Generalized manifolds (ANR's and AR's) and null decompositions of manifolds

by

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Abstract. We prove the following theorem: THEOREM. For each topological n-manifold M^n, $n \geqslant 3$, there exists an uncountable family $\mathcal{M}^n$ of n-dimensional absolute nbd. retracts such that each X in $\mathcal{M}^n$ satisfies (1) X has the (proper) homotopy type of M^n, and (2) X does not contain any strongly movable proper subset of dimension $\geqslant 2$; furthermore, if M^n is a manifold without boundary, then each X in $\mathcal{M}^n$ is a generalized n-manifold satisfying $X \times E^1$ is homeomorphic to $M^n \times E^1$ in addition to (1) and (2). Moreover, each X in $\mathcal{M}^n$ contains movable subsets of dimension $\geqslant 2$ and the space X is obtained from M^n as a decomposition space associated with a null decomposition of M^n into arcs and singletons. Some other related matters are discussed; for instance, it is shown that a strongly movable continuum has UV^1 for small loops (actually more is shown, see the statement at the end of the proof of Proposition (6.9.2)).

1. Introduction, notation and terminology

(1.1) Introduction. Suppose G is a cell-like upper semicontinuous decomposition of a compact and connected n-manifold M^n without boundary such that the decomposition space M^n/G is finite dimensional. It is well-known that M^n/G is a generalized n-manifold which is not, in general, an n-manifold. Recently, J. W. Cannon [13] has used a disjoint disk property (DDP) in his remarkable solution to the double suspension problem and later R. D. Edwards has proved the following far reaching extension of [13] and [36]: If $n \geqslant 5$ and M^n/G has DDP, then the projection $p\colon M^n \to M^n/G$ can be approximated (arbitrarily close) by a homeomorphism. Thus, DDP appears to be the definitive (and minimal) condition whose presence guarantees that the generalized n-manifold M^n/G is an n-manifold. On the other hand, the failure of DDP can be successfully used to produce generalized n-manifolds with rather exotic topological structure; the results of this note may be interpreted in this context. A recent deep theorem of F. Quinn [33] states that, in fact, every generalized n-manifold is a cell-like image of a topological n-manifold; therefore, the failure of DDP is the only obstacle for any generalized n-manifold to be an n-manifold, see Lacher [27] and J. W. Cannon [12] for historical and other details. The failure of DDP can cause enough damage that the generalized n-manifold, $n \geqslant 3$, does not contain any proper compact subset of dimension $\geqslant 2$ which looks like a polyhedron (or ANR). More specifically, we prove the following theorem:

THEOREM. *For each topological n-manifold M^n, $n \geq 3$, there exists a family $\mathcal{M}^n$ of topologically distinct n-dimensional absolute nbd. retracts such that each X in $\mathcal{M}^n$ satisfies* (1) *X has the homotopy type of M^n, and* (2) *X does not contain any strongly movable proper subset of dimension ≥ 2; furthermore, if M^n is a manifold without boundary, then each X in $\mathcal{M}^n$ is a generalized n-manifold satisfying $X \times E^1$ is homeomorphic to $M^n \times E^1$ in addition to* (1) *and* (2).

Each space X in the theorem is constructed as a decomposition space associated with a certain upper semicontinuous decomposition G of the manifold M^n such that the set of all the nondegenerate elements of G form a (countable) null collection of arcs; therefore, the decomposition G is the minimal in the sense that it is a null collection and G is the simplest since each of its nondegenerate elements is an arc which is the simplest cell-like continuum. Furthermore, it is shown that if G is an arbitrary decomposition of an n-manifold M^n whose nondegenerate elements form a null collection of arcs, then the associated decomposition space X contains movable proper subsets of dimension ≥ 2. Therefore, the results of this note are the best possible.

We have relied heavily on [37] for many technical details on linking, and we have also depended on Wright [46] whenever possible. We have tried to preserve the geometric flavor of its predecessors [5, 38, 39, 40, 41] and most notably the work of Bing and Borsuk [5]. The main ingredient is a Cantor set construction of Daverman and Edwards (we often refer to Daverman and Edwards by DE) which plays a crucial role in our construction, see Daverman [16] for an exposition. Another ingredient is the existence of certain wild arcs in S^n whose complements have certain specific fundamental groups, see M. Brown [11] and Roslaniec [34]. We also use several results and techniques from the shape theory, K. Borsuk [8], the theory of retracts, K. Borsuk [7], and the theory of cell-like decompositions, Lacher [27].

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(1.2) Notation and terminology. We denote by B^n the closed ball of unit radius in the n-dimensional Euclidean space E^n and we denote by S^{n-1} the boundary sphere of B^n. A set X is *uncountable* if X has the cardinality of the set of the real numbers. Any Cantor set constructed in E^n, S^n, or B^n by the Daverman and Edwards construction [16, 17] will be called a DE *Cantor set* (or a DE *embedding of the Cantor set*). We refer to Daverman and Edwards by DE whenever convenient. A *generalized n-manifold X* is an ENR (Euclidean neighborhood retract) such that $H_*(X, X-\{x\}; Z) \approx H_*(E^n, E^n-\{0\}; Z)$ for any $x \in X$. One may consult J. W. Cannon [14] for an interesting discussion of generalized n-manifolds where many other references may also be found. By a *proper subset A* of a space X we mean $A \neq X$ and $A \neq \emptyset$. We shall use the terminology ANR, AR, FANR, and strongly movable only for compact metric space (cf. [7, 8]). We refer to Borsuk [7, 8], Cannon [12], Daverman [16], and Lacher [27] for specific results and also for references to the work of many others. A collection of subsets of a metric space X is called a *null collection* if for $\varepsilon > 0$ all but finitely many sets in the collection have diameter $> \varepsilon$ (in X). All spaces are assumed to be metrizable. Two isomorphic groups are sometimes considered to be equal.

2. Sequences of groups

Throughout this section we shall be concerned with sequences of groups. By $\{G_{n_i}\}_i$ we mean the sequence $G_{n_1}, G_{n_2}, \ldots$

(2.1) DEFINITION. A group G has *property* (*) if there exists a strictly increasing sequence $[n_i]_i$ of positive integers such that $G_{n_i} \neq G_{n_j}$, whenever $i \neq j$, where G_{n_k} is the free product (cf. [29]) of n_k copies of G for $k = 1, 2, \ldots$ The sequence of groups $\{G_{n_i}\}_i$, as above, will be called a *-*sequence* for G.

(2.2) THE *-SET $\mathcal{S}$. Let $\{G_{n_i}\}_i$ be a *-sequence for G. We observe that every subsequence $\{G_{m_i}\}_i$ of $\{G_{n_i}\}_i$ is also a *-sequence for G. Two subsequences $\{G_{m_i}\}_i$ and $\{G_{l_i}\}_i$ are defined to be *distinct* if and only if the sets $\{G_{m_i}\colon 1 \leq i < \infty\}$ and $\{G_{l_i}\colon 1 \leq i < \infty\}$ are distinct. It is easy to see that the set $\mathcal{S}$ consisting of all the distinct subsequences of the *-sequence $\{G_{n_i}\}_i$ has the power of the continuum (an uncountable set). The set $\mathcal{S}$ will be called the *-*set* for G corresponding to the *-sequence $\{G_{n_i}\}_i$.

(2.3) MAIN EXAMPLE. Rosłaniec [34] has shown that the group G having the presentation

$$\Big[\bigcup_{\pm i=0}^{\infty} \{C_i\}\colon \bigcup_{\pm i=0}^{\infty} \{C_{i-1}C_iC_{i+1} = C_iC_{i+1}C_{i-1}C_i\}\Big]$$

has a *-sequence $\{G_{n_i}\}_i$ for a suitable sequence $\{n_i\}_i$ of positive integers, see [34] for more details. The *-set for G corresponding to the sequence $\{G_{n_i}\}_i$ will serve as an important example for us. Recall that the group G is the fundamental group of the complement of the celebrated Artin–Fox arc in the 3-sphere S^3 [21].

3. Arcs in S^n, $n \geq 3$

(3.1) Preliminary results on arcs. We shall use the following results concerning arcs in S^n.

(3.1.1) THEOREM (M. Brown [11]). *For every arc $\alpha \subset S^n$, there is an arc $\alpha^* \subset S^{n+1}$ such that $(S^n - \alpha)$ has the homotopy type of $(S^{n+1} - \alpha^*)$.*

(3.1.2) THEOREM (Roslaniec [34]). *There exists a null sequence $\{A_j\}_j$ of disjoint arcs in S^3 and a group G with a *-sequence $\{G_{n_i}\}_i$ such that for each j, $1 \leq j < \infty$, $\pi_1(S^3 - A_j) \approx G_j$, where G_j is the free product of j-many copies of G.*

(3.1.3) THEOREM (Brown–Rosłaniec). *For each n, $n \geq 3$, there exists a null sequence $\{A_j\}_j$ of arcs in S^n and a group G with a *-sequence $\{G_{n_i}\}_i$ such that for each j, $1 \leq j < \infty$, $\pi_1(S^n - A_j) \approx G_j$, where G_j is the free product of j-many copies of G.*

(3.1.4) Remark. Theorem (3.1.3) follows from Theorems (3.1.1) and (3.1.2). Roslaniec [34] has given a specific group G, see (2.3), and a specific *-sequence

$\{G_{n_i}\}_i$ for G satisfying the conclusions of Theorems (3.1.2) and (3.1.3). *For our purposes, any group G with a $*$-sequence $\{G_{n_i}\}_i$ satisfying the conclusions of Theorem* (3.1.3) *will suffice.*

(3.2) A method of attaching an arc to a given arc. By $\langle x_1 x_2 \dots x_k\rangle$ we mean an arc in E^n, $n \geqslant 3$, such that the arc starts at x_1, goes through $x_2, x_3, \dots, x_{k-1}$ in this order, and ends at x_k. Suppose M^n is a compact and connected P.L. n-manifold in E^n with boundary ∂M^n. Suppose $\langle ab\rangle$ is an arc contained in $(M^n - \partial M^n)$. Choose a point c in ∂M^n. It is easy to see that there exists a P.L. arc $\langle bc\rangle \subset M^n$ such that $\langle ab\rangle \cap \langle bc\rangle = \{b\}$ and $\langle bc\rangle \cap \partial M^n = \{c\}$. Let d be a point in $(E^n - M^n)$ such that d is in the component of $(E^n - M^n)$ whose closure contains c. Choose two balls B_1 and B_2 centered at d such that $B_2 \subset B_1$ and $B_1 \cap M^n = \emptyset$. It is easy to see that there exists a P.L. arc $\langle cd\rangle$ such that $\langle cd\rangle \cap M^n = \{c\}$ and $\langle cd\rangle$ meets $[B_1 - \mathrm{Int}(B_2)]$ in a subarc $\langle c'd'\rangle$ of $\langle cd\rangle$. Given an arc $\langle de\rangle \subset B_2$ with $\langle abcd\rangle \cap \langle de\rangle = \{e\}$. *The arc $\langle abcde\rangle$ is obtained by attaching $\langle de\rangle$ to $\langle ab\rangle$ by the* P.L. *arc* $\langle bcd\rangle$. Put $U = E^n - (B_2 \cup \langle abcc'd'\rangle)$ and $V = \mathrm{Int}(B_1) - \langle c'd'de\rangle$. By the Seifert and Van Kampen Theorem (cf. [29]), we have shown that $\pi_1(E^n - \langle abcde\rangle)$ is the free product of $\pi_1(U) \approx \pi_1(E^n - \langle c'd'de\rangle) \approx \pi_1(E^n - \langle de\rangle)$ with $\pi_1(V) \approx \pi_1(E^n - \langle abcc'd'\rangle) \approx \pi_1(E^n - \langle ab\rangle)$ since $(U \cap V)$ is simply connected.

4. Preliminary results

(4.1) The DE embeddings of the Cantor set and arcs in E^n, $n \geqslant 3$. A closed $(n-2)$-manifold M in E^n, $n \geqslant 3$, has a (topological) *tubular neighborhood* M_α in E^n if M_α is homeomorphic to $M \times B^2$ under a homeomorphism which carries M onto $M \times \{0\}$. For notational convenience, we shall identify M_α with $M \times B^2$ and forget the homeomorphism. We shall also refer to M_α as a *tube* with *center* M and *normal disk* B^2. *All manifolds considered in this note will be* P.L. *and orientable. All the closed manifolds in* E^n *will have (topological) tubular neighborhoods.* We begin with some results from [16,17].

(4.1.1) THEOREM (DE [16, 17]). *Suppose, M, M_α and E^n as above in* (4.1). *Then for each $\varepsilon > 0$, there exists a finite set $\{M_{\alpha i}\colon 1 \leqslant i \leqslant n_\alpha\}$ of disjoint n-manifolds such that*: (a) *For each i, $1 \leqslant i \leqslant n_\alpha$, $M_{\alpha i} \subset M_\alpha$, the diameter of $M_{\alpha i}$ is less than ε, and $M_{\alpha i}$ is a tube with a center which is an $(n-2)$-manifold*; and (b) *a loop γ in $(E^n - M_\alpha)$ is nullhomotopic in $(E^n - M)$ if and only if γ is nullhomotopic in* $(E^n - \bigcup_{i=1}^{n_\alpha} M_{\alpha i})$.

Theorem (4.1.1) may be considered as a generalization of the classic Antoine's construction [1]. Indeed, the goal is to produce a wild Cantor set, analogous to the celebrated "Antoine's necklace" [1, 6, 32], by iterating this construction for each $M_{\alpha i}$, $1 \leqslant i \leqslant n_\alpha$.

(4.1.2) THEOREM (DE [16, 17]). *Suppose M and M_α as in Theorem* (4.1.1). *Then there exists a Cantor set C in* $\mathrm{Int}(M_\alpha)$ *such that any loop γ in $(E^n - M_\alpha)$ is nullhomotopic in $(E^n - C)$ if and only if γ is nullhomotopic in $(E^n - M)$.*

(4.2) Definitions and terminology. A set $\{X_i\colon 1 \leqslant i \leqslant n\}$ consisting of disjoint subsets of a metric space X will be called a δ-*chain* provided each X_i has diameter $< \delta$ and each of the sets $(X_1 \cup X_2)$, $(X_2 \cup X_3)$, ..., $(X_{n-1} \cup X_n)$, and $(X_n \cup X_1)$ has diameter less than 2δ. The finite set $\{M_{\alpha i}\colon 1 \leqslant i \leqslant n_\alpha\}$ given in Theorem (4.1.1) will be called an ε-*chain of n-manifolds substituting for M_α* at the first stage of DE construction provided (1) this set is an ε-chain, and (2) each loop in $M_{\alpha i}$ is nullhomotopic in $(E^n - M_{\alpha(i+1)})$ and each loop in $M_{\alpha(i+1)}$ is nullhomotopic in $(E^n - M_{\alpha i})$ where $1 \leqslant i < n_\alpha$ and choose $i+1 = 1$ if $i = n_\alpha$ (or consider i modulo n_α). The construction of an ε-chain substituting for M_α can be done, for instance, as an added feature of DE construction; this follows from "the standard replacement technique" of DE. The concept of an ε-chain substituting for M_α is simply a way of ordering the manifolds and that there is very little geometry between the consecutive two manifolds other than the assertion (2) in the definition. We omit details and we refer the reader to Wright [46; p. 123–124] for a relevent discussion. Furthermore, we require that in the first stage of DE construction we use only an ε-chain substituting for the manifold M_α, for a suitable $\varepsilon > 0$, whenever we use DE construction; while, the manifolds used at any successive stage will be called a *chain* and which may or may not be an ε-chain substituting for the manifolds under consideration.

(4.3) The parallel DE Cantor sets. Suppose the first stage of the DE construction is finished, i.e., suppose an ε-chain of n-manifolds $\{M_{\alpha i}\colon 1 \leqslant i \leqslant n_\alpha\}$ substituting for M_α is given. Since each $M_{\alpha i}$ is a tube we identify it with $N_{\alpha i} \times B^2$, see (4.1). Now, choose two disjoint tubes $T_{\alpha i} = N_{\alpha i} \times B_{\alpha i}$ and $T'_{\alpha i} = N_{\alpha i} \times B'_{\alpha i}$ satisfying: (1) the tubes miss the center of $N_{\alpha i} \times B^2$, (2) there exist b and b' inside $\mathrm{Int}(B^2)$ such that $N_{\alpha i} \times \{b\}$ and $N_{\alpha i} \times \{b'\}$ are the respective centers for $T_{\alpha i}$ and $T'_{\alpha i}$, (3) $B_{\alpha i}$ and $B'_{\alpha i}$ are two subdisks of B^2 with respective centers b and b'. We stipulate that the chain inside each $M_{\alpha i}$, required in the second stage of the DE construction, is contained in $T_{\alpha i}$. This requirement will apply only to the second stage. Now, iterate the usual DE construction to construct the third stage, ..., nth stage, ..., and a Cantor set C inside M_α. Clearly, C is the disjoint union of the Cantor sets $\{C_{\alpha i} = C \cap M_{\alpha i}\}_{i=1}^{n_\alpha}$. Note that for each i, $C_{\alpha i}$ is contained in $T_{\alpha i}$. Observe that there exists an isotopy of $M_{\alpha i}$ taking $T_{\alpha i}$ onto $T'_{\alpha i}$ such that the disk $B_{\alpha i}$ goes onto $B'_{\alpha i}$ inside B^2 and the boundary of $M_{\alpha i}$ remains fixed throughout the isotopy; we shall refer to the isotopy of this type as a *vertical isotopy*. Therefore, the Cantor set C goes onto a Cantor set C' where C' is the disjoint union of the Cantor sets $\{C'_{\alpha i}\}_{i=1}^{n_\alpha}$ and $C'_{\alpha i}$ is the image of $C'_{\alpha i}$ at the end of the isotopy. We say $C_{\alpha i}$ and $C'_{\alpha i}$ (C and C') are two *parallel Cantor sets* in $M_{\alpha i}$ (in M_α). Let $A_{\alpha i}$ be an arc inside $T_{\alpha i}$ containing $C_{\alpha i}$ such that $A_{\alpha i}$ is PL modulo $C_{\alpha i}$. The image arc $A'_{\alpha i}$ of $A_{\alpha i}$, under the end of the isotopy, certainly contains $C'_{\alpha i}$. We say $A_{\alpha i}$ and $A'_{\alpha i}$ are *parallel arcs* in $M_{\alpha i}$ and the set

$$\{A_{\alpha i} \cup A'_{\alpha i}\colon 1 \leqslant i \leqslant n_\alpha\}$$

will be called the ε-*chain of parallel arcs substituting for* M_α.

(4.4) The dyadic arcs. For each two consecutive elements $M_{\alpha i}$ and $M_{\alpha(i+1)}$ we choose a simple closed curve S such that (1) S lies in the complement of $M_{\alpha(i+1)}$, (2) S and the boundary Σ of a normal disk of $M_{\alpha(i+1)}$ bound an annulus does not meet $M_{\alpha(i+1)}$ away from Σ, and (3) $(S \cap M_{\alpha i})$ equals to an arc J contained in $M_{\alpha i}$. Let $A_{\alpha i} = \langle xy \rangle$ and $A'_{\alpha i} = \langle x'y' \rangle$ where x goes to x' and y goes to y' under the vertical isotopy which carries $A_{\alpha i}$ onto $A'_{\alpha i}$. Our notation $\langle abc \rangle$ means an oriented arc running from a to b and then to c. By extending the arcs if necessary we assume $J = \langle yy' \rangle$. Choose z and z' in S such that $S = \langle yzy' \rangle \cup \langle y'z'y \rangle$ where $J = \langle yzy' \rangle$. Let $L_{\alpha i} = (A_{\alpha i} \cup A'_{\alpha i}) \cup \langle y'z'y \rangle$. The arc $L_{\alpha i}$ will be called a *dyadic arc* substituting for $M_{\alpha i}$. Observe that $(L_{\alpha i} \cup M_{\alpha i})$ contains a loop which links $M_{\alpha(i+1)}$, or $L_{\alpha i} \cup \langle yzy' \rangle$ contains a loop which links $M_{\alpha(i+1)}$. It is clear that $\{L_{\alpha i}\colon 1 \leqslant i \leqslant n_\alpha\}$ is an 2ε-chain of arcs which will be called the 2ε-*chain of dyadic arcs* (or *dyadic chain*) *substituting for* M_α.

5. A family of decompositions

(5.1) A family of manifolds in E^n, $n \geqslant 3$. Let $\mathscr{F}^{(n-2)}$ denote the family of all the PL $(n-2)$-manifolds in E^n with rational vertices and with tubular neighborhoods, see (4.1). It follows from the results in [37] and from the connected sum construction that for each continuum $A \subset E^n$ of dimension $\geqslant 2$ and for each open set $U \subset E^n$ with $(U \cap A) = \emptyset$, there exists $M \in \mathscr{F}^{(n-2)}$ such that A *h-links* M, i.e., each nbd. of A in E^n contains a simple closed curve which is linked with M (see [37] for more details) and $(M \cap U)$ is nonempty. Let $M_1, M_2, M_3, \ldots$ be an enumeration of the family $\mathscr{F}^{(n-2)}$ such that each manifold in $\mathscr{F}^{(n-2)}$ appears an infinite number of times. It is clear that the above discussions remain valid when E^n is replaced by S^n.

(5.2) Some choices. Fix an integer $n \geqslant 3$. Choose a group G with a $*$-sequence $\{G_{n_i}\}_i$ such that for each i, $1 \leqslant i < \infty$, there exists an arc $A_i \subset S^n$ of diameter $< 1/i$ satisfying $\pi_1(S^n - A_i) = G_{n_i}$, see Theorem (3.1.3). Let $\mathscr{S}$ denote the $*$-set for G corresponding to $\{G_{n_i}\}_i$, see (2.1.1). Choose an enumeration $M_1, M_2, M_3, \ldots$ of the family $\mathscr{F}^{(n-2)}$ satisfying the assertions of (5.1) and fix this enumeration. The group G and the $*$-sequence $\{G_{n_i}\}_i$ will remain fixed throughout; however, we do not make fixed choices for the arcs A_i's. For each element $\lambda \in \mathscr{S}$, where $\mathscr{S}$ is the $*$-set given above, we shall construct a decomposition space S^n_λ of S^n such that $\lambda \neq \mu$ implies S_λ is not homeomorphic to S_μ (details will follow).

(5.3) The construction. Suppose $\lambda = \{G_{m_i}\}_i$ in $\mathscr{S}$ is given. Choose a tube T_1 containing M_1 such that T_1 is contained inside $N(M_1, 1) = \{x \in S^n\colon d(x, m) < 1$ for some m in $M_1\}$. Choose a 1-chain $\{M_{1i}\colon 1 \leqslant i \leqslant n_1\}$ substituting for M_1, see (4.1.3). Let $\{L_{1i}\colon 1 \leqslant i \leqslant n_1\}$ denote the 2-chain of dyadic arcs substituting for T_1, see (4.4). Put $H_{1i} = \pi_1(S^n - L_{1i})$. There are two cases and we consider them separately.

Case I. For each i the group H_{1i} is not isomorphic to any group in the $*$-sequence $\{G_{n_i}\}_i$.

Case II. There exists a group H_{1j} such that $H_{1j} = G_{n_i}$ for some i.

Suppose Case I is true. We construct an arc $A_{m_1} \subset S^n$ such that $\pi_1(S^n - A_{m_1}) = G_{m_1}$ and A_{m_1} is disjoint from the arcs in $\{L_{1i}\colon 1 \leqslant i \leqslant n_1\}$, and the diameter of A_{m_1} is < 1. This finishes our construction for Case I.

Suppose Case II is true, i.e., H_{1j} is isomorphic G_{n_i}, as above. Choose an index $m_k > n_i$. Now G_{m_k} equals to the free product G_{n_i} with $G_{(m_k - n_i)}$ where $G_{(m_k - n_i)}$ equals to the free product of $(m_k - n_i)$-many copies of G. By (3.2) we modify our arc L_{1j} to obtain a new arc $\hat{L}_{1j}$ sufficiently near to L_{1j} such that $\pi_1(S^n - \hat{L}_{1j}) = G_{m_k}$. As a consequence, in either case we may assume without loss of generality that the chain of $\{L_{1i}\colon 1 \leqslant i \leqslant n_1\}$ satisfies the property:

(P) *For each i the group $H_{1i} = \pi_1(S^n - L_{1i})$ is either isomorphic to some G_{m_i} or it is not isomorphic to any G_{n_j}.*

In addition, we construct an arc $A_m \subset S^n$ of diameter < 1 such that $\pi_1(S^n - A_{m_1}) = G_{m_1}$ and A_{m_1} is disjoint from the arcs in $\{L_{1i}\colon 1 \leqslant i \leqslant n_1\}$.

Suppose the construction has been performed inside the tubes $T_1, T_2, \ldots, T_{(j-1)}$. We next describe this construction inside T_j where T_j is a tube contained in $N(M_j, 1/j) = \{x \in S^n\colon d(x, m) < 1/j$ for some $m \in M_j\}$. Choose $1/j$-chain $\{M_{ji}\colon 1 \leqslant i \leqslant n_j\}$ substituting for T_j and let $\{L_{ji}\colon 1 \leqslant i \leqslant n_j\}$ denote the $2/j$-chain of dyadic arcs substituting for M_j satisfying (1) these arcs are disjoint from the finitely many arcs employed in the chains of arcs substituting for $T_1, \ldots,$ and $T_{(j-1)}$, (2) we require that the chain $\{L_{ji}\colon 1 \leqslant i \leqslant n_j\}$ satisfies the property (P); furthermore, we construct an arc $A_{m_j} \subset S^n$ of diameter $\leqslant 1/j$ disjoint from all the arcs previously employed such that $\pi_1(S^n - A_{m_j}) = G_{m_j}$. The requirement (1) in the previous sentence can be fulfilled by requiring the DE Cantor sets to be disjoint which is a feature of the DE construction and then general positioning the arcs modulo the Cantor sets, see Wright [46] for a related discussion. This finishes our inductive step in the construction. Let S^n_λ denote the decomposition of S^n such that the set N_λ of all the nondegenerate elements of S^n_λ is the union of all the chains of arcs, $\{L_{ij}\colon 1 \leqslant i \leqslant n_j\}$ where $1 \leqslant i \leqslant \infty$, with the set of arcs $\{A_{m_i}\colon 1 \leqslant i < \infty\}$. Clearly, for each $\varepsilon > 0$ all but finitely arcs in N_λ have diameters $> \varepsilon$, i.e., the nondegenerate elements of the decomposition S^n_λ form a null collection. This finishes our construction of the decomposition S^n_λ. In order to avoid proliferation of symbols we denote the associated decomposition space again by S^n_λ.

6. Decompositions of S^n, $n \geqslant 3$

Suppose $n \geqslant 3$ is arbitrary but fixed for the following discussions. We assume several results from the context of cell-like mappings and we refer the reader to Lacher's excellent survey article [27] for the discussion of these results and other related matters. We start with the following well-known proposition (cf. [27]):

(6.1) Proposition. *For each λ in $\mathscr{S}$ the decomposition space S^n_λ is a generalized n-manifold* (*definition later*) *such that the projection $p_\lambda\colon S^n \to S^n_\lambda$ is a* (*simple/fine*) *homotopy equivalence.*

A finite dimensional ANR (ENR) X is a *generalized n-manifold* if $H_*(X, X - \{x\}; Z)$ is isomorphic to $H_*(E^n, E^n - \{0\}; Z)$ for each x in X where Z is the group of the integers (under addition). A generalized n-manifold is locally orientable, i.e., the orientation sheaf (generated by the presheaf $U \to H_n(X, X-U; Z)$ where the homology can be taken in the sense of Borel–Moore or Čech) is locally constant, see Cannon [14] for an interesting discussion of these matters where the reference to the work of Bredon and others may also be found.

(6.2) The Groups $\pi_1(S^n_\lambda - \{x\})$'s. Suppose $\lambda = \{G_{m_i}\}_i$ is an (fixed) element of $\mathcal{S}$. We observe that the group $\pi_1(S^n_\lambda - \{x\}) = 0$, G_{m_i}, or H_{ij}, respectively, when $p_\lambda^{-1}(x)$ consists of exactly one point, $p_\lambda^{-1}(x) = A_{m_i}$, or $p_\lambda^{-1}(x) = L_{ij}$. We reindex and arrange the groups H_{ij}'s into a sequence which is denoted by $\{H^m_i\}_i$. Hence, every nonzero group $\pi_1(S^n_\lambda - \{x\})$ appears in the sequence $\{H^m_i\}_i$ or in $\{G_{m_i}\}_i$. Let $\hat{\lambda} = (\{H^m_i\}_i, \{G_{m_i}\}_i)$ denote the (ordered) pair of these sequences. The pair $\hat{\lambda}$ has the following property:

(P_*) *Each group H^m_i is either isomorphic to some group in the sequence $\{G_{m_i}\}_i$ or it is not isomorphic to any group in the sequence $\{G_{n_i}\}_i$*. This follows from the property (P) of (5.3).

Suppose $\hat{\lambda} = (\{H^m_i\}_i, \{G_{m_i}\}_i)$ and $\hat{\mu} = (\{H^l_i\}_i, \{G_{l_i}\}_i)$ are given where $\lambda = \{G_{m_i}\}_i$ and $\mu = \{G_{l_i}\}_i$ are two distinct elements of the $*$-set $\mathcal{S}$, see Section 2. Since $\lambda \neq \mu$, we let without loss of generality G_{m_j} be a group which is not isomorphic to any group in the sequence $\{G_{l_i}\}_i$. In this setting, we have proved the following Lemma (6.3).

(6.3) Lemma. *The distinct pairs $\hat{\lambda}$ and $\hat{\mu}$, given above, have the property that the group G_{m_j} does not appear in either of the component sequences $\{H^l_i\}_i$ or $\{G_{l_i}\}_i$ of $\hat{\mu}$.*

(6.4) Lemma. *If λ and μ are two distinct elements of $\mathcal{S}$, then the decomposition spaces S^n_λ and S^n_μ are topologically distinct.*

Proof. Let λ, μ, $\hat{\lambda}$, and $\hat{\mu}$ be given as above satisfying the assertions of Lemma (6.3). We do not lose any generality by making this assumption. This means that the group $G_{m_j} = \pi_1(S^n_\lambda - \{x_0\})$, for some x_0 in S^n_λ, is not isomorphic to $\pi_1(S^n_\mu - \{x\})$ for any x in S^n_μ. This finishes the proof of Lemma (6.4).

For each $n \geq 3$ and a $*$-set $\mathcal{S}$ we denote by $\mathcal{S}^n$ the set $\{S^n_\lambda: \lambda \in \mathcal{S}\}$ containing the decomposition spaces. The following theorem summarizes our results thus far:

(6.5) Theorem. *For each $n \geq 3$, the set $\mathcal{S}^n$ consists of uncountably many topologically distinct generalized n-manifolds such that each S^n_λ in $\mathcal{S}^n$ has the (simple/fine) homotopy type of S^n and this (simple/fine) homotopy equivalence is induced by the projection $p_\lambda: S^n \to S^n_\lambda$.*

Since the nondegenerate elements for our decomposition is a null collection of arcs, the following proposition is a consequence of some results of Meyer [30]:

(6.6) Proposition. *For each $n \geq 3$, any space S^n_λ in $\mathcal{S}^n$ is an $(n+1)$-manifold factor. More precisely, $S^n_\lambda \times S^1 (S^n_\lambda \times E^1)$ is homeomorphic to $S^n \times S^1$ $(S^n \times E^1)$.*

(6.7) Remark. It is well-known that every generalized n-manifold is an n-manifold when $n \leq 2$; moreover, every (compact) generalized n-manifold with the homotopy type of S^n is homeomorphic to S^n where $n = 1$ or 2 (cf. [45]). This may be contrasted with Theorem (6.5) and some other results of this note. The following observation may also be useful in the sequel: *An open subset of a generalized n-manifold is itself a generalized n-manifold.*

(6.8) The subsets of S^n_λ and "The backing-up technique". We fix a decomposition space S^n_λ throughout the following discussion. We illustrate the backing-up technique in the following simpler but important setting.

(6.8.1) Setting. Given a closed proper subset A of S^n_λ satisfying (1) A has UV^1, (2) A has dimension ≥ 2, and (3) A contains a simple closed curve C such that $C' = p_\lambda^{-1}(C)$ is a simple closed curve in $A' = p_\lambda^{-1}(A)$ where $p_\lambda: S^n \to S^n_\lambda$ is the projection.

(6.8.2) Goal. Our immediate goal is to use "the backing-up argument" to reach a contradiction and thereby prove that S^n_λ does not contain any proper closed subset of dimension ≥ 2 with UV^1.

It follows from a theorem of Sher [35] that A' and A have the same shape. Since UV^1 is a shape invariant (cf. [8]), it follows that A' has UV^1. Let W_0 be an open subset of $(S^n - A')$ such that the complement $(S^n - W_0)$ is a nbd. of A'. Now apply UV^1 of A' to find a nest $V_1 \supset V_2 \supset \ldots$ of open saturated nbds. of A' in S^n such that (1) V_1 does not intersect W_0 (choose V_1 inside $(S^n - W_0)$), (2) $A' = \bigcap_{i=1}^{\infty} V_i$, and (3) each loop inside V_{i+1} is nullhomotopic inside V_i. There exists a manifold M belonging to the family $\mathcal{F}^{(n-2)}$ such that C' links M and M intersects W_0. This means that there exists a tube $T_j = M \times D^2$ with center $M_j = M$ such that a normal disk $\{m\} \times D^2$ is contained in W_0, see Sections 4 and 5. We put $M_\alpha = T_j$ so that we may use the notation of Section 4. Let $\{M_{\alpha i}: 1 \leq i \leq n_\alpha\}$ and $\{L_{\alpha i}: 1 \leq i \leq n_\alpha\}$ denote, respectively, ε_0-chain of manifolds and $2\varepsilon_0$-chain of dyadic arcs substituting for M_α (where $\varepsilon_0 = 1/j$), see Section 4. The simple closed curve C' bounds a PL singular disk $f: D^2 \to V_{n_\alpha}$ since C' is contained in $V_{n_\alpha+1}$. Since C' and the center M of M_α are linked, it follows from [16] that there exists a punctured disk $P \subset D^2$ and a map $f': (P, \partial P) \to (M_{\alpha i}, \partial M_{\alpha i})$ for some i, $1 \leq i \leq n_\alpha$, such that f' is I-essential in the sense of [16], and $f'(P) \subset V_{n_\alpha}$. It follows from [16] that each arc $A_{\alpha i}$ and $A'_{\alpha i}$ intersects $f'(P)$; furthermore, it follows that the dyadic arc $L_{\alpha i}$ is contained in V_{n_α} since V_{n_α} is saturated. We have set this up so that we may use the notation and terminology of (4.3) and (4.4). In this setting, we state and prove the following:

(6.8.3) "The Backing-up Lemma". *The hypothesis $L_{\alpha i}$ contained in V_{n_α}, as above, implies $L_{\alpha(i+1)}$ is contained in $V_{(n_\alpha - 1)}$.*

Proof of (6.8.3). Choose a point a in $[A'_{\alpha i} \cap f'(P)]$ and a point a' in $[A'_{\alpha i} \cap f'(P)]$. Our notation $\langle xyzw \rangle$ etc. denotes an arc or a simple closed curve starting at x and traversing through y, z and ending at w in the order these letters appear in $\langle xyzw \rangle$. Let $\langle aa' \rangle$ be an arc inside $f'(P)$. We choose two subarcs $\langle ay \rangle$ and $\langle a'y' \rangle$ of arcs $A_{\alpha i}$ and $A'_{\alpha i}$, respectively. Since the loop (or the simple closed curve) $\langle yzy'a'ay\rangle$ (see (4.1.5)) is contained in $M_{\alpha i}$, it is nullhomotopic in $(E^n - M_{\alpha(i+1)})$ by (4.1.3). Therefore, the arcs $\langle yaa'y'\rangle$ and $\langle yzy'\rangle$ are homotopic inside $(E^n - M_{\alpha(i+1)})$ with endpoints fixed. Now the loop $\langle aa'y'z'y'a\rangle$ links $M_{\alpha(i+1)}$ since $S = \langle yzy'z'y\rangle$ links $M_{\alpha(i+1)}$; furthermore, since this loop $\langle aa'y'z'ya\rangle$ is contained inside V_{n_α} it is nullhomotopic in $V_{(n_\alpha - 1)}$. This means that we may apply our earlier arguments, see the argument following the statement (6.8.2), to conclude that $L_{\alpha(i+1)}$ is contained in $V_{(n_\alpha-1)}$. This finishes our proof of (6.8.3).

By iteratively applying (6.8.3), we have that $\bigcup_{i=1}^{n_\alpha} L_{\alpha i}$ is contained inside V_1. Since W_0 contains a normal disk, see above, it follows from [16] that this normal disk intersects $\bigcup_{i=1}^{n_\alpha} L_{\alpha i}$, i.e., W_0 intersects V_1. We reach a contradiction to our assumption $(V_1 \cup W_0) = \emptyset$. This finishes our proof and our Goal (6.8.2) is reached. We now start with the following setting and our Goal is the same as (6.8.2).

(6.8.4) Setting. Given a closed proper subset A of S_λ^n such that A has dimension $\geqslant 2$ and A have UV^1.

We recall some facts from [37] concerning linking. It is shown in [37] that there exists a manifold M belonging to the family $\mathscr{F}^{(n-2)}$ such that each nbd. of A contains a simple closed curve which links M. This allows us to apply the backing-up argument given above and we conclude that no subsets of S_λ^n satisfies (6.8.4). The following theorem can be regarded as a preliminary version of our main theorem which is stated here to summarize our discussions given above and to make the transition to our main result easier:

(6.8.5) Theorem (Preliminary Version). *For each integer $n \geqslant 3$ the family $\mathscr{S}^n$ consisting of uncountably many topologically distinct generalized n-manifolds satisfies the following: Each S_λ^n belonging to $\mathscr{S}^n$ satisfies* (1) *$S_\lambda^n \times S^1$ is homeomorphic to $S^n \times S^1$,* (2) *S_λ^n has the homotopy type of S^n, and* (3) *S_λ^n does not contain any closed proper subset A of dimension $\geqslant 2$ such that A has* UV^1.

(6.8.6) Remark. We observe that the class of compacta with UV^1 contains all compacta with trivial shape (FAR's), and hence, all the compact metric absolute retracts. This implies that S_λ^n, as above in Theorem (6.8.5), does not contain any cell-like set of dimension $\geqslant 2$ and hence any cell of dimension $\geqslant 2$. Furthermore, one may replace (3) in Theorem (6.8.5) by the following: (3′) *If A is any compactum with* UV^1 *such that A has dimension $\geqslant 2$, then A cannot be embedded in S_λ^n as a proper subset of S_λ^n.*

(6.9) A property $\mathrm{UV}^1(sl)$ and strong movability. We shall extend our results of (6.8.5); for instance, we shall show that S_λ^n does not contain any proper subset A of dimension $\geqslant 2$ which is an ANR. We begin with some preliminary notions.

(6.9.1) Definition. A continuum A contained in an ANR X has the *property* UV^1 *for small loops* (Abbreviate: $\mathrm{UV}^1(sl)$) if there exists a positive number δ satisfying: For each nbd. U of A in X there exists a nbd. $V \subset U$ of A such that each δ-loop (i.e., a loop of diameter $<\delta$) in V is nullhomotopic in U.

(6.9.2) Proposition. *Let A be a continuum in an* ANR *X such that A is shape dominated by a compact polyhedron P. Then A has the property $\mathrm{UV}^1(sl)$ in X.*

Proof. Since P shape dominates A, we let $f\colon A \to P$ and $g\colon P \to A$ denote two fundamental sequences in the sense of Borsuk [8] such that $gf\colon A \to A$ is equivalent to the identity fundamental sequence $i_A\colon A \to A$. Since P is an ANR, the shape map $f\colon A \to P$ is represented by a map which we denote by $f\colon A \to P$; and furthermore, our definition of $f\colon A \to P$ and $g\colon P \to A$ considers P embedded in the ANR P itself rather than taking an embedding of P in some larger AR-space which is usually done in defining fundamental sequences, this does not affect anything. We now choose a nest $V_1 \supset V_2 \supset V_3 \supset \ldots$ of nbds. of A in X such that (1) $A = \bigcap_{i=1}^{\infty} V_i$, and (2) for each i the composite map $g_i f\colon V_{i+1} \to V_i$ is homotopic to the inclusion $V_{i+1} \to V_1$, where $f\colon V_{i+1} \to P$ and $g_i\colon P \to V_i$, respectively, are the maps in the definitions of $f\colon A \to P$ and $g\colon P \to A$. Let η be a positive number such that each η-subset of P (i.e., a subset of diameter $<\eta$) is contractible to a point in P. Choose a number $\delta > 0$ such that the image of each δ-subset of V_2 is an η-subset of P. It is now easy to see that this suffices to prove the result and our proof is finished. *We observe that this proof also shows that each δ-subset of A contracts to a point inside each V_i, $1 \leqslant i < \infty$.*

(6.9.3) Remark. A continuum A which is shape dominated by a compact polyhedron is referred to as "FANR" or "strongly movable" (cf. [8]). The class $\mathscr{C}$ of continua each of which is shape dominated by a compact polyhedron is strictly larger than the class of continua whose members are shape equivalent to compact polyhedra (cf. [8]). It follows that $\mathrm{UV}^1(sl)$ is enjoyed by members of $\mathscr{C}$. This suffices for our applications and we do not pursue whether $\mathrm{UV}^1(sl)$ is a shape invariant. Easy examples show that a movable continuum may not have $\mathrm{UV}^1(sl)$; as an example, it is easy to see that the Hawaiian ear ring in S^2 (this is an infinite wedge of circles whose diameters converge to zero) is movable but does not have $\mathrm{UV}^1(sl)$.

(6.10) Strongly movable subset of S_λ^n. Suppose A is a proper closed and connected subset of S_λ^n such that A is strongly movable and dimension of A is $\geqslant 2$. The subset $A' = p_\lambda^{-1}(A)$ of S^n has the shape of A and furthermore A' is a proper continuum contained in S^n of dimension $\geqslant 2$. Since strong movability is a shape invariant (cf. [8]), it follows that A' is strongly movable. Suppose W_0 is an open subset of $(S^n - A')$. By Proposition (6.9.2), we find a nest $V_1 \supset V_2 \supset V_3 \supset \ldots$ of saturated open nbds. of A in S^n and a number $\delta > 0$ such that (1) $A' = \bigcap_{i=1}^{\infty} V_i$, (2) for each i each δ-loop in V_{i+1} is nullhomotopic inside V_i, and (3) V_1 is contained in $(S^n - W_0)$. Choose a $\frac{1}{2}\delta$-subset B of A' such that the dimension of B is $\geqslant 2$. The fact that B exists is elementary (cf. [25]). Our arguments given in [37] can be applied without change to find a manifold M in $\mathscr{F}^{(n-2)}$ such that B h-links M in the sense of [37] and M intersects W_0. It follows that each V_i contains a simple closed curve C_i of diameter $<\delta$ such that C_i and M are linked. Choose a tube M_α

such that a normal disk of M_α is contained in W_0 and the first stage chain $\{M_{\alpha i}: 1 \leqslant i \leqslant n_\alpha\}$ is a $\frac{1}{2}\delta$-chain of n-manifolds substituting for M_α. The "backing-up technique" applies and we conclude that S^n_λ does not contain any strongly movable proper closed subset of dimension $\geqslant 2$. The following theorem summarizes our results:

(6.10.1) THEOREM. *For each integer $n \geqslant 3$ the family $\mathscr{S}^n$ consisting of uncountably many topologically distinct generalized n-manifolds satisfies the following: Each S^n_λ belonging to $\mathscr{S}^n$ satisfies* (1) *$S^n_\lambda \times S^1$ is homeomorphic to $S^n \times S^1$*, (2) *S^n_λ has the (simple/fine) homotopy type of S^n, and* (3) *S^n_λ does not contain any closed proper subset of dimension $\geqslant 2$ which is strongly movable (see the remark below concerning* (3)).

(6.10.2) Remark. The class $\mathscr{C}$ defined in (6.9.3) is precisely the class of strongly movable continua (cf. [8]). Every ANR-space (AR-space) belongs to $\mathscr{C}$, and hence, $\mathscr{C}$ contains all compact polyhedra; furthermore, $\mathscr{C}$ contains all cell-like sets (compacta of trivial shape), and more generally, $\mathscr{C}$ contains all continua of the shape of a compact polyhedron. Theorem (6.8.5) follows immediately from Theorem (6.10.1). The assertion (3) in Theorem (6.10.1) can be replaced by the following: (3′) *if A is any strongly movable continuum (compactum) of dimension $\geqslant 2$ then there is no embedding $\varphi: A \to S^n_\lambda$ such that $\varphi(A)$ is a proper subset of S^n_λ* (it is obvious that we cannot rule out embeddings of A onto S^n_λ!).

7. Decompositions of B^n and n-manifolds, $n \geqslant 3$

Let S^{n-1} denote the boundary of B^n. We consider S^n as the one point compactification of the interior $\mathring{B}^n$ of B^n. Suppose $n \geqslant 4$. For each λ, we consider the decompositions S^n_λ and S^{n-1}_λ (recall that we frequently identify decomposition and the decomposition space) of S^n and S^{n-1}, respectively. Let $\mathring{B}^n_\lambda$ denote the decomposition of B^n which is induced from the decomposition S^n_λ. We denote by $\hat{B}^n_\lambda$ the decomposition of B^n obtained from the union of decompositions $\mathring{B}^n_\lambda$ and S^{n-1}_λ. It is clear that $\hat{B}^n_\lambda$ is a null collection, and hence, an upper semicontinuous decomposition. The associated decomposition space which we again denote by $\hat{B}^n_\lambda$ contains an $(n-1)$-dimensional ANR S^{n-1}_λ. This is not desirable. This can be easily corrected by a method of "attaching chords."

(7.1) A method of attaching chords. Throughout the following discussions we let $n \geqslant 4$. Let B^n, $\mathring{B}^n$, and S^{n-1} as above. A chord in B^n is a PL arc $\langle pq \rangle$ in B^n such that the end points p and q belong to S^{n-1}. Choose distinct points $p_1, q_1, p_2, q_2, p_3, q_3, \ldots$ such that the sequence $\{d(p_i, q_i)\}_i$ of distances converges to zero and choose a sequence $\{\langle p_i q_i \rangle\}_i$ of disjoint chords such that the sequence of diameters of these chords converges to zero. We shall refer to the sequence $\{\langle p_i q_i \rangle\}_i$ as a *sequence of chords attached to* S^{n-1} (in B^n). It is clear that one can construct a sequence of chords attached to S^{n-1}. For B^3, a rather specific sequence of chords is desired which is discussed in several other places, see [5, 38]. We observe that the fundamental group of the complement of a chord in B^n is trivial (remember $n \geqslant 4$).

(7.2) Modifying the decomposition $\hat{B}^n_\lambda$. We construct a decomposition B^n_λ of B^n such that the set of all the nondegenerate elements of B^n_λ is the union of the set of all the nondegenerate elements of $\hat{B}^n_\lambda$ and a set consisting of sequence of chords attached to S^{n-1}. There is no difficulty in choosing these chords so that all the nondegenerate arcs of B^n_λ are disjoint; for instance, this can be accomplished by inductively constructing chords alongs with other arcs in the interior $\mathring{B}^n$ which will yield chords and the induced decomposition of $\mathring{B}^n$ from S^n_λ simultaneously. Clearly, the decomposition B^n_λ is a null collection consisting of arcs and singletons and hence B^n_λ is an upper semicontinuous decomposition of B^n.

(7.3) Properties of the decomposition space B^n_λ. Let $q_\lambda: B^n \to B^n_\lambda$ denote the projection. We observe that the image $q_\lambda(S^{n-1})$ is not an ANR. This can be easily seen be observing that the fundamental group of $q_\lambda(S^{n-1})$ is not finitely generated since each of the sets $\{p_1, q_1\}, \{p_2, q_2\}, \ldots$ goes to a point under q_λ. This is elementary and we omit details. Suppose A is a strongly movable proper subset of B^n_λ where dimension of A is $\geqslant 2$. The case when $q_\lambda^{-1}(A)$ is contained in S^{n-1} or $\mathring{B}^n$ is treated by applying the backing-up technique in S^{n-1} or $\mathring{B}^n$, respectively. The case when $q_\lambda^{-1}(A)$ meets S^{n-1} and $\mathring{B}^n$ needs some consideration. Since the dimension of A is $\geqslant 2$, it follows that either $A_1 = [q_\lambda^{-1}(A) \cap S^{n-1}]$ or $A_2 = [q_\lambda^{-1}(A) \cap \mathring{B}^n]$ has dimension $\geqslant 2$ [25]. Suppose A_1 has dimension $\geqslant 2$. Apply the backing-up technique inside S^{n-1} as follows. Choose a manifold $M \subset S^{n-1}$ such that A_1 h-links M. Choose a nest $V_1 \supset V_2 \supset \ldots$ of nbds. of $q_\lambda^{-1}(A)$ inside B^n satisfying the properties given in (6.10) and $W_0 \subset S^{n-1}$ where $W_0 \cap V_1 = \varnothing$, $W_0 \cap M \neq \varnothing$ and W_0 is an open subset of S^{n-1}. The backing-up technique applies and we are done. The case when A_2 has dimension $\geqslant 2$ is handled exactly the same way. We shall now state the following:

(7.3.1) THEOREM. *For each integer $n \geqslant 3$, there is an uncountable family $\mathscr{B}^n$ of topologically distinct n-dimensional AR's such that each B^n_λ in $\mathscr{B}^n$ satisfies:* (1) *$B^n_\lambda \times I$ is not homeomorphic to $B^n \times I$*, (2) *$B^n_\lambda \times I^2$ is homeomorphic to $B^n \times I^2$, and* (3) *B^n_λ does not contain any strongly movable proper subset of dimension $\geqslant 2$.*

We have limited our discussions to $n \geqslant 4$. The case $n = 3$ follows from discussions in [38]. The techniques of [41] are different from this note and the families of AR's are also different. Our assertions (1) and (2) in Theorem (7.3.1) follows from [19]. We now state some easy extensions of our results.

(7.4) Decompositions of manifolds. We may extend our results to obtain the following:

(7.4.1) THEOREM. *For each topological n-manifold M^n with $n \geqslant 3$, there exists an uncountable family $\mathscr{M}^n$ of topologically distinct n-dimensional ANR's such that each M^n_λ in $\mathscr{M}^n$ does not contain any strongly movable proper subset of dimension $\geqslant 2$ and M^n_λ has the (proper/simple/fine) homotopy type of M^n. Furthermore, if M^n is a manifold without boundary, then each M^n_λ in $\mathscr{M}^n$ is a generalized n-manifold satisfying $M^n_\lambda \times E^1 \approx M^n \times E^1$ ($M^n_\lambda \times S^1 \approx M^n \times S^1$).*

A proof can be easily constructed based on our arguments for S^n and B^n. We shall indicate this in the following discussion. Choose a starfinite covering of the manifold consisting of coordinate neighborhoods and apply the construction inductively in these coordinate neighborhoods. The details are elementary but lengthy and we omit them. These details can be carried out in several different ways for a manifold with a triangulation or one may use a handle-body decomposition (cf.[26]).

(7.5) Rigid generalized n-manifolds. A topological space X is *rigid* if the only homeomorphism of X onto itself is the identity homeomorphism. The following results follows by combining arguments of this note with the technique of [34] (see also [41]).

(7.5.1) THEOREM. *The family $\mathscr{F}$, $\mathscr{F} = \mathscr{S}^n$, $\mathscr{B}^n$, or $\mathscr{M}^n$, can be constructed such that each element of $\mathscr{F}$ is rigid in addition to the properties stated in Theorems* (6.10.1), (7.3.1), *or* (7.4.1), *respectively.*

This shows that the group of homeomorphisms for a generalized n-manifold can be very small. This may be contrasted with some well-known results concerning the group of homeomorphisms for a manifold.

(7.6) $UV^1(sl)$ **revisited.** Suppose G is a cell-like upper semicontinuous decomposition of a metric space X and suppose A is a closed subset of the decomposition space X/G with $UV^1(sl)$ in X/G. The usual lifting arguments (cf. [27]) show that the set $p^{-1}(A)$ has $UV^1(sl)$ in X where $p: X \to X/G$ is the projection map. We observe that we have actually proved the following more general result:

(7.6.1) THEOREM. *The family $\mathscr{F}$, $\mathscr{F} = \mathscr{S}^n$, $\mathscr{B}^n$, or $\mathscr{M}^n$, satisfies, in addition to the properties stated in Theorems* (6.10.1), (7.3.1), *or* (7.4.1), *that each X belonging to $\mathscr{F}$ does not contain any proper (closed) subset of dimension $\geqslant 2$ with $UV^1(sl)$ in X. The same applies to the family given in Theorem* (7.5.1).

(7.7) A question of John Walsh (communicated to us by R. J. Daverman). *Does there exist a cell-like map from a generalized n-manifold onto an n-manifold?*

The answer to this question, in general, is negative. Consider any cell-like map $f: X \to N^n$ from a generalized n-manifold X belonging to the family $\mathscr{F}$, $\mathscr{F} = \mathscr{S}^n$ or $\mathscr{M}^n$, onto an n-manifold N^n. Let A be a strongly movable proper subset N^n such that $\dim(A) \geqslant 2$. It is easy to see that $p^{-1}(A)$ is strongly movable (in fact, $p^{-1}(A)$ is shape equivalent to A [35]) and $p^{-1}(A)$ is a proper subset of X of dimension $\geqslant 2$. This is a contradiction. We have actually proved the following:

(7.7.1) THEOREM (Stability under cell-like mappings). *Given X as above. If $f: X \to Y$ is a surjective cell-like mapping, then Y does not contain any proper subset A of dimension $\geqslant 2$ with* $UV^1(sl)$ *in Y.*

(7.7) Movable subsets. The following theorem shows that our decomposition spaces contain enough movable subsets:

(7.8.1) THEOREM. *If G is an upper semicontinuous decomposition of an n-manifold M^n with $n \geqslant 2$ such that the nondegenerate elements of G form a null collection of arcs, then the decomposition space M^n/G contains a movable proper subset of dimension k where $0 \leqslant k \leqslant n$.*

Proof. Choose a closed k-cell D inside M^n. Let $\hat{D} = p^{-1}[p(D)]$ denote the saturation of D where $p: M^n \to M^n/D$ denotes the projection. It is easy to see that $\hat{D}$ is a continuum of dimension k. It remains to show that $\hat{D}$ is movable. This can be easily seen by shrinking $D \subset \hat{D}$ and observing that $\hat{D}/D$ is a 1-dimensional planar continuum (cf. [8]). This suffices to prove the theorem.

(7.8.2) COROLLARY. *Suppose X belongs to the family $\mathscr{F}$, $\mathscr{F} = \mathscr{B}^n$, $\mathscr{S}^n$, or $\mathscr{M}^n$, and suppose k is an integer satisfying $0 \leqslant k \leqslant n$. Then, X contains a movable proper subset of dimension k.*

(7.8.3) Concluding remark. Each compactum A inside X, X as in Corollary (7.8.2), can be approximated by a locally connected compactum in the following rather strong sense: For each $\varepsilon > 0$ there exists a map $\varphi: A \to A_\varepsilon$ onto a locally connected subset A_ε of X such that $d[a, \varphi(a)] < \varepsilon$ for each a in A [41]. Observe that A_ε is pointed 1-movable. Theorem (7.8.1) is also a consequence of our more general results which are too technical to discuss here.

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