Δ-extension and Hanf-numbers

by

Jouko Väänänen (Helsinki)

Abstract. It is proved that while Δ preserves the Hanf-numbers of most logics, it fails, in a suitable Boolean extension, to preserve the Hanf-number of the logic with the Härtig-quantifier. The preservation of some general Löwenheim-Skolem properties under Δ and variants of Δ is discussed.

§ 1. Introduction. This paper is concerned with implicit definability in extensions of elementary logic. Our main interest is in the preservation of Löwenheim–Skolem type properties under various extension operations based on projective classes.

The logic LI with the Härtig-quantifier

$$IxyA(x)B(y) \leftrightarrow \mathrm{card}(A) = \mathrm{card}(B)$$

turns out to be of particular interest. LI is one of the very few well known logics the Hanf-number of which is not (provably) preserved under the Δ-operation. A proof of this is the main contribution of this paper and occupies Chapter 4. Chapter 2 introduces the relevant notions of projective definability. Chapter 3 is concerned with the provable cases of the various possible preservation results.

We use the following notation: *Many-sorted similarity types* (or just *types*) consist of sort symbols, predicate symbols, function symbols and constant symbols. If L is a type, a *model* $\mathfrak{A}$ of type L consists of a domain $|\mathfrak{A}|$, which is the union of the domains of the different sorts of L, and interpretations of the symbols of L in $|\mathfrak{A}|$. The class of all models of type L is denoted $\mathrm{Str}(L)$. The *reduct* $\mathfrak{A}\restriction L$ of a model $\mathfrak{A}$ of type L' to a type $L \subseteq L'$ is defined as usual. $\mathrm{Card}(\mathfrak{A})$ is the cardinality of the domain of $\mathfrak{A}$. The notion of an *abstract logic* is used frequently but the exact definition is not essential. The reader may consult [7] or think of an abstract logic L^* as a family of model classes with certain simple closure properties. If $\varphi \in L^*$ is such a class of models of type L and $\mathfrak{A} \in \mathrm{Str}(L)$, we write $\mathfrak{A} \models \varphi$ for $\mathfrak{A} \in \varphi$. If $K \subseteq \mathrm{Str}(L)$ is a model class, then $\overline{K}$ is $\mathrm{Str}(L) - K$. If $\varphi \in L^*$, $\mathrm{Mod}(\varphi)$ denotes the class of all models of φ (and is equal to φ if the above definition is used). *Second order logic* L^{II} quantifies over n-ary relations for any $n < \omega$. $\alpha, \beta, \gamma, \delta$ and ε refer to ordinals; $\varkappa, \lambda, \mu$ refer to cardinals. $R(\alpha)$ is the set of all sets of rank $<\alpha$. $\exp(\varkappa)$ is the cardinality of the power set of $\varkappa$.

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§ 2. Projective classes. Projective classes have been considered in the literature for some time already. The new aspect we pursue in this paper is the cardinality of the domains along which the projection is taken. It turns out that a reasonable restriction on this cardinality leads to a smoother theory than no restriction at all. In this chapter we introduce the relevant notions of projective definability, in particular the new notion of bounded projective definability, and discuss some examples of the use of these notions.

Suppose K is a class of models of type L. If $L' \subseteq L$ and $\mathfrak{A}$ is a model of type L', we write

$$E(\mathfrak{A}, K) = \{\mathfrak{B} \in K: \mathfrak{B} \upharpoonright L' = \mathfrak{A}\}.$$

Note that if L has sorts others than those of L', then $E(\mathfrak{A}, K)$ may contain arbitrary large models. On the other hand, if L and L' have the same sorts, then every model in $E(\mathfrak{A}, K)$ has the same domain as $\mathfrak{A}$.

The *projection of K along $L-L'$* is defined as usual:

$$\mathrm{Proj}_{L'}(K) = \{\mathfrak{A} \in \mathrm{Str}(L'): E(\mathfrak{A}, K) \neq \emptyset\}\ (^1).$$

If L^* is an abstract logic and $\varphi \in L^*$, we write $\mathrm{Proj}_L(\varphi)$ for $\mathrm{Proj}_L(\mathrm{Mod}(\varphi))$. K is a projective class of L^* if $K = \mathrm{Proj}_L(\varphi)$ for some $\varphi \in L^*$. The *family of projective classes of* L^* is denoted $\Sigma(L^*)$ and it can be viewed as an abstract logic itself. If K and $\overline{K}$ are in $\Sigma(L^*)$ then K is said to be *Δ-definable* in L^*. The family of model classes which are Δ-definable in L^* is denoted $\Delta(L^*)$. This sublogic of $\Sigma(L^*)$ has been extensively studied in [7] where also other references are given.

The model class K is a *simple projective class of* L^*, in symbols $K \in \Sigma_1^1(L^*)$, if $K = \mathrm{Proj}_{L'}(\varphi)$ for some $\varphi \in L^*$ such that L and L' have the same sorts. $\Delta_1^1(L^*)$ denotes the family of model classes K such that K and $\overline{K}$ are in $\Sigma_1^1(L^*)$. Clearly $\Delta_1^1(L^*) \subseteq \Delta(L^*)$. Note that $\Sigma_1^1(L^{\mathrm{II}}) = L^{\mathrm{II}}$ and therefore $\Delta_1^1(L^{\mathrm{II}}) = L^{\mathrm{II}}$, but $L^{\mathrm{II}} \neq \Delta(L^{\mathrm{II}})$.

One of the many applications of Δ is the following: Many logics are defined by adding a new quantifier to $L_{\omega\omega}$, like LQ_1 and LI for example. However, the expressive power of such logics may be quite unbalanced. For example, LQ_1 is not able to say that an equivalence-relation has $\aleph_1$ many classes (see [6]). This defect can be removed by using the Δ_1^1- and Δ-operations.

The Δ_1^1-operation seems to be sufficient to make the logic LQ_1 a reasonably closed logic, but $\Delta_1^1(LI)$ still suffers from the disadvantage that it is not able to define the notion of well-ordering, because in countable domains $\Delta_1^1(LI)$ can be translated into $L_{\omega_1\omega}$. As the notion of well-ordering is definable in $\Delta(LI)$ (see below), we conclude that $\Delta(LI)$ represents the strength of LI better than $\Delta_1^1(LI)$. The next examples emphasize further the difference between Δ_1^1 and Δ.

2.1. Examples. A. This example is from [4]. Let L consist of two sorts M_1 and M_2, the binary predicate $<$ and the ternary predicate F. Let K be the LI-definable class of L-models in which

(¹) To preserve an analogy with second order logic, we assume that $L-L'$ is finite. This is not essential for our results however.

1) $<$ linearly orders $M_2 \supseteq M_1$, M_1 cofinal in M_2.

2) $\mathrm{card}\{x \in M_2 |\ x<a\} < \mathrm{card}\{x \in M_2 |\ x<b\}$ whenever $a<b$ are in M_1.

3) If a_0 is the $<$-smallest element of M_1, then the set $\{x \in M_2 |\ x<a_0\}$ is countable.

4) If $a \neq a_0$ is in M_1, then every proper initial segment of $\{x \in M_2 |\ x<a\}$ is mapped one-one into some $\{x \in M_2 |\ x<b\}$, $a>b \in M_1$, by a mapping coded by F.

Let L' be the subtype of L consisting of M_1 and $<$. It is easily seen that the class of well-ordered models is $\mathrm{Proj}_{L'}(K)$. If $\mathfrak{A} \in \mathrm{Str}(L')$ has power $\varkappa$, then every model in $E(\mathfrak{A}, K)$ has power less than $\aleph_{\varkappa^+}$.

B. Let L' consist of just one sort M_1 and let K be the class of models $\mathfrak{A}$ of type $L \supset L'$ such that $\mathrm{card}(M_1^{\mathfrak{A}}) < \aleph_2$, $\mathrm{card}(\mathfrak{A}) = \aleph_2$, and $\mathfrak{A}$ has a linear ordering every proper initial segment of which can be mapped one-one into $M_1^{\mathfrak{A}}$. K is LQ_2-definable. Let $K' = \mathrm{Proj}_{L'}(K)$. K' is the class of models $\langle M \rangle$, where $\mathrm{card}(M) = \aleph_1$. By definition, K' is $\Sigma(LQ_2)$-definable. However, K' is not even $\Sigma_1^1(L_{\omega_1\omega}Q_2)$-definable, because in models of power $\leqslant \aleph_1$ the logic $\Sigma_1^1(L_{\omega_1\omega}Q_2)$ translates into $\Sigma_1^1(L_{\omega_1\omega})$, and K' is not $\Sigma_1^1(L_{\omega_1\omega})$-definable. Note that K' is $\Delta(LQ_0 Q_2)$- but not $\Delta_1^1(LQ_0 Q_2)$-definable.

C. Let us consider the following predicate of set theory: $C(\alpha)$ if and only if there is an ordinal β such that

$$(*) \qquad \exp(\aleph_{\beta+\gamma+1}) \geqslant \aleph_{\beta+\gamma+3} \quad \text{for} \quad \gamma<\alpha.$$

Let K be the class of well-ordered structures the order type α of which satisfies $C(\alpha)$. To see that K is $\Sigma(LI)$-definable, observe the following: By Example A, the class of well-founded models of any finite part of ZFC is $\Sigma(LI)$-definable. We may furthermore restrict ourselves to models which have just the real cardinals. Now, there is a finite part T of ZFC such that $C(\alpha)$ holds if and only if there is a transitive model M of T with real cardinals such that $\alpha \in M$ and $M \models C(\alpha)$. Using this fact it is easy to prove that K is $\Sigma(LI)$-definable. We shall later construct a model of set theory in which K is neither Δ- nor Σ_1^1-definable in LI. The $\Sigma(LI)$-definition of K differs from those in Examples A and B above in that there is no obvious way of bounding the cardinality of the new sorts, that is, if $C(\alpha)$ then there may be arbitrary large β such that $(*)$.

The above examples indicate that there is a notion of projective definability which lies strictly between Σ_1^1 and Σ. For a rigorous definition, let K be a model class of type L. K is a *bounded projective class of* L^*, in symbols $K \in \Sigma^B(L^*)$, if there is a $\varphi \in L^*$ such that

$$K = \{\mathfrak{A} \in \mathrm{Str}(L): E(\mathfrak{A}, \varphi) \neq \emptyset\}$$

and

$$\forall \mathfrak{A} \in \mathrm{Str}(L)\, \exists \varkappa \forall \mathfrak{B} (\mathfrak{B} \in E(\mathfrak{A}, \varphi) \rightarrow \mathrm{card}(\mathfrak{B}) \leqslant \varkappa).$$

If K and $\overline{K}$ are Σ^B-definable in L^*, we write $K \in \Delta^B(L^*)$.

Trivially $\Delta_1^1(L^*) \subseteq \Delta^B(L^*) \subseteq \Delta(L^*)$. Examples A and B above show that

$\Delta_1^1(LQ_0Q_2) \neq \Delta^B(LQ_0Q_2)$ and $\Delta_1^1(LI) \neq \Delta(LI)$. Later we shall construct a model of set theory in which $\Delta(LI) \neq \Delta^B(LI)$.

It is a triviality that any logic L^* with the property

(+) If $\varphi \in L^*$, $\mathfrak{B} \models \varphi$ and $\mathfrak{A} \subseteq \mathfrak{B}$ is infinite, then there is $\mathfrak{C} \models \varphi$ such that $\mathfrak{A} \subseteq \mathfrak{C} \subseteq \mathfrak{B}$ and $\mathrm{card}(\mathfrak{C}) = \mathrm{card}(\mathfrak{A})$,

also satisfies the property

$$(++) \qquad \Sigma_1^1(L^*) = \Sigma(L^*)$$

provided that only infinite models are considered. Examples of logics satisfying (+) are $L_{\omega_1\omega}$ and LQ for any continuous $\mathfrak{Q}$. The logic $L_{\omega_1\omega}$ satisfies (++) (without the infiniteness provisio). For stronger results see [5] and [8].

LQ_0 is an example of a logic which satisfies (+) but fails to satisfy (++) if finite models are included.

More generally, the following *interpolation-properties* of an abstract logic L^* can be considered in connection with the operations Σ_1^1, Σ^B and Σ:

$$\text{(I1)} \qquad L^* = \Sigma_1^1(L^*),$$
$$\text{(I2)} \qquad \Sigma_1^1(L^*) = \Sigma^B(L^*),$$
$$\text{(I3)} \qquad \Sigma^B(L^*) = \Sigma(L^*),$$
$$\text{(I4)} \qquad L^* = \Delta_1^1(L^*),$$
$$\text{(I5)} \qquad \Delta_1^1(L^*) = \Delta^B(L^*),$$
$$\text{(I6)} \qquad \Delta^B(L^*) = \Delta(L^*).$$

Trivially, (I2) → (I5) and (I3) → (I6). Moreover, (I1) → (I4). $\Sigma_1^1(LI)$ is a logic which satisfies (I1) but not (I5) whereas LQ_0 satisfies (I2) and (I3) but not (I4). L^{II} satisfies (I3) and (I6) (see below) but not (I5). $\Delta_1^1(LI)$ satisfies (I4) but not (I1). $\Delta^B(L^{\mathrm{II}})$ satisfies (I5) but not (I2). Finally, $L_{\omega\omega}$ satisfies (I3), (I4), (I5) and (I6) but fails to satisfy (I1).

This analysis still leaves open a few possible implications between (I1)-(I6) but these cases are shown in Chapter 4 to be unprovable in ZFC.

Concerning (I3) we have the following result:

2.2. Proposition. *Suppose L^* satisfies the following two conditions for some cardinal $\varkappa \geqslant \omega$:*

(1) *If $\varphi \in L^*$, $\mathfrak{B} \models \varphi$ and $\mathfrak{A} \subseteq \mathfrak{B}$, then there is a $\mathfrak{C} \models \varphi$ such that $\mathfrak{A} \subseteq \mathfrak{C} \subseteq \mathfrak{B}$ and* $\mathrm{card}(\mathfrak{C}) = \max(\mathrm{card}(\mathfrak{A}), \varkappa)$,

(2) *There is a $\theta \in L^*$ such that θ has a model of power $\varkappa$ but does not have arbitrary large models.*

Then L^* satisfies (I3), *i.e.* $\Sigma^B(L^*) = \Sigma(L^*)$.

Proof. Suppose $\varphi \in L^*$ and

$$K = \{\mathfrak{A} \in \mathrm{Str}(L')\colon E(\mathfrak{A}, \varphi) \neq \emptyset\}.$$

Suppose $\theta \in L_0^*$ for some L_0 such that $L_0 \cap L = \emptyset$, and θ has no models of power $>\lambda$. Let ψ be the conjunction of φ, θ and a sentence saying that the union of the domains of $L' \cup L_0$ has the same power as that of $L \cup L' \cup L_0$. Now

$$K = \{\mathfrak{A} \in \mathrm{Str}(L')\colon E(\mathfrak{A}, \psi) \neq \emptyset\}$$

and

$$\forall \mathfrak{A} \in \mathrm{Str}(L')\, \forall \mathfrak{B} \in E(\mathfrak{A}, \psi)\big(\mathrm{card}(\mathfrak{B}) \leqslant \max(\mathrm{card}(\mathfrak{A}), \lambda)\big).$$

Therefore $K \in \Sigma^B(L^*)$. ■

This result implies that the following logics satisfy (I3) and (I6): LQ_α, $L_{\varkappa\lambda}$, $LQ^{MM(n)}$ (the Magidor-Malitz-logics). Another category of logics satisfying (I3) is the very strong logics. We have the following result:

2.3. Proposition. *Suppose $L^{\mathrm{II}} \subseteq \Sigma^B(L^*)$. Then $\Sigma^B(L^*) = \Sigma(L^*)$.*

Proof. Suppose $\varphi \in L^*$ and

$$K = \{\mathfrak{A} \in \mathrm{Str}(L')\colon E(\mathfrak{A}, \varphi) \neq \emptyset\}.$$

There is an L^{II}-sentence ψ which defines the class of models isomorphic to $\langle R(\alpha), \varepsilon\rangle$ for some α. For any $\mathfrak{A} \in K$ let $\varkappa(\mathfrak{A})$ be the least ordinal α such that there is an $\mathfrak{A}' \cong \mathfrak{A}$ in $R(\alpha)$ such that $E(\mathfrak{A}', \varphi) \cap R(\alpha) \neq \emptyset$. Using φ and the $\Sigma^B(L^*)$-definition of ψ one can write an L^*-sentence θ such that

$$K = \{\mathfrak{A} \in \mathrm{Str}(L')\colon E(\mathfrak{A}, \theta) \neq \emptyset\}$$

and

$$\forall \mathfrak{A} \in K\, \forall \mathfrak{B} \in E(\mathfrak{A}, \theta)\big(\mathrm{card}(\mathfrak{B}) = \beth_{\varkappa(\mathfrak{A})}\big).$$

This proves that K is in $\Sigma^B(L^*)$. ■

Let H be the Henkin-quantifier ([3]). It follows from [3] that $L^{\mathrm{II}} \subseteq \Sigma^B(LH)$. Hence by 2.3, $\Sigma^B(LH) = \Sigma(LH)$ and also $\Delta^B(LH) = \Delta^B(L^{\mathrm{II}})$. Let S be the similarity-quantifier

$$S = \{\langle M, R, P\rangle\colon R, P \subseteq M^2, \langle M, R\rangle \cong \langle M, P\rangle\}.$$

By [11], $L^{\mathrm{II}} \subseteq \Sigma^B(LS)$ and therefore $\Sigma^B(LS) = \Sigma(LS)$ and $\Delta^B(LS) = \Delta(L^{\mathrm{II}})$. Finally, if $V = L$, then $L^{\mathrm{II}} \subseteq \Sigma^B(LI)$ (see e.g. [10]) whence $\Sigma^B(LI) = \Sigma(LI)$ and $\Delta^B(LI) = \Delta^B(L^{\mathrm{II}})$.

§ 3. Preservation results.

We introduce a very general Löwenheim–Skolem property and investigate its preservation under the various operations of § 2. Particular attention is given to Hanf-numbers.

Let A and B be classes of cardinals. An abstract logic L^* has the property

$$LS_\mu(A, B)$$

if every set T of L^*-sentences of cardinality $\leqslant \mu$ which has a model of power $\varkappa \in A$ has a model of power $\lambda \in B$. We use the following obvious notation:

$$[\varkappa, \lambda] = \{\mu\colon \mu \text{ is a cardinal and } \varkappa \leqslant \mu \leqslant \lambda\},$$
$$[\varkappa, \infty[= \{\mu\colon \mu \text{ is a cardinal and } \varkappa \leqslant \mu\}.$$

For example, $L_{\omega\omega}$ satisfies $LS_\mu(A, B)$ whenever $\mu \geqslant \omega$ and $B \subseteq [\mu, \infty[$, and LQ_1 satisfies $LS_\omega(A, \{\aleph_1\})$ if $A \subseteq [\aleph_1, \infty[$. For logics like LI and L^{II} it is very difficult to find interesting classes A and B such that even $LS_1(A, B)$ would hold.

The following obvious proposition is part of the folklore of the subject, but the fact that it should be formulated for Σ_1^1 rather than Σ has not been emphasized in the literature, rather the contrary (see e.g. [6]).

3.1. Proposition. *The following are equivalent for any* L^*:

(i) L^* *satisfies* $LS_\mu(A, B)$,

(ii) $\Sigma_1^1(L^*)$ *satisfies* $LS_\mu(A, B)$.

Proof. Suppose $T = \{\varphi_\alpha \colon \alpha < \mu\}$ is a set of $\Sigma_1^1(L^*)$-sentences such that T has a model $\mathfrak{A}$, $\operatorname{card}(\mathfrak{A}) \in A$. For $\alpha < \mu$ let $\Phi_\alpha \in L^*$ give the simple projective definition of φ_α. Let $S = \{\Phi_\alpha \colon \alpha < \mu\}$. We may assume that the types of the sentences Φ_α are so chosen that $\mathfrak{A}$ expands to a model $\mathfrak{A}'$ of S, such that $\operatorname{card}(\mathfrak{A}') \in A$. By (i), S has a model $\mathfrak{B}$ of power $\lambda \in B$. A reduct of $\mathfrak{B}$ is a model of T of power $\lambda \in B$. ∎

If Σ_1^1 is replaced by Σ above, the proof breaks down in two places: $\operatorname{card}(\mathfrak{A}')$ may be greater than $\operatorname{card}(\mathfrak{A})$ and the reduct of $\mathfrak{B}$ may have power less than λ. These difficulties disappear if A is a final segment and B an initial segment. Therefore we have the following result:

3.2. Proposition. *The following are equivalent for any* L^*:

(i) L^* *satisfies* $LS_\mu([\varkappa, \infty[, [1, \lambda])$,

(ii) $\Sigma(L^*)$ *satisfies* $LS_\mu([\varkappa, \infty[, [1, \lambda])$.

By 2.1. B, the class K of models $\langle A \rangle$ where $\operatorname{card}(A) = \aleph_1$ is $\Delta^B(LQ_0 Q_2)$-definable. Therefore $\Delta^B(LQ_0 Q_2)$ does not satisfy $LS_1(\{\aleph_1\}, \{\aleph_0\})$. But in models of power $\leqslant \aleph_1$ $LQ_0 Q_2$ translates into LQ_0, and therefore Proposition 3.1 is false if Σ_1^1 is replaced by Σ^B (or by one of Δ^B, Σ, Δ).

The *Löwenheim-number* $l(L^*)$ of an abstract logic L^* is usually defined as the least $\varkappa$ such that L^* satisfies $LS_1(\varkappa, \infty[, [1, \varkappa])$. By 3.2 Σ preserves Löwenheim-numbers. The *Hanf-number* $h(L^*)$ of L^* is usually defined as the least $\varkappa$ such that L^* satisfies $LS_1(\varkappa, \infty[, [\lambda, \infty[)$ for all λ. Proposition 3.2 does not apply in connection with Hanf-numbers, but Σ^B has the following more helpful preservation property:

3.3. Proposition. *The following are equivalent for any* L^*:

(i) L^* *satisfies* $LS_\mu([\varkappa, \infty[, [\lambda, \infty[)$ *for all* $\lambda \geqslant \lambda_0$,

(ii) $\Sigma^B(L^*)$ *satisfies* $LS_\mu([\varkappa, \infty[, [\lambda, \infty[)$ *for all* $\lambda \geqslant \lambda_0$.

Proof. We proceed as in the proof of 3.1. By definition, for every model $\mathfrak{C}$ of T there is a cardinal $\varkappa(\mathfrak{C})$ such that every extension of $\mathfrak{C}$ to a model of S has power less than $\varkappa(\mathfrak{C})$. Suppose $\lambda \geqslant \lambda_0$ is given. Let

$$\lambda_1 = \sup\{\varkappa(\mathfrak{C}) \colon \mathfrak{C} \models T \text{ and } |\mathfrak{C}| \leqq \lambda\}.$$

As L^* satisfies $LS_\mu([\varkappa, \infty[, [\lambda_0 \cdot \lambda_1^+, \infty[)$, S has a model $\mathfrak{B}$ of power $\geqslant \lambda_1^+$. Let $\mathfrak{D}$ be a reduct of $\mathfrak{B}$ such that $\mathfrak{D} \models T$. Let $\mathfrak{D}' \cong \mathfrak{D}$ such that $|\mathfrak{D}'| \subseteq \operatorname{card}(\mathfrak{D})$. Let $\mathfrak{B}' \cong \mathfrak{B}$ such that $\mathfrak{D}'$ is a reduct of $\mathfrak{B}'$. If $\operatorname{card}(\mathfrak{D}') \leqslant \lambda$, then $\operatorname{card}(\mathfrak{B}') \leqslant \varkappa(\mathfrak{D}') \leqslant \lambda_1$ which contradicts $\operatorname{card}(\mathfrak{B}) \geqslant \lambda_1^+$. Therefore $\operatorname{card}(\mathfrak{D}') > \lambda$ and we are through. ∎

From the above result it follows that Σ^B (and Δ^B) preserve Hanf-numbers. This may be interesting as Δ^B seems to be at least as useful as Δ (they in fact coincide in most cases). The above result also seems to capture the true content behind the claim (e.g. in [7]) that Δ preserves Hanf-numbers. In the next chapter we show that this claim is false in a suitable Boolean extension.

There is an artificial way of extending 3.1 to the Σ-operation (partly pursued in [9]): For any model $\mathfrak{A}$ let $\operatorname{card}^m(\mathfrak{A})$ be the least of the powers of the sorts of $\mathfrak{A}$. Define $LS_\mu^+(A, B)$ as $LS_\mu(A, B)$ but using card^m instead of card. Then for any L^*, L^* satisfies $LS_\mu^+(A, B)$ is and only if $\Sigma(L^*)$ satisfies $LS_\mu^+(A, B)$. Let $h^+(L^*)$ be the least $\varkappa$ such that L^* satisfies $LS_1^+([\varkappa, \infty[, [\lambda, \infty[)$ for all λ. It follows that Σ preserves h^+ ([9]).

The following rather curious Hanf-number occurs in [7]: Let $h^-(L^*)$ be the least $\varkappa$ such that L^* satisfies $LS_1(\{\varkappa\}, [\lambda, \infty[)$ for all λ. Clearly $h^-(L^*) \leqslant h(L^*) \leqslant h^+(L^*)$. h^- is preserved by Σ_1^1 but not by Σ^B. The following result shows that h^- and h^+ may be of some interest as characteristic numbers of abstract logics:

3.4. Proposition. *Let* L^* *be an abstract logic.*

(i) $h^+(L^*) = h(\Sigma(L^*))$,

(ii) $h(L^*) \;= h^-(\Sigma^B(L^*))$.

Proof. (i) Trivially, $h(\Sigma(L^*)) \leqslant h^+(\Sigma(L^*)) = h^+(L^*)$, so we only have to prove $h^+(L^*) \leqslant h(\Sigma(L^*))$. Suppose $\varphi \in L^*$, φ has a model $\mathfrak{A}$ such that $\operatorname{card}^m(\mathfrak{A}) \geqslant h(\Sigma(L^*))$, and $\varkappa$ is an arbitrary cardinal. Let ψ be the conjunction of φ and a sentence which says that the sort M_1 can be mapped one-one into every other sort. Let $L = \{M_1\}$ and

$$K = \{\mathfrak{A} \in \operatorname{Str}(L) \colon E(\mathfrak{A}, \psi) \neq \emptyset\}.$$

K is in $\Sigma(L^*)$ and has a model of power $\geqslant h(\Sigma(L^*))$. Therefore K has a model $\mathfrak{B}$ of power $\geqslant \varkappa$. Suppose $\mathfrak{C} \in E(\mathfrak{B}, \psi)$. $\mathfrak{C}$ is a model of φ and $\operatorname{card}^m(\mathfrak{C}) \geqslant \varkappa$. This ends the proof of (i).

(ii) Trivially, $h^-(\Sigma^B(L^*)) \leqslant h(\Sigma^B(L^*)) = h(L^*)$, so we only have to prove $h(L^*) \leqslant h^-(\Sigma^B(L^*))$. Suppose $\varphi \in L^*$ has a model of power $\geqslant h^-(\Sigma^B(L^*))$. If φ has arbitrary large models we are through. Suppose then φ has no models of power greater than $\varkappa$. Let K be the class of models $\langle A \rangle$ such that $\langle A \rangle$ can be expanded by a model of φ of power $\geqslant \operatorname{card}(A)$. K is $\Sigma^B(L^*)$-definable because the powers of models of φ are bounded by $\varkappa$. K has a model of power $h^-(\Sigma^B(L^*))$ and therefore a model of power greater than $\varkappa$. This model gives rise to a model of φ of power greater than $\varkappa$, which contradicts the choice of $\varkappa$. ∎

§ 4. The main results.

In this chapter we prove the following theorems:

4.1. Theorem. *Let* M *be a countable model of* GB+GCH+*Global Choice. There is a countable extension* N *of* M *to a model of* GB+*Global Choice such that* M *and* N *have the same ordinals, cardinals and cofinalities, and*

$$N \models h(LI) < h(\Delta(LI)).$$

In this model also $h^-(LI) < h(LI)$ *holds.*

4.2. THEOREM. *Let M be a countable model of* GB+GCH+*Global Choice. There is a countable extension N of M to a model of* GB+*Global Choice such that M and N have the same ordinals, cardinals and cofinalities, and*

$$N \models h(\Delta(LI)) < h(\Sigma(LI)).$$

In this model also $h^-(LI) < h(LI)$ holds.

By the main result of [2], the above theorems have the following corollary:

4.3. COROLLARY. *If* Con(ZF), *then* $\mathrm{Con}(\mathrm{ZFC}+h^-(LI)<h(LI)<h(\Delta(LI)))$ *and* $\mathrm{Con}(\mathrm{ZFC}+h^-(LI)<h(LI)+h(\Delta(LI))<h(\Sigma(LI))$. *In particular* $\mathrm{Con}(\mathrm{ZFC}+h^-(LI)<h(LI)<h^+(LI))$.

4.4. COROLLARY. *Apart from the trivial* (I1) → (I4), (I2) → (I5) *and* (I3) → (I6) *no implication among* (I1)-(I6) *is provable in* ZFC.

Proof. Recall that most of the possible implications were already shown to, be provably false. If $h(LI)<h(\Delta(LI))$, then $\Sigma_1^1(LI)$ satisfies (I1) but not (I3) or (I6) $\Sigma^B(LI)$ satisfies (I2) but not (I3) or (I6), $\Delta_1^1(LI)$ satisfies (I4) but not (I3) or (I6) and $\Delta^B(LI)$ satisfies (I5) but not (I3) or (I6). If $h(\Delta(LI))<h(\Sigma(LI))$, then $\Delta(LI)$ satisfies (I6) but not (I3). ■

For the proofs of the above theorems we recall some facts from Easton-style forcing with classes. Suppose $\alpha\in On$ and F is a function (class) defined on On-α with values in On such that

$$\text{(E1)} \qquad \forall\beta\gamma(\alpha<\beta\leqslant\gamma \to F(\beta)\leqslant F(\gamma)),$$

$$\text{(E2)} \qquad \forall\beta>\alpha(\aleph_\beta<\mathrm{cf}(F(\beta))).$$

By [1] there is a class $\boldsymbol{P}(F)$ of forcing conditions such that if GCH is assumed, then

$$\boldsymbol{P}(F) \Vdash \forall\beta>\alpha(\beta \text{ regular} \to \exp(\aleph_\beta) = \aleph_{F(\beta)})$$

and $\boldsymbol{P}(F)$ preserves cardinals, cofinalities and $R(\alpha)$. Here $\boldsymbol{P}(F)\Vdash\varphi$ means that every condition weakly forces φ. Moreover, $\boldsymbol{P}(F)$ is homogeneous and therefore, if $\varphi(x_1,\ldots,x_n)$ is a formula of set theory and $a_1,\ldots,a_n$ are hereditarily ordinal definable, then

$$\boldsymbol{P}(F)\Vdash\varphi(a_1,\ldots,a_n) \quad\text{or}\quad \boldsymbol{P}(F)\Vdash\neg\varphi(a_1,\ldots,a_n).$$

Recall the definition of the predicate $C(\alpha)$ from 2.1.C. Let

$$S(\alpha)\leftrightarrow\alpha>1\ \&\ C(\alpha)\ \&\ \forall\beta(C(\beta)\to\beta\leqslant\alpha),$$
$$D(\alpha)\leftrightarrow\exists\varkappa>\alpha(\exp(\aleph_{\varkappa+\alpha+1})\geqslant\aleph_{\varkappa+\alpha+4}),$$
$$P(\alpha)\leftrightarrow D(\alpha)\ \&\ \forall\beta(D(\beta)\to\beta=\alpha),$$
$$H(\alpha)\leftrightarrow S(\alpha)\ \&\ P(\alpha).$$

4.5. LEMMA. *Suppose $H(\alpha)$. Then $\alpha<h(\Delta(LI))$.*

Proof. Let K be the class of well-ordered sets the order type of which is $\leqslant\alpha$. Note that $\beta\leqslant\alpha\leftrightarrow C(\beta)$. Therefore K is $\Sigma(LI)$-definable (see 2.1.C). Note also that

$$\beta>\alpha\leftrightarrow\exists\gamma<\beta\,\exists\varkappa>\gamma(\exp(\aleph_{\varkappa+\gamma+1})\geqslant\aleph_{\varkappa+\gamma+4}).$$

Using this it is not difficult to prove that $\overline{K}$ is $\Sigma(LI)$-definable. Therefore $K\in\Delta(LI)$. K has no models of power $>\mathrm{card}(\alpha)$. Hence $\alpha<h(\Delta(LI))$. ■

To prove Theorem 4.1 we define a function F such that

$$\boldsymbol{P}(F)\Vdash H(\alpha)\ \&\ h(LI)\leqslant\alpha$$

for some α. Suppose α and β are infinite ordinals. Define $F_{\alpha\beta}(\gamma)$ for $\gamma>\beta$ by:

$$F_{\alpha\beta}(\gamma+1)=\begin{cases}\gamma+4 & \text{if } \exists\varkappa>\alpha(\gamma=\varkappa+\alpha),\\ \gamma+3 & \text{if } \exists\varkappa>\alpha(\varkappa\cdot2\leqslant\gamma<\varkappa\cdot2+\alpha),\\ \max(F_{\alpha\beta}(\gamma),\gamma+2) & \text{otherwise},\end{cases}$$

$$F_{\alpha\beta}(\nu)=\nu+1 \quad \text{for limit } \nu.$$

Note that $F_{\alpha\beta}$ satisfies the conditions (E1) and (E2) and if $\beta<\beta'$ then $\forall\gamma>\beta'(F_{\alpha\beta'}(\gamma)=F_{\alpha\beta}(\gamma))$ and therefore every $\boldsymbol{P}(F_{\alpha\beta'})$-term (term of the $\boldsymbol{P}(F_{\alpha\beta'})$-forcing language) is also a $\boldsymbol{P}(F_{\alpha\beta})$-term.

4.6. LEMMA. *Suppose* GCH. *For every α and β $\boldsymbol{P}(F_{\alpha\beta})\Vdash H(\alpha)$.*

Proof. The claim follows immediately from the definition of $F_{\alpha\beta}$. ■

4.7. LEMMA. *There are α and β such that $\boldsymbol{P}(F_{\alpha\beta})\Vdash h(LI)\leqslant\alpha$.*

Proof. Suppose the contrary, that is, for all α and β $\boldsymbol{P}(F_{\alpha\beta})\Vdash\alpha<h(LI)$. Let $\mathrm{Mod}(\varphi,\varkappa)$ be the formula "$\varphi\in LI$, $\varkappa$ is a cardinal and φ has a model of power $\geqslant\varkappa$". Let $\varkappa_0=h(LI)$ and $\boldsymbol{P}_1=\boldsymbol{P}(F_{\varkappa_0\omega})$. Then $\boldsymbol{P}_1\Vdash\varkappa_0<h(LI)$ and therefore there are $\varphi_1\in LI$ and $\varkappa_1$ such that

$$\boldsymbol{P}_1\Vdash\mathrm{Mod}(\varphi_1,\varkappa_0)\ \&\ \neg\mathrm{Mod}(\varphi_1,\varkappa_1).$$

If $\beta<\omega_1$ and $\varkappa_\gamma$ has been defined for $\gamma<\beta$, let $\varkappa=\sup\{\varkappa_\gamma:\gamma<\beta\}$ and $\boldsymbol{P}_\beta=\boldsymbol{P}(F_{\varkappa_0\varkappa})$. Then $\boldsymbol{P}_\beta\Vdash\varkappa_0<h(LI)$, whence there are $\varphi_\beta\in LI$ and $\varkappa_\beta>\varkappa$ such that

$$(*) \qquad \boldsymbol{P}_\beta\Vdash\mathrm{Mod}(\varphi_\beta,\varkappa_0)\ \&\ \neg\mathrm{Mod}(\varphi_\beta,\varkappa_\beta).$$

This procedure defines a sequence $\{\varphi_\beta:\beta<\omega_1\}$ of LI-sentences. As LI is countable there are β and $\gamma<\omega_1$ such that $\beta<\gamma$ and $\varphi_\beta=\varphi_\gamma$. Now we have

$$\boldsymbol{P}_\gamma\Vdash\mathrm{Mod}(\varphi_\gamma,\varkappa_0).$$

Suppose $\mathfrak{U}$ is a $\boldsymbol{P}_\gamma$-term and $p\in\boldsymbol{P}_\gamma$ such that

$$p\Vdash^*\mathfrak{U}\models\varphi_\gamma\ \&\ \mathrm{card}(\mathfrak{U})\geqslant\varkappa_0.$$

As $\boldsymbol{P}_\gamma$ and $\boldsymbol{P}_\beta$ preserve cardinals and $\beta<\gamma$, there is a q in $\boldsymbol{P}_\beta$ such that $q\Vdash^*\mathfrak{U}\models\varphi_\gamma$. As $\boldsymbol{P}_\beta\Vdash\neg\mathrm{Mod}(\varphi_\beta,\varkappa_\beta)$, we have $q\Vdash^*\mathrm{card}(\mathfrak{A})<\varkappa_\beta$. In fact we may assume that $q\Vdash^*\mathfrak{A}\in R(\varkappa_\beta)$. As $\boldsymbol{P}_\gamma$ preserves $R(\varkappa_\beta)$, there is a model $\mathfrak{B}$ such that $q\Vdash^*\mathfrak{U}=\mathfrak{B}$.

If $\mathfrak{B}$ does not satisfy φ_γ, then $P_\gamma \Vdash \mathfrak{B} \models \neg\varphi_\gamma$, a contradiction. Therefore $\mathfrak{B}$ is a model of φ_γ of power $\geqslant h(LI)$. Hence φ_γ has arbitrary large models. Because P_β preserves cardinals,

$$P_\beta \Vdash \forall\lambda \mathrm{Mod}(\varphi_\beta, \lambda),$$

which contradicts (*) ■

Proof of 4.1. Apart from the claim concerning $h^-(LI)$ the claim of the theorem is easily proved using 4.5, 4.6 and 4.7: We let N be a $P(F_{\alpha\beta})$-generic extension of M, where α and β are determined by 4.7. To satisfy $h^-(LI)<h(LI)$ in the extension we have to choose the extension more carefully. Let $I(\alpha)$ be the predicate $\exp(\aleph_\alpha)\geqslant\aleph_{\alpha+4}$. If $\beta=\sup\{\alpha\colon I(\alpha)\}$ exists, then clearly $\beta<h(LI)$. Suppose now for a moment $V = L$ and $\alpha = h^-(LI)$. Let B be the usual Cohen-algebra which gives the sentence $I(\alpha^+)$ value 1 without introducing new subsets of α. Then in V^B there are no non-constructible models of power α (up to isomorphism) and therefore $V^B \models h^-(LI) = \alpha<h(LI)$. To prove 4.1 we now combine this construction with that of 4.7. So, let M be gives as in 4.1. Let $\varkappa = h^-(LI)$ in M and let N_1 be a generic extension of M obtained by adding $\aleph_{\varkappa\cdot 3+5}$ new subsets of $\aleph_{\varkappa\cdot 3+1}$ but no new subsets of $\aleph_{\varkappa\cdot 3}$. Then $N_1 \models h^-(LI)<h(LI)$. We cannot apply 4.6 directly inside N_1 because N_1 does not satisfy GCH. However, if $\beta>\aleph_{\varkappa\cdot 3+1}$, then $P(F_{\alpha\beta}) \Vdash H(\alpha)$ in N_1. So we only have to make sure that $\beta>\aleph_{\varkappa\cdot 3+1}$ in the proof of 4.7 and we get α and β such that $P(F_{\alpha\beta}) \Vdash H(\alpha)\ \&\ h(LI)\leqslant\alpha$ in N_1. Note that still $P(F_{\alpha\beta}) \Vdash h^-(LI)<h(LI)$. Let G be $P(F_{\alpha\beta})$-generic over N_1 and $N=N_1[G]$. For details of the construction of G and $N_1[G]$ see [1]. ■

The proof of Theorem 4.2 is somewhat similar to the above. The role of the predicate $H(\alpha)$ is now played by $D(\alpha)$. The following result has been essentially proved already:

4.8. Lemma. *If $D(\alpha)$, then $\alpha<h(\Sigma(LI))$.*

For any α, β and $\gamma>\beta$ we define $F'_{\alpha\beta}(\gamma)$ as follows:

$$F'_{\alpha\beta}(\gamma+1) = \begin{cases} \gamma+4 & \text{if } \exists\alpha'\leqslant\alpha\,\exists\varkappa>\alpha'(\varkappa\leqslant\gamma\leqslant\varkappa+\alpha'), \\ \max(F'_{\alpha\beta}(\gamma), \gamma+2) & \text{otherwise}, \end{cases}$$

$$F'_{\alpha\beta}(\nu) = \nu+1 \quad \text{for limit } \nu.$$

The functions $F'_{\alpha\beta}$ have similar properties as the functions $F_{\alpha\beta}$ but in addition, $F'_{\alpha\beta}(\gamma)\leqslant F'_{\alpha'\beta}(\gamma)$ if $\alpha\leqslant\alpha'$.

4.9. Lemma. *Suppose* GCH. *For every α and β $P(F'_{\alpha\beta}) \Vdash D(\alpha)$.*

Proof. The claim follows immediately from the definition of $F'_{\alpha\beta}$. ■

4.10. Lemma. *There are α and β such that $P(F'_{\alpha\beta}) \Vdash h(\Delta(LI))\leqslant\alpha$.*

Proof. Suppose the contrary, that is $P(F'_{\alpha\beta}) \Vdash \alpha<h(\Delta(LI))$ for all α and β. The logic $\Delta(LI)$ has the disadvantage that its syntax depends heavily on the underlying model of set theory. For the sake of clarity, let us assume that $\Delta(LI)$-sentences are triples $\langle\varphi,\psi,L\rangle$ such that $\mathrm{Proj}_L(\varphi)$ is the complement (in $\mathrm{Str}(L)$) of $\mathrm{Proj}_L(\psi)$ and $\varphi,\psi\in LI$. Let $\mathrm{Mod}(\theta,\varkappa)$ be now the formula "$\theta = \langle\varphi,\psi,L\rangle$, $\varphi,\psi\in LI$, $\varkappa$ is a cardinal and there is a model of power $\geqslant\varkappa$ in $\mathrm{Proj}_L(\varphi)$". Let $\pi(LI)$ denote the family of model classes K such that $\overline{K}$ is in $\Sigma(LI)$.

Now we use induction on $\alpha<\omega_1$ to construct cardinals $\varkappa_\alpha$, λ_α and $\Delta(LI)$-sentences φ_α such that if α is a successor ordinal then for arbitrary large γ

$$(**)\qquad P(F'_{\varkappa_\alpha\gamma}) \Vdash \mathrm{Mod}(\varphi_\alpha,\varkappa_\alpha)\ \&\ \neg\mathrm{Mod}(\varphi_\alpha,\lambda_\alpha).$$

Case 1. $\alpha=\delta+1$. Let $\varkappa_\alpha=\lambda_\delta^+$. Let γ be any infinite ordinal. Then $P(F'_{\varkappa_\alpha\gamma}) \Vdash \varkappa_\alpha<h(\Delta(LI))$ whence there are ψ_γ and μ such that

$$(*)\qquad P(F'_{\varkappa_\alpha\gamma}) \Vdash \psi_\gamma\in\Delta(LI)\ \&\ \mathrm{Mod}(\psi_\gamma,\varkappa_\alpha)\ \&\ \neg\mathrm{Mod}(\psi_\gamma,\mu).$$

Let μ_γ be the least μ such that (*) holds. By the Replacement Axiom there is a ψ such that $\psi_\gamma=\psi$ for arbitrary large γ. We let $\varphi_\alpha=\psi$. To define λ_α we need the following auxiliary argument: Let C be the class of γ such that $\psi_\gamma=\psi$ and let $\nu=\min(C)$. Let $\gamma\in C$ be arbitrary. We prove that $\mu_\gamma\leqslant\mu_\nu$. For this end, suppose $\mathfrak{U}$ is a $P(F'_{\varkappa_\alpha\gamma})$-term and $p\in P(F'_{\varkappa_\alpha\gamma})$ such that

$$p \Vdash^* (\mathfrak{U}\models\psi\ \&\ \mathrm{Card}(\mathfrak{U})\geqslant\mu_\nu).$$

As $\nu\leqslant\gamma$, $\mathfrak{U}$ is a $P(F'_{\varkappa_\alpha\nu})$-term and there is a $q\in P(F'_{\varkappa_\alpha\nu})$ such that

$$q \Vdash^* (\mathfrak{U}\models\psi\ \&\ \mathrm{Card}(\mathfrak{U})\geqslant\mu_\nu),$$

whence by homogeneity, $P(F'_{\varkappa_\alpha\nu}) \Vdash \mathrm{Mod}(\psi_\nu,\mu_\nu)$, a contradiction with (*). Therefore $P(F'_{\varkappa_\alpha\gamma}) \Vdash \neg\mathrm{Mod}(\psi_\gamma,\mu_\nu)$ and $\mu_\gamma\leqslant\mu_\nu$ by the minimality of μ_γ. Now we let $\lambda_\alpha=\max(\mu_\nu,\varkappa_\alpha)$.

Case 2. α is limit. Let $\varkappa_\alpha=\lambda_\alpha=\sup\{\varkappa_\beta\colon \beta<\alpha\}$ and $\varphi_\alpha=\exists x(x=x)$. This ends the construction.

Choose successor ordinals α and β such that $\varphi_\alpha=\varphi_\beta$ and $\alpha<\beta$. Let $\gamma>\delta>\lambda_\beta$ such that

$$P(F'_{\varkappa_\alpha\gamma}) \Vdash \neg\mathrm{Mod}(\varphi_\alpha,\lambda_\alpha)$$

and

$$P(F'_{\varkappa_\beta\delta}) \Vdash \mathrm{Mod}(\varphi_\alpha,\varkappa_\beta).$$

Let $\mathfrak{U}$ be a $P(F'_{\varkappa_\beta\delta})$-term and $p\in P(F'_{\varkappa_\beta\delta})$ such that

$$p \Vdash^* \mathfrak{U}\models\varphi_\alpha\ \&\ \mathrm{Card}(\mathfrak{U})\geqslant\varkappa_\beta\ \&\ \mathfrak{U}\in R(\lambda_\beta).$$

As $\delta>\lambda_\beta$, there is a $\mathfrak{B}$ such that $p\Vdash^*(\mathfrak{B}=\mathfrak{U}\ \&\ \mathfrak{B}\models\varphi_\alpha)$. Note that $F'_{\varkappa_\alpha\gamma}(\varepsilon)\leqslant F'_{\varkappa_\beta\delta}(\varepsilon)$ for all $\varepsilon>\gamma$. Hence if $q\in P(F'_{\varkappa_\alpha\gamma})$ and $q\Vdash^*(\mathfrak{B}\models\neg\varphi_\alpha)$ then there is an $r\in P(F'_{\varkappa_\alpha\delta})$ such that $r\Vdash^*(\mathfrak{B}\models\neg\varphi_\alpha)$, a contradiction with $p\Vdash^*(\mathfrak{B}\models\varphi_\alpha)$ (by homogeneity). Therefore $P(F'_{\varkappa_\alpha\gamma}) \Vdash (\mathfrak{B}\models\varphi_\alpha)$, but as $\lambda_\alpha<\varkappa_\beta$, this contradicts (**). ■

Proof of 4.2. If the claim concerning $h^-(LI)$ is again ignored, we can let N be a $P(F'_{\alpha\beta})$-generic extension of M, where α and β are determined by 4.10. $h^-(LI)$ is then taken care of using the same trick as in the proof of Theorem 4.1. ■

The only fact we used in the above Lemmata 4.7 and 4.10 about LI was that satisfaction of LI-sentences is absolute with respect to cardinals preserving extensions. Let us say that a generalized quantifier Q (defined without parameters) is *Easton-absolute*, if for any F and G satisfying (E1), (E2) and the condition $\forall\alpha(F(\alpha)\leqslant G(\alpha))$, for any $P(F)$-term t and for any $p\in P(F)$ p weakly forces $Q(t)$ in $P(G)$ only if p weakly forces $Q(t)$ in $P(F)$. In rough terms, this means that the predicate Q is preserved under forcing à la Easton. The quantifier I and the quantifier

$RxyA(x,y) \leftrightarrow \{(a,b)\colon A(a,b)\}$ has the order type of a regular cardinal

are Easton-absolute. Theorems 4.1 and 4.2 have now the following more general forms:

4.11. Theorem. *Let M be a countable model of* GB+GCH+*Global Choice and Q an Easton-absolute generalized quantifier in M. Then there is a countable extension N of M to a model of* GB+*Global Choice such that M and N have the same ordinals, cardinals and cofinalities, and*

$$N\models h(LQ)<h(\Delta(LI)).$$

Another similar model N' can be found such that

$$N'\models h(\Delta(LQ))<h(\Sigma(LI)).$$

Combining 4.11 and the compactness theorem yields:

4.12. Corollary. *If* Con(ZF), *then* Con(ZFC+*for every provably Easton-absolute Q, $h(LQ)<h(\Delta(LI))$), and* Con(ZFC+*for every provably Easton-absolute Q, $h(\Delta(LQ))<h(\Sigma(LI))$).*

4.13. Open problem. Is there a generalized quantifier Q such that ZFC $\vdash h(LQ)<h(\Delta(LQ))$?

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DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF HELSINKI
Helsinki, Finland

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