ir}{2}+1\right)}\,r\,h(r)\,dr,$$

which is valid in the region $0<\operatorname{Re}s<2$. Denote the integral on the right-hand side of (3.24) by $J(s)$. It is not defined on the lines $\operatorname{Re}s=0$ and $\operatorname{Re}s=2$ but has a meaning in the other parts of the complex plane. The function arising from the analytical continuation of $J(s)$ to the region $\operatorname{Re}s>2$ is not equal to the integral $J(s)$ given by (3.24) there.

The *analytic continuation* can be done as follows. For the suitable path



the pole of $\Gamma\left(-\frac{s}{4}+\frac{ir}{2}+\frac{1}{2}\right)$ at $r=i\left(1-\frac{s}{2}\right)$ lies between R and P. Let us consider

$$J_P(s) = \int_P F(r)\,dr \tag{3.25}$$

with the integrand

$$F(r) = \frac{\Gamma\left(\frac{s}{4}+\frac{ir}{2}\right)\Gamma\left(-\frac{s}{4}+\frac{ir}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{s}{4}+\frac{ir}{2}+\frac{1}{2}\right)\Gamma\left(-\frac{s}{4}+\frac{ir}{2}+1\right)} h(r) r.$$

From the fact that

$$\operatorname*{res}_{r=i(1-s/2)} \Gamma\left(-\frac{s}{4}+\frac{ir}{2}+\frac{1}{2}\right) = -2i$$

one deduces the value of

$$\operatorname*{res}_{r=i(1-s/2)} F(r) = \frac{\Gamma\left(\frac{s-1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{s}{2}\right)}(2-s)h\left(i\frac{2-s}{2}\right).$$

Having in mind the equality

$$J(s)-J_P(s) = 2\pi i \operatorname*{res}_{r=i(1-s/2)} F(r)$$

we find that the analytical continuation of $J(s)$ is given in some neighbourhood U of $s=2$ by the expression

$$J(s) = J_P(s)+2i\sqrt{\pi}\,(2-s)\frac{\Gamma\left(\frac{s-1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}h\left(i\frac{2-s}{2}\right) \tag{3.26}$$

where the function $J_P(s)$ is analytical in U. The modified function

$$I^*_{1,\mathrm{ell}(4)}(s) = \zeta^*_k(s) I_{1,\mathrm{ell}(4)}(s),$$

after inserting the formulae (3.20) and (3.24), can be expressed in the region $0<\operatorname{Re}s<2$ as

$$I^*_{1,\mathrm{ell}(4)}(s) = \frac{1}{2^7\pi}\zeta^*_k\left(\frac{s}{2}\right)^2 l\left(\frac{s}{2}\right)2^s iJ(s).$$

Its analytical continuation is given by

PROPOSITION 10. *It is true in a neighbourhood U of the point $s=2$ that*

$$I^*_{1,\mathrm{ell}(4)}(s) = \frac{1}{2^7\pi}\zeta^*_k\left(\frac{s}{2}\right)^2 l\left(\frac{s}{2}\right)2^s iJ_P(s)+$$

$$+\frac{1}{2^6\sqrt{\pi}}\zeta^*_k\left(\frac{s}{2}\right)^2 l\left(\frac{s}{2}\right)2^s(s-2)\frac{\Gamma\left(\frac{s-1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}h\left(i\frac{s-2}{2}\right)$$

with $l(s/2)$ given by (3.21) *and $J_P(s)$ by* (3.25).

The principal part of $I^*_{1,\mathrm{ell}(4)}(s)$. Let us consider the second term on the right-hand side of (3.26). Utilizing the Laurent expansion for

$$\zeta^*_k\left(\frac{s}{2}\right)^2 = \frac{4}{(s-2)^2}+\frac{4}{s-2}\left(\frac{1}{\pi}\gamma_{Q(i)}-\gamma-\log\pi\right)+O(1)$$

and the value $l(1)=1$ we obtain that the principal part of this term is equal to

$$\tfrac{1}{4}h(0)(s-2)^{-1}.$$

Considering the first term on the right-hand side of (3.26) we must use the Taylor expansion around $s=2$;

$$J_P(s) = J_P(2)+(s-2)J'_P(2)+O\left((s-2)^2\right)$$

$$= 2i\int_P h(r)\,dr+(s-2)\left\{i\int_P\left(\frac{\Gamma'}{\Gamma}\left(1+\frac{ir}{2}\right)-\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+\frac{ir}{2}\right)\right)h(r)\,dr-\right.$$

$$\left.-\int_P\frac{h(r)}{r}\,dr\right\}.$$

In the first and in the second integral one can move the path of integration from P to R. The third integral can be calculated using the contour $P\cup\{-P\}$; thus

$$\int_P\frac{h(r)}{r}\,dr = -\pi i h(0).$$

The principal part of the first term is equal to

$$\frac{1}{2^7\pi}\left\{\frac{4}{(s-2)^2}+\frac{4\gamma^*}{(s-2)}\right\}4\,\{1+(s-2)\log 2\}\left\{1-\frac{3}{8}(s-2)\log 2\right\}\times$$

$$\times\left\{4\pi g(0)-(s-2)\int_{-\infty}^{\infty}\left(\frac{\Gamma'}{\Gamma}\left(1+\frac{ir}{2}\right)-\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+\frac{ir}{2}\right)\right)h(r)\,dr-\pi(s-2)h(0)\right\}+$$

$$+O(1)$$

$$=\frac{1}{2}g(0)(s-2)^{-2}+\left\{-\frac{1}{8}h(0)+\left(\frac{\gamma^*}{2}+\frac{5}{16}\log 2\right)g(0)-\right.$$

$$\left.-\frac{1}{8\pi}\int_{-\infty}^{\infty}\left(\frac{\Gamma'}{\Gamma}\left(1+\frac{ir}{2}\right)-\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+\frac{ir}{2}\right)\right)h(r)\,dr\right\}(s-2)^{-1}+O(1),$$

where

$$\gamma^*=\frac{1}{\pi}\gamma_{Q(i)}-\gamma-\log\pi.$$

This enables us to write down the final formula for the residue

$$\operatorname*{res}_{s=2}I^*_{1,\mathrm{ell}(4)}(s)=\frac{1}{8}h(0)+\left(\frac{\gamma^*}{2}+\frac{5}{16}\log 2\right)g(0)- \tag{3.27}$$

$$-\frac{1}{8\pi}\int_{-\infty}^{\infty}\left(\frac{\Gamma'}{\Gamma}\left(1+\frac{ir}{2}\right)-\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+\frac{ir}{2}\right)\right)h(r)\,dr.$$

3.11. The integral $I_{1,\mathrm{par}}(s)$. We now calculate the contribution to the integral $I(s)$ given by parabolic elements

$$\gamma=\pm\begin{pmatrix}1 & \\ c & 1\end{pmatrix}\quad\text{with}\quad c\in\mathfrak{a},\ c\neq 0.$$

It has the form

$$I_{1,\mathrm{par}}(s)=\sum_{[\gamma]}\sum_{\substack{\sigma\in\Gamma_\gamma\setminus\Gamma/\Gamma_\infty\\ c(\sigma^{-1}\gamma\sigma)\neq 0}}k(u,(\sigma^{-1}\gamma\sigma)\cdot u)v^s d\mu(u),$$

where the first sum is over all conjugacy classes of parabolic elements in the group $\mathrm{PSL}(2,\mathfrak{a})$. In the integral

$$\int_H k(u,\gamma\cdot u)v^s d\mu(u)=\int_H\varphi(g^{-1}\gamma g)\,d\mu(u)\quad\text{with}\quad\gamma=\begin{pmatrix}1 & \\ c & 1\end{pmatrix},\quad c\neq 0, \tag{3.28}$$

we make the change of variable $g\mapsto Tg$, where $T=\begin{pmatrix} & 1\\ -1 & \end{pmatrix}$. The element γ, when considered as a conformal mapping of the Riemann sphere $\bar{C}$, has the unique fixed point 0 and T takes 0 to ∞; hence $\tilde{\gamma}=T^{-1}\gamma T=\begin{pmatrix}1 & c\\ & 1\end{pmatrix}$ has the fixed point ∞ and

$$T\cdot v=\frac{v}{|z|^2+v^2}.$$

Thus (3.28) transforms into

$$\int_H\varphi(g^{-1}\tilde{\gamma}g)\left(\frac{v}{|z|^2+v^2}\right)^s d\mu(u)=\int_H\varphi\left(\frac{|c|^2}{v^2}\right)\left(\frac{v}{|z|^2+v^2}\right)^s\frac{dx\,dy\,dv}{v^3}$$

$$=\frac{1}{|c|^2}\,\frac{\pi\Gamma(s-1)}{\Gamma(s)}\int_0^{\infty}\varphi(u^2)u^{s-1}du=\frac{1}{|c|^2}V(s,2)$$

for $0<\operatorname{Re}s<2$; we have followed the similar way to that of [24]. The final formula for

$$I^*_{1,\mathrm{par}}(s)=\zeta^*_k(s)I_{1,\mathrm{par}}(s)$$

is related to that of $I^*_2(s)$;

$$I^*_{1,\mathrm{par}}(s)=I^*_2(2-s)$$

$$=\frac{\zeta^*_k(2-s)\,\zeta^*_k\left(\frac{s}{2}\right)}{8\cdot 2^{2-s}\pi^{\frac{3-s}{2}}\Gamma\left(\frac{3-s}{2}\right)}\cdot\int_{-\infty}^{\infty}\frac{\Gamma\left(\frac{2-s}{2}+ir\right)\Gamma\left(\frac{2-s}{2}-ir\right)}{\Gamma(ir)\Gamma(-ir)}h(r)\,dr,$$

which is valid in the region $0<\operatorname{Re}s<2$. We take a suitable path of integration P (as for the elliptic elements of trace zero) and consider the integral

$$J_P(s)=\int_P\frac{\Gamma\left(\frac{2-s}{2}+ir\right)\Gamma\left(\frac{2-s}{2}-ir\right)}{\Gamma(ir)\Gamma(-ir)}h(r)\,dr.$$

We collect the results on $I_{1,\mathrm{par}}(s)$ in

PROPOSITION 11. *The analytic continuation of $I^*_{1,\mathrm{par}}(s)$ is given by*

$$I^*_{1,\mathrm{par}}(s)=\frac{\zeta^*_k(s-1)\,\zeta^*_k(s/2)}{8\cdot 2^{2-s}\pi^{\frac{3-s}{2}}\Gamma\left(\frac{3-s}{2}\right)}J_P(s)+\frac{\zeta^*_k(s-1)\,\zeta^*_k(s/2)}{8\pi^{\frac{2-s}{2}}\Gamma\left(\frac{s-2}{2}\right)}h\left(i\,\frac{s-2}{2}\right)$$

for s in some neighbourhood of the point $s=2$, and

(3.29) $$\operatorname*{res}_{s=2} I^*_{1,\mathrm{par}}(s)$$

$$= \frac{1}{8}h(0)+\left(\frac{3}{4\pi}\gamma_{Q(i)}-\gamma-\frac{1}{2}\log\pi\right)g(0)-\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\Gamma'}{\Gamma}(1+ir)h(r)\,dr.$$

4. The Selberg trace formula. We have considered the integral

$$I(s) = \int_{\Gamma\setminus H} K_0(u,u)E(u;s)\,d\mu(u),$$

where K_0 is "the discrete part" of the kernel function K given by (2.9), and the normalized one

(4.1) $$I^*(s) = \zeta_k^*(s)I(s) = \int_{\Gamma\setminus H} K_0(u,u)E^*(u;s)\,d\mu(u).$$

Since $\operatorname*{res}_{s=2}E^*(u;s)=1$, we have that $\operatorname*{res}_{s=2}I^*(s) = \int_{\Gamma\setminus H}K_0(u,u)\,d\mu(u)$. The last equality is the starting point for our derivation of Selberg's trace formula. Using (2.10) we can write now

(4.2) $$\sum_{j=1}^{\infty} h(r_j) = \operatorname*{res}_{s=2} I^*(s).$$

In the preceding section we have calculated the integral $I(s)$ in the regions of its convergence and investigated the analytic continuation together with the formulae for the principal parts of its separate terms. We have left only the problem of absolute convergence of the series defining $I_1(s)$ in the region $2<\operatorname{Re}s<A$. The validity of the formulae for $I_n(s)$, $n=2,3,4$, in this region has been asserted in Propositions 5, 4 and 3. Since $I(s)$, as given by (4.1), is well defined (except for poles of the Eisenstein series involved) this gives a posteriori proof of the absolute convergence of the infinite sum (3.10) defining the function $I_1(s)$. Inserting into (4.2) the decomposition of $I(s)$ written down in the preceding section and the formulae (3.2), (3.3), (3.4), (3.13), (3.16), (3.19), (3.27) and (3.29) for the residues of its separate terms we obtain finally

THEOREM. *For the discontinuous transformation group* $\mathrm{PSL}(2, Z[i])$ *of the symmetric space* $H=\mathrm{SL}(2,C)/\mathrm{SU}(2)$ *and the corresponding eigenvalues* $\{r_j\}_{j=1}^{\infty}$ *of the Laplace–Beltrami operator there is the equality:*

$$\sum_{j=1}^{\infty} h(r_j) = 2\pi^{-2}\Bigg\{\sum_{t\in Z,|t|>2} Q(t^2-4)\operatorname*{res}_{s=2}\zeta\left(\frac{s}{2},t^2-4\right)+$$

$$+\sum_{t\in Z[i],\operatorname{Im}t\neq 0} Q(\Delta_{\mathrm{hyp}}(t))\operatorname*{res}_{s=2}\zeta\left(\frac{s}{2},t^2-4\right)\Bigg\}+$$

$$+\frac{\zeta_k(2)}{2\pi^4}\int_{-\infty}^{\infty} r^2h(r)\,dr+\frac{1}{2}h(0)+\Big\{\frac{4}{9}\log(2+\sqrt{3})+\frac{5}{16}\log 2-$$

$$-\frac{1}{2}\gamma+\frac{1}{4\pi}\gamma_{Q(i)}\Big\}g(0)-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\zeta_k^{*\prime}}{\zeta_k^*}(1+ir)h(r)\,dr-$$

$$-\frac{1}{8\pi}\int_{-\infty}^{\infty}\left\{\frac{\Gamma'}{\Gamma}\left(1+\frac{ir}{2}\right)-\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+\frac{ir}{2}\right)+2\frac{\Gamma'}{\Gamma}(1+ir)\right\}h(r)\,dr.$$

Added in proof. Concerning Proposition 2: J. Elstrodt and J. Mennicke obtained the similar estimate $\lambda_1>\frac{2}{5}\pi^2$. Professor J. Mennicke informed me about the exact numerical calculation of $\lambda_1=1+\mu_1^2$ with $\mu_1\cong 8.555i$.

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Added in proof

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Addendum and corrigendum to the paper "On sums of powers and a related problem", Acta Arith. 36(1980), pp. 125-141

by

K. Thanigasalam (Monaca, Penn.)

Improving on the results of the paper, it is possible to prove

THEOREM. *All sufficiently large integers N are representable in the form*

$$N = \sum_{s=1}^{20} x_s^{s+1}$$

(x's being non-negative integers).

The proof will appear in Portugaliae Math.

The following misprints are also noted:

p. 129, in (30) $= f_2\Big(\prod_{k=2}^{23} f_k\Big)$ should be replaced by $= \prod_{k=2}^{23} f_k$,

p. 136, in (53) K_3 should be replaced by K_4,

p. 137, in Lemma 24 $N^{\mu_5'}$ should be replaced by $N^{\mu_5'/5}$

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