foncteur $E^{(\ )}$ soit une multialgèbre s'exprime par la relation $E^I = \coprod_{\mathscr{M}} E^{(I,\mathscr{M})}$, c'est-à-dire que, pour toute famille $(x_i) \in E^I$, il existe un unique $\mathscr{M}$ tel que $(x_i) \in E^{(I,\mathscr{M})}$, à savoir $\mathscr{M} = \{I' \subset I\colon I' \neq \emptyset$ et $(x_i)_{i\in I'}$ majoré$\}$. Si E, F sont ordonnés complets, une application sup-continue propre $f\colon E \to F$ détermine une transformation naturelle $f^{(\ )}\colon E^{(\ )} \to F^{(\ )}$ définie par $f^{(I,J)}(x_i) = (f(x_i))$. Réciproquement une multialgèbre $F\colon M \to \mathrm{Ens}$ détermine un ensemble ordonné complet $E = F1$ dont l'ordre est défini de la façon suivante: on considère le cardinal $2 = \{0, 1\}$, l'ensemble $\mathscr{M}_0 = \{\{0\}, \{1\}, \{0, 1\}\}$ de parties de 2, l'objet $(2, \mathscr{M}_0)$ de M et le morphisme $\sigma = \{0, 1\}\colon (2, \mathscr{M}_0) \to 1$. On définit alors $x \leqslant y$ par: $(x, y) \in F(2, \mathscr{M}_0)$ et $F(\sigma(x, y)) = y$. En outre, une transformation naturelle $t\colon F \to G$ entre deux multialgèbres définit une application sup-continue propre $t_1\colon F1 \to G1$.

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Classical hierarchies from a modern standpoint

Part I. C-sets

by

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Abstract. The C-sets form the smallest family of sets of reals containing the open sets and stable under complementation and operation $\mathscr{A}$. The theory of C-sets is surveyed, and a new selection result proved: A G_δ-valued, C-measurable multifunction whose graph is a C-set admits a C-measurable selector.

Chapter A. Introduction

§ 1. Motivation. Scattered through the literature of measure theory, general topology, logic, and probability are several definitions of families of sets of reals properly including the Borel sets and provably included in the Lebesgue measurable sets. Their interest derives from two circumstances: On the one hand, in certain situations in everyday mathematical practice, particularly in connection with selection problems of the sort surveyed in [20], non-Borel sets arise naturally. On the other hand, for ordinary mathematical purposes nonmeasurable sets are worthless, and many working mathematicians would even rather avoid sets whose measurability can only be established using, say, Martin's Axiom or large cardinals.

As we go beyond the Borel, the first natural stopping place is Selivanovski's [15] family of *ensembles criblés* or *C-sets*, the smallest family containing the open sets and stable under complementation and what is called *Suslin's operation* or *operation* $\mathscr{A}$. The most basic properties of C-sets can be derived from theorems in the classical text [9]: The C-sets form a σ-field containing the analytic sets (continuous images of Borel sets) and stable under inverse image by C-measurable functions and containing only Lebesgue measurable sets. Together with a celebrated theorem on the uniformization of analytic sets due to Yankov/von Neumann (v. § 8 below), these properties suffice for many applications, e.g. the following (cf. [20, § 9] for finer results of Brown & Purves et al.):

Let $f\colon R^2 \to [0, 1]$ be a Borel function, and suppose that for each x the function $f(x, \cdot)$ achieves its infimum, which we denote $f^*(x)$. For many purposes it would

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be desirable to have a Lebesgue measurable function g satisfying $f(x, g(x)) = f^*(x)$, but a Borel g need not exist. Now the basic properties cited above imply that f^* is C-measurable, since the inverse images of enough open sets are analytic: $(f^*)^{-1}[[0, r)]$ — projection $(\operatorname{graph} f \cap (R^2 \times [0, r)))$. Now the celebrated theorem cited above implies the existence of a C-measurable function h defined on all pairs (x, y) for which there exists a point z with $f(x, z) = y$, and assigning each such pair one such point. We then get a suitable C-measurable g by setting $g(x) = h(x, f^*(x))$.

Modern game-theoretic methods lead to many refinements and extensions of basic classical results about C-sets, among them the selection theorem announced in the Abstract. The present paper provides a survey of the theory of C-sets from the modern game-theoretic viewpoint, including much "folklore" and culminating in the proof of our selection result, which uses much machinery developed in connection with other theorems.

The author's chief debt is to Professors R. L. Vaught and D. E. Miller for cooperation and intellectual stimulation over many years. The influence of ideas of Y. N. Moschovakis and A. S. Kechris will be evident on every page. Helpful communications have been received from C. Dellachérie and from R. D. Mauldin.

§ 2. Notation & terminology. We collect some conventions used throughout this paper, to be glanced over and referred back to as needed.

(a) Set theory. An ordinal *is* its set of predecessors: $0 = \emptyset$, $2 = \{0, 1\}$, $\omega = \{0, 1, 2, \ldots\}$, $\Omega = \{\alpha\colon \alpha \text{ a countable ordinal}\}$. A *sequence* is simply a function ξ whose domain is some ordinal α, called the *length* of ξ. The value $\xi(\beta)$ at $\beta < \alpha$ is called the βth *term* of ξ. The restriction $\xi|\beta$ to arguments $<\beta$ is called an *initial segment* of ξ; $\eta \lhd \xi$ denotes that η is an initial segment of ξ. $\xi \oplus i$ denotes ξ with i adjoined at the end, the sequence of length $\alpha+1$ whose βth term for $\beta<\alpha$ is $\xi(\beta)$ and whose αth term is i. $i \oplus \xi$ is similarly defined.

$Q(I)$ (resp. $Q^*(I)$) will denote the set of sequences from I of finite (resp. finite *even*) length, including the empty one ( ). We fix an enumeration of $Q(\omega)$ in a sequence of length ω, in which the initial segments of any sequence precede the sequence itself. The *code number* $\# s$ of $s \in Q(\omega)$ is its place in this enumeration. I^J always denotes the functions from J to I. In case I, J are nonempty and countable, this set is to be given the topology having as subbasis the sets $\{\xi\colon \xi(j) = i\}$ for $j \in J$ and $i \in I$. The *characteristic function* χ_A of a set A takes the value 1 on A, 0 off A. The power set $\mathscr{P}(J)$ is to be given the topology making it $\simeq 2^J$ under the map associating a set with its characteristic function. We define a bijection π from ω^2 to ω by $\pi(i,j) = 2^i(2j+1)-1$; then we define a bijection Π from $\omega \times \omega^\omega$ to ω^ω by $\Pi(i, \xi)(j) = \xi(\pi(i,j))$.

(b) Game theory. We wish to assign a meaning to formulas of type:

(i) $\exists i_0 \in I\, \forall j_0 \in I\, \exists i_1 \in I\, \forall j_1 \in I \ldots \Phi(i_0, j_0, i_1, j_1, \ldots)$.

To this end we consider an infinite game for two persons, the $\exists$-player PRO and the $\forall$-player CON: PRO opens by choosing $i_0 \in I$, CON responds by choosing $j_0 \in I$, then PRO chooses i_1, then CON chooses j_1, ... each player having at each stage perfect information about the opposing player's earlier moves. PRO wins the game if the sequence $(i_0, j_0, \ldots) \in I^\omega$ thus generated satisfies condition Φ. A *strategy* for a player is a rule telling that player how to move on each turn as a function of the opposing player's earlier moves. Formally, for the first player it is a function $Q(I) \to I$; for the second, a function $Q(I) - \{( )\} \to I$. If φ is a strategy for, say, the first player, then a (finite or infinite) sequence $(i_0, j_0, \ldots)$ constitutes a (partial or complete) play *agreeing* with φ if the i's are what φ says they should be: $i_0 = \varphi(( ))$, $i_1 = \varphi((j_0))$, $i_2 = \varphi((j_0, j_1))$, ... A strategy is a *winning* one for a player iff that player can be assured of winning the game by using the strategy. E.g. a strategy for PRO is a winning one iff every complete play agreeing with it satisfies Φ. *We define* (i) *to mean that* PRO *has a winning strategy*. Our game is called *determined* if *some* player has a winning strategy. In that case, if (i) does not hold then the following does:

(ii) $\forall i_0 \in I\, \exists j_0 \in I\, \forall i_1 \in I\, \exists j_1 \in I \ldots \neg\Phi(i_0, j_0, i_1, j_1, \ldots)$.

Sometimes we consider modified games in which, when play so far has produced a finite sequence s, the set of moves permitted to the next player is some $I(s) \subseteq I$ depending on s. Games of this kind reduce to the original kind by modifying the winning condition Φ so that the first player (if any) to move outside the appropriate $I(s)$ forfeits the game. Later we will consider games where the sequence of moves has length $>\omega$.

(c) Topology. By a *standard* space we mean an uncountable *Polish* space or space admitting a countable basis and a complete metric. The usual reference for such spaces is [9] and elementary results from that source will be used without comment. Deeper results from [9] will be referred to as *classical theorems* and cited by location in that text. E.g. [9, 33 III] contains the result that the spaces I^J are standard; [9, 33 VI] that an uncountable G_δ subspace of a standard space is standard; [9, 36 III] that all standard spaces are Borel-isomorphic. [9] contains ample historical references and no attempt to duplicate them will be made here.

A function f from an arbitrary space $\mathscr{X}$ to another space $\mathscr{Y}$ will be called *measurable* w.r.t. a σ-field $\mathscr{H}$ of subset of $\mathscr{X}$ if $f^{-1}[U] \in \mathscr{H}$ for all Borel (equivalently: open) $U \subseteq \mathscr{Y}$. For $\mathscr{Y} = \mathscr{X}$ the class of $\mathscr{H}$-measurable functions need *not* be stable under composition. It will be if $\mathscr{H}$ is stable under inverse image by $\mathscr{H}$-measurable functions. For $\mathscr{X}$ standard, a σ-field $\mathscr{H}$ satisfying this last condition and containing the Borel sets will be called *uniform*. For such an $\mathscr{H}$ there is a "uniform" way to extend the notion of $\mathscr{H}$-set to arbitrary standard spaces $\mathscr{Y}$, since all Borel-isomorphisms $f\colon \mathscr{Y} \to \mathscr{X}$ give rise to the same class $\{f^{-1}[U]\colon U \in \mathscr{H}\}$.

Let I be a denumerable set, $\mathscr{X}$ an arbitrary space. An *I-indexed system* of subsets of $\mathscr{X}$ is a function $A = (A(i)\colon i \in I)$ from I to $\mathscr{P}(\mathscr{X})$. Its *indicator* is the function $\iota_A(x) = \{i\colon x \in A(i)\}$ from $\mathscr{X}$ to $\mathscr{P}(I)$, which is measurable w.r.t. any σ-field containing all the $A(i)$. Any $B \subseteq \mathscr{P}(I)$ gives rise to a *set operation* Γ with *truth table* B, defined on I-indexed systems by $\Gamma(A) = \iota_A^{-1}[B]$. The *dual* operation is co-$\Gamma(A)$.

$= \mathscr{X} - \Gamma((\mathscr{X} - A(i)\colon i \in I))$. If $\mathscr{H}$ is a uniform family of subsets of $\mathscr{P}(I)$ and $B \in \mathscr{H}$, Γ is called an *$\mathscr{H}$-operation*. Note that this implies $\mathscr{H}$ is stable under Γ, and since $B = \Gamma((\{y \in \mathscr{P}(I)\colon i \in y\}\colon i \in I))$ the converse implication also holds. One set operation Γ is *reducible* to another Γ, if there is a systematic method — we do not stop to define this with perfect rigor — of rewriting any $\Gamma(A)$ in the form $\Gamma'(A')$ where each $A'(j)$ is a finite Boolean combination of the $A(i)$, i.e. belongs to the field of sets they generate. Plainly reducibility is transitive, is preserved by dualization, etc., and such facts will be used tacitly below.

Chapter B. The C-hierarchy

§ 3. Operation $\mathscr{A}$. In this section we introduce the operation $\mathscr{A}$ which gives rise to the family of C-sets. It is defined on $Q(\omega)$-indexed systems by:

$$\mathscr{A}(A) = \bigcup_{\xi \in \omega^\omega} \bigcap_{n \in \omega} A(\xi|n).$$

Thus we have $x \in \mathscr{A}(A)$ iff:

$$\exists i_0 \in \omega\ \exists i_1 \in \omega\ \exists i_2 \in \omega \ldots \forall n [x \in A((i_0, i_1, i_2, \ldots)|n)]$$

and in the terminology of the preceding section, the truth table of A amounts to:

$$\{s \subseteq Q(\omega)\colon \exists T \subseteq S[(\ ) \in T \,\&\, \forall s \in T \exists i \in \omega\ (s \oplus i \in T)]\}.$$

Countable intersection or *meet*, countable union or *join*, and double application of $\mathscr{A}$ are all reducible to $\mathscr{A}$. Indeed:

$$\bigcap_{i \in \omega} A(i) = \bigcup_{\xi} \bigcap_{n} B(\xi|n)$$

where $B(s) = A(\text{length } s)$,

$$\bigcup_{i \in \omega} A(i) = \bigcup_{\xi} \bigcap_{n} B(\xi|n)$$

where $B((\ )) = \mathscr{X}$ and $B((i, \ldots)) = A(i)$,

$$\bigcup_{\eta} \bigcap_{m} \bigcup_{\zeta} \bigcap_{k} A(\eta|m, \zeta|k) = \bigcup_{\xi} \bigcap_{n} B(\xi|n)$$

where $B(\xi|n) = A(\Pi(0, \xi)|m, \Pi(1+m, \xi)|k)$ for the largest value of $\pi(m, k)$ for which these items can be computed from the data supplied by $\xi|n$. Now over any space $\mathscr{X}$ we introduce a hierarchy of subsets thus:

$\mathscr{B}^1_0 =$ Borel sets,
$\mathscr{A}^1_{\beta+1} =$ sets obtainable by $\mathscr{A}$ from indexed systems of elements of $\mathscr{B}^1_\beta$,
$\mathscr{C}^1_{\beta+1} =$ complements of elements of $\mathscr{A}^1_{\beta+1}$,
$\mathscr{D}^1_{\beta+1} = \mathscr{A}^1_{\beta+1} \cap \mathscr{C}^1_{\beta+1}$,
$\mathscr{B}^1_{\beta+1} = \sigma$-field generated by $\mathscr{A}^1_{\beta+1}$,
$\mathscr{B}^1_\alpha = \sigma$-field generated by $\bigcup_{\beta<\alpha} \mathscr{B}^1_\beta$ at limits $\alpha < \Omega$,
$C\text{-sets} = \bigcup_{\alpha<\Omega} \mathscr{B}^1_\alpha$.

Plainly all these classifications are stable under Borel-isomorphism. Since inverse image commutes with complementation, meet, join, and $\mathscr{A}$, the C-sets are stable under inverse image by C-measurable functions. The C-sets form the smallest family containing the open sets and stable under complementation and $\mathscr{A}$, this family automatically being a σ-field by the reducibility of meet and join to $\mathscr{A}$. Each $\mathscr{A}^1_{\beta+1}$ is stable under $\mathscr{A}$ by the reducibility of double application of $\mathscr{A}$ to single, and each $\mathscr{C}^1_{\beta+1}$ is stable under co-$\mathscr{A}$; hence these classes are stable under meet and join, and $\mathscr{D}^1_{\beta+1}$ is a σ-field.

Further argument using the reducibilities above shows that $\mathscr{A}^1_1$ consists of the sets obtainable by $\mathscr{A}$ from open and closed sets. If $\mathscr{X}$ is metrizable (so that open sets are F_σ and closed sets G_δ) we can improve this to "from closed sets" and equally to "from open sets". For standard $\mathscr{X}$, by a classical theorem [9, 39 II], $\mathscr{A}^1_1$ coincides with the *analytic* sets or continuous images of Borel sets. The sets in $\mathscr{C}^1_1$ are then called *co-analytic*, and $\mathscr{B}^1_1$-measurable functions *analytically* measurable. Further argument using the reducibilities shows that $\mathscr{A}^1_{\gamma+2}$ consists of the sets obtainable by $\mathscr{A}$ from $\mathscr{C}^1_{\gamma+1}$-sets, and at limits α, $\mathscr{A}^1_{\alpha+1}$ of the sets obtainable by $\mathscr{A}$ from the union of the $\mathscr{B}^1_\beta$ for $\beta < \alpha$. Such facts will be used tacitly below.

§ 4. Operation $\mathscr{G}$. In this section we introduce a game-theoretic equivalent $\mathscr{G}$ of $\mathscr{A}$. The material in this section has been extracted from a much more general setting in [11]. Operation $\mathscr{G}$ is defined on $Q^*(\omega)$-indexed systems by letting x belong to $\mathscr{G}(B)$ iff:

$$(*) \qquad \exists i_0 \in \omega\ \forall j_0 \in \omega\ \exists i_1 \in \omega\ \forall j_1 \in \omega \ldots \forall n [x \in A((i_0, j_0, i_1, j_1, \ldots)|2n)].$$

Visibly, $\mathscr{A}$ is reducible to $\mathscr{G}$. Conversely, $\mathscr{G}(B) = \mathscr{A}(A)$ where $A((\ )) = B((\ ))$ and for other s, $A(s)$ is determined as follows: Let length $s-1 = \#(j_0, \ldots, j_m)$. Let $i_0 = s(\#(\ ))$, $i_1 = s(\#(j_0))$, $i_2 = s(\#(j_0, j_1))$, ... and $t = (i_0, j_0, \ldots)$. Set $A(s) = B(t)$. In this way to any $\xi \in \omega^\omega$ corresponds a strategy $\varphi(t) = \xi(\# t)$ which is a winning strategy for the game $(*)$ associated with membership in $\mathscr{G}(B)$ just in case x belongs to all $A(\xi|n)$ as required by membership in $\mathscr{A}(A)$. So $\mathscr{A}$ and $\mathscr{G}$ are interreducible.

A major tool in classical studies of analytic and C-sets is an inductive analysis of $\mathscr{A}$, which has a modern parallel in an inductive analysis of $\mathscr{G}$. Let A and B be respectively $Q(\omega)$- and $Q^*(\omega)$-indexed systems. Define:

$$A^0(s) = \bigcap_{\substack{u \in Q(\omega)\\ u \lhd s}} A(u), \qquad B^0(t) = \bigcap_{\substack{v \in Q^*(\omega)\\ v \lhd t}} B(v),$$

$$A^{\beta+1}(s) = \bigcup_{i \in \omega} A^\beta(s \oplus i), \qquad B^{\beta+1}(t) = \bigcup_{i \in \omega} \bigcap_{j \in \omega} B(t \oplus i \oplus j),$$

$$A^\alpha(s) = \bigcap_{\beta<\alpha} A^\beta(s), \qquad B^\alpha(t) = \bigcap_{\beta<\alpha} B^\beta(t) \text{ at limits } \alpha \leqslant \Omega.$$

It is readily verified that for $u \lhd s$ and $\beta \leqslant \alpha$ we have $A^\alpha(s) \subseteq A^\beta(u)$, and similarly for B. For $\alpha < \Omega$ the $A^\alpha(s)$ and $B^\alpha(t)$ belong to the σ-field generated by the $A(s)$ or $B(t)$. Classically [9, 3 XIV] it was known that $\mathscr{A}(A) = A^\Omega((\ ))$. As a modern parallel we have (b) below. Note that the classical result essentially follows from the modern one by regarding $\mathscr{A}$ as a special case of $\mathscr{G}$.

(a) DETERMINATENESS THEOREM (Gale & Stewart). *The game* $(*)$ *associated with membership in* $\mathscr{G}(\boldsymbol{B})$ *is determined.*

(b) INDUCTIVE ANALYSIS THEOREM (Moschovakis). $\mathscr{G}(\boldsymbol{B}) = \boldsymbol{B}^{\Omega}((\ ))$.

Proofs. For any x and s, let the x-*rank* of s be the least β with $x \notin \boldsymbol{B}^{\beta+1}(s)$ is such exists. Otherwise, let it be Ω. To prove (a) and (b) it will suffice to show that if $x\text{-rank}((\ )) = \Omega$ (resp. is $<\Omega$) then PRO (resp. CON) has a winning strategy for $(*)$. We begin by noting some immediate consequences of the definitions:

(i) $0 \leqslant x\text{-rank}(t) < \Omega \to \forall i\ \exists j\ x\text{-rank}(t\oplus i\oplus j) < x\text{-rank}(t)$,

(ii) $\alpha < x\text{-rank}(t) \to \exists i\ \forall j\ \alpha \leqslant x\text{-rank}(t\oplus i\oplus j)$.

In (i), let $j(t\oplus i)$ be the least suitable j; and if $x\text{-rank}(t) = -1$, let $j(t\oplus i) = 0$. In (ii), if $x\text{-rank}(t) = \Omega$, then for *every* $\alpha<\Omega$ there exists *some* suitable i. But in fact, some *one* i must work for uncountably many and hence *all* α, so that $x\text{-rank}(t\oplus i\oplus j) = \Omega$ for all j. Let $i(t)$ be the least such.

Now if $x\text{-rank}((\ )) = \Omega$, we define the *canonical strategy* φ for PRO as follows: If play so far has produced t, as next move choose $i(t)$. In this way PRO ensures that each partial play t produced has $x\text{-rank} = \Omega > -1$, so that $x \in \boldsymbol{B}(t)$. Thus φ is a winning strategy for PRO.

If by contrast $x\text{-rank}((\ ))<\Omega$, we define the *canonical strategy* ψ for CON as follows: If play so far has produced t and PRO as next move chooses i, choose as next move $j(t\oplus i)$. In this way CON ensures that the x-ranks of the partial plays produced decrease until a t is reached with x-rank -1, so $x \notin B(v)$ for some $v \lhd t$. Thus ψ is a winning strategy for CON. ■

(c) COMPARISON LEMMA (Moschovakis). *If the* $\boldsymbol{B}(t)$ *are* C-*sets, then so is* $\{x\colon x\text{-}rank(t)<x\text{-}rank(v)\}$ *for each* v *and* t.

(d) CANONICAL STRATEGY THEOREM (Moschovakis). *If the* $\boldsymbol{B}(t)$ *are* C-*sets, then the function associating to each* x *the canonical strategy* φ *or* ψ *for* PRO *or* CON *as the case may be is* C-*measurable.*

Proofs. In (d) note that the space of strategies is standard, being the disjoint union of $\omega^{Q(\omega)}$ (the strategies for PRO) and ω^J for $J = Q(\omega) = \{(\ )\}$ (the strategies for CON). (d) is immediate from (c), since the definition of the canonical strategy only involved comparison of ranks.

As for (c), we begin by noting a consequence of (i) and (ii) above, viz. the equivalence of $0\leqslant x\text{-rank}(t)<x\text{-rank}(v)$ with:

(iii) $\exists i\ \forall j\ \forall m\ \exists n[x\text{-rank}(t\oplus i\oplus j)<\min(x\text{-rank}(t), x\text{-rank}(v\oplus m\oplus n))]$.

It follows that $x\text{-rank}(t)<x\text{-rank}(v)$ iff:

(iv) $\exists i_0\ \forall j_0\ \forall m_0\ \exists n_0\ \exists i_1\ \forall j_1\ \forall m_1\ \exists n_1 \ldots \exists k[x \in \boldsymbol{B}^0(v+\eta|2k) - \boldsymbol{B}^0(t+\zeta|2k)]$

where $\zeta = (m_0, n_0, \ldots)$ and $\eta = (i_0, j_0, \ldots)$.

Now (iv) shows how to obtain the set mentioned in (c) by co-$\mathscr{G}$, or equivalently by co-$\mathscr{A}$, from sets in the field generated by the $B(u)$. ■

§ 5. **Universal sets.** A famous classical theorem [9, 39 III] tells us that for standard spaces Borel = analytic $\cap$ co-analytic, or in our notation $\mathscr{B}_0^1 = \mathscr{D}_1^1$. In the present section we show that with this one exception all the classifications introduced in § 3 are distinct. (a) below is essentially to be found in [9, 38 VI]; (b) comes from [8]. Throughout this section let $\mathscr{X}$ be a standard space.

(a) FIRST HIERARCHY THEOREM. $\mathscr{A}_{\alpha+1}^1 \neq \mathscr{B}_{\alpha+1}^1$ *for all* $\alpha<\Omega$.

Proof. It suffices to construct an $\mathscr{A}_{\alpha+1}^1$ subset A of $\mathscr{X}\times\omega^\omega$ which is *universal* for the class of $\mathscr{A}_{\alpha+1}^1$ subsets of $\mathscr{X}$, i.e. such that each such set has the form of a cross-section $\{x\colon (x,\xi)\in A\}$ of A at some $\xi\in\omega^\omega$. For then given a Borel-isomorphism $f\colon \mathscr{X}\to\mathscr{X}\times\omega^\omega$ the set $f^{-1}[A]$ belongs to $\mathscr{A}_{\alpha+1}^1$ but cannot belong to $\mathscr{C}_{\alpha+1}^1$, else given a Borel-isomorphism $g\colon \mathscr{X}\to\omega^\omega$, $\{x\colon (x, g(x))\notin A\}$ would have to be an $\mathscr{A}_{\alpha+1}^1$ set and hence a cross-section of A, a manifest absurdity.

It suffices to take as A, $\mathscr{A}(A)$ where $A(s)$ is the set of (x,ξ) with $(x, \Pi(\#s,\xi))\in C$, and C is as follows:

Case (i). $\alpha = 0$. Let $(W(i)\colon i\in\omega)$ enumerate a basis for $\mathscr{X}$ and let

$$C = \{(x,\xi)\colon \exists i(\xi(i) = 0 \,\&\, x\in W(i))\},$$

an open set universal for the open sets.

Case (ii). $\alpha = \beta+1$. Assume a suitable set has been constructed for $\beta+1$ and let C be its complement, a $\mathscr{C}_{\beta+1}^1$ set universal for the $\mathscr{C}_{\beta+1}^1$ sets.

Case (iii). α a limit. Let $(\beta(i)\colon i\in\omega)$ enumerate the ordinals $<\alpha$, and assume that for each i a suitable set A_i has been constructed for $\beta(i)+1$. Let C be the union of the $\{(x, i\oplus\xi)\colon (x,\xi)\in A_i\}$, a $\mathscr{B}_\alpha^1$ set universal for the union of the $\mathscr{B}_\beta^1$ for $\beta<\alpha$. ■

(b) SECOND HIERARCHY THEOREM (Kunugui). $\mathscr{B}_\alpha^1 \neq \mathscr{D}_{\alpha+1}^1$ *for* $0<\alpha<\Omega$.

Proof. It suffices to construct for some standard $\mathscr{X}$ a $\mathscr{D}_{\alpha+1}^1$ subset of $\mathscr{X}\times\mathscr{X}$ universal for the $\mathscr{B}_\alpha^1$ subsets of $\mathscr{X}$. This we do for $\alpha = \beta+1$. The limit case is similar. At 0 we only get a *co-analytic* set universal for Borel sets.

As a preliminary, define an I-*tree* to be a $T\subseteq Q^*(I)$ such that $s\oplus i\oplus j\in T$ always imply $s\in T$ and $s\oplus i'\oplus j'\in T$ for all i' and j'. Call s *terminal* for T iff $s\in T$ but $s\oplus i\oplus j\notin T$ for some/any i and j. The ω-trees form a closed, hence standard, subspace $\mathscr{T}$ of $\mathscr{P}(Q^*(\omega))$. $T\in\mathscr{T}$ is *wellfounded* iff $\forall\xi\in\omega^\omega\ \exists n\in\omega\ \xi|2n\notin T$. The wellfounded trees form a co-analytic set $W = \mathscr{T}-\mathscr{A}(A)$ where $A(s) = \{T\colon s\in T\}$.

As to the actual construction, let Z be an $\mathscr{A}_{\beta+1}^1$ subset of $\mathscr{X}\times\omega^\omega$ universal for the $\mathscr{A}_{\beta+1}^1$ subsets of $\mathscr{X}$. Let Y consist of the $(x, 0\oplus\xi)$ for $(x,\xi)\notin Z$ and the $(x, 1\oplus\xi)$ for $(x,\xi)\in Z$, so its cross-sections give $\mathscr{A}_{\beta+1}^1\cup\mathscr{C}_{\beta+1}^1$. Let D be the $\mathscr{C}_{\beta+2}^1$ subset of $\mathscr{X}\times\omega^\omega\times\mathscr{T}$ which is the complement of $\mathscr{G}(\boldsymbol{B})$, where $\boldsymbol{B}(s)$ is the $\mathscr{B}_{\beta+1}^1$ set of those triples (x,ξ,T) such that the proper initial segment t of s (if any) which is terminal for T satisfies $(x, \Pi(\#t,\xi))\notin Y$. (So $s\in T$ implies $(x,\xi,T)\in\boldsymbol{B}(s)$ trivially.)

CLAIM. *D is a* $\mathscr{D}_{\beta+2}^1$ *set.*

To see this consider the inductive analyses:

$$A^0(s) = A(s)\,, \qquad B^0(s) = B(s)\,,$$
$$A^{\delta+1}(s) = \bigcup_{i,j} A^\delta(s\oplus i\oplus j)\,,$$
$$B^{\delta+1}(s) = \bigcup_i \bigcap_j B^\delta(s\oplus i\oplus j)\,,$$
$$A^\gamma(s) = \bigcap_{\delta<\gamma} A^\delta(s)\,, \qquad B^\gamma(s) = \bigcap_{\delta<\gamma} B^\delta(s) \text{ at limits}\,,$$

W and D are respectively the complements of $A^\Omega((\ ))$ and $B^\Omega((\ ))$. Comparing we see that $T\in A^\gamma(s)$ implies $(x,\xi,T)\in B^\gamma(s)$. Hence $(x,\xi,T)\in D$ implies $T\in W$. But for $T\in W$ we have $\forall\xi\in\omega^\omega\,\exists!n\in\omega\ \xi|2n$ is terminal for T. Hence for $T\in W$ and $y\subseteq Q^*(\omega)$ the following are equivalent:

(i) $\neg\exists i_0\,\forall j_0\,\exists i_1\,\forall j_1\ldots\forall n[t=(i_0,j_0,\ldots)|2n$ terminal for $T\to t\notin y]$,

(ii) $\forall i_0\,\exists j_0\,\forall i_1\,\exists j_1\ldots\forall n[t$ as above terminal for $T\to t\in y]$.

But the game associated with (x,ξ,T) being a member of D has the form (i) for $y=\{s\colon (x,\Pi(\#s,\xi))\in Y\}$. So (ii) shows how D can be obtained as the intersection of a co-analytic set $(\mathscr{X}\times\omega^\omega\times W)$ with a set obtainable by $\mathscr{G}$ from the $\mathscr{B}^1_{\beta+1}$ sets $B(s)$. Hence D is an $\mathscr{A}^1_{\beta+2}$ set.

CLAIM. *D is universal for the $\mathscr{B}^1_{\beta+1}$ sets.*

To see this note that every such set is obtainable from cross-sections of Y by iterated application of $\bigcap_i\bigcup_j$. Now the cross-section of Y at η is the cross-section of D at (T,ξ) where $T=\{(\ )\}$ and $\Pi(\#(\ ),\xi)=\eta$. And if each E_{ij} is the cross-section of D at some (ξ_{ij},T_{ij}), then $\bigcap_i\bigcup_j E_{ij}$ is the cross-section at (ξ,T) where T consists of $(\ )$ plus all $i\oplus j\oplus s$ for $s\in T_{ij}$, and ξ satisfies $\Pi(\#(i\oplus j\oplus s),\xi) = \Pi(\#s,\xi_{ij})$. So the cross-sections of D do indeed include all $\mathscr{B}^1_{\beta+1}$ sets. ∎

Chapter C. Regularity properties

§ 6. Measure. Game-theoretic analyses of category and measure go back a long way (to the work of Mazur & Banach reconstructed in [13, chapter 6], and to [12]), but the analyses to be presented in this section and the next derive from much more recent work of Vaught [18], [19], of Kechris [5], [6], [7], and of their collaborators. Classically it was known that $\mathscr{A}$ preserves measurability in quite general circumstances; here we consider only the most important special case.

Let $\mathscr{X}$ be a standard space, μ a complete regular probability measure on $\mathscr{X}$. So μ is the completion of a Borel measure satisfying $\mu(\mathscr{X})=1$, and there exists a countable field $\mathscr{U}$ of μ-measurable sets such that any μ-measurable set can be approximated up to arbitrarily small measure by an element of $\mathscr{U}$. Let A be a $Q(\omega)$-indexed system of μ-measurable sets. In showing $\mathscr{A}(A)$ is μ-measurable we may without loss of generality assume A *nested*, so that $A(s)\subseteq A(t)$ whenever $t\lhd s$.

By a *relevant triple* we mean a $w=(q,U,s)$ consisting of a rational $q\in[0,1)$, a $U\in\mathscr{U}$, and an $s\in Q(\omega)$. By an *extension* of w we mean another relevant triple (r,V,t) such that $V\subseteq U$ and t is of form $s\oplus i$. By a *w-display* we mean a finite set of extensions of w such that (i) their V's are disjoint, and (ii) their r's sum to more than q.

(a) MEASURE FORMULA (Kechris). *$\mathscr{A}(A)$ is μ-measurable, and for any rational $p\in[0,1)$ we have $\mu(\mathscr{A}(A))>p$ iff for $w_0=(p,\mathscr{X},(\ ))$ we have:*

(∗) $\exists w_0$*-display* $W_0\ \forall w_1\in W_0\ \exists w_1$*-display* $W_1\ \forall w_2\in W_1\ldots\forall n[w_n=(r,V,t)\to$ $\to\mu(V\cap A(t))>r]$.

Proof. One half of the proof consists in showing that if PRO has a wining strategy φ for the game (∗), then $\mathscr{A}(A)$ has inner measure $>p$. To see this we define displays $W(u)$ for some $u\in Q(\omega)$: Let $W((\ ))=\varphi((\ ))$. If $W(u)$ is defined, let $\big(w(u\oplus i)\colon i<\operatorname{card}W(u)\big)$ enumerate it, and apply φ to the sequence of $w(v)$, v an initial segment of $u\oplus i$, to obtain $W(u\oplus i)$. Let S_n be the set of u of length n for which $w(u)=\big(r(u),V(u),t(u)\big)$ is defined. For $n>0$ let C_n be the union of the $V(u)\cap A\big(t(u)\big)$ for $u\in S_n$; let C be the intersection of the C_n. C is μ-measurable, and it will suffice to establish:

CLAIM. *$C\subseteq\mathscr{A}(A)$ and $\mu(C)>p$.*

Indeed, it follows from the disjointness condition (i) in the definition of display that for each $x\in C$ there is a *unique* $\xi\in\omega^\omega$ with $x\in V(\xi|n)\cap A\big(t(\xi|n)\big)$ for all n. Since the $t(\xi|n)$ are precisely the $\eta|n$ for a certain $\eta\in\omega^\omega$, this shows $x\in\bigcap_n A(\eta|n)$ and $C\subseteq\mathscr{A}(A)$. Moreover, it follows from the summation condition (ii) in the definition of display that if we let r_n be the sum of the $r(u)$ for $u\in S_n$, then $p<r_1<r_2<\ldots$ Since plainly $\mu(C_n)>r_n$, this shows $\mu(C)>p$, proving the claim.

The other half of the proof consists in showing that if the outer measure $\mu^*(\mathscr{A}(A))$ is greater than p, then PRO has a winning strategy. To see this, call a relevant triple (r,V,t) *acceptable* if $\mu(V\cap A(t))>r$, and *desirable* if $\mu^*(V\cap B(t))>r$ where $B(t)=\{x\colon \exists\xi\in\omega^\omega(t\lhd\xi\ \&\ \forall n\in\omega\ x\in A(\xi|n)\}$. In this language, the hypothesis of this half of the proof is that w_0 is desirable, and to win (∗) PRO has to ensure that each w_n produced is acceptable. Since $B(t)\subseteq A(t)$, desirability implies acceptability, and so to show PRO has a winning strategy it will suffice to show that if w is desirable, then there exists a w-display all whose elements are desirable. Since $B(t)$ is the union of the $B(t\oplus i)$ it will in fact suffice to establish:

CLAIM. *If $\mu^*(\bigcup_{i\in\omega}C_i)>q$, then there exists a finite set of triples $(r,V,(i))$ satisfying the disjointness and summation conditions of the definition of display and such that $\mu^*(V\cap C_i)>r$.*

Indeed, we could cover each C_i by an element B_i the σ-field generated by $\mathscr{U}$ with the property that $\mu(V\cap B_i)=\mu^*(V\cap C_i)$ for every $V\in\mathscr{U}$. So we may assume the C_i are μ-measurable, and then we may assume them disjoint. There exist a finite N and a positive e such that the measure of the union of the C_i for $i<N$ exceeds q by

more then N^2e. For $i<N$ let $U_i\in\mathscr{U}$ be such that the symmetric difference $U_i\triangle C_i$ is of measure $<e$, and disjointify them, setting $V_i=U_i-\bigcup_{j<i}U_j$. It is easily computed that $\mu(\bigcup_{i<N}(V_i\cap C_i))>q$. Hence we can find r_i so that the finite set

$$\{(r_i, V_i, (i)):\ i<N\ \&\ \mu(V_i\cap C_i)>0\}$$

is as required to complete the proof. ■

There is a natural way, described in [14], to make the set of all complete regular probability measures on the standard space $\mathscr{X}$ into a standard space $\mathscr{M}(\mathscr{X})$. Call $A\subseteq\mathscr{X}$ *universally* measurable if it is μ-measurable for each $\mu\in\mathscr{M}(\mathscr{X})$. For such A and for rational $p\in[0,1)$, let $A^{>p}=\{\mu\in\mathscr{M}(\mathscr{X}):\ \mu(A)>p\}$. The topology of [14] is such that this set is open in $\mathscr{M}(\mathscr{X})$ whenever A is open in $\mathscr{X}$. We always have:

$$\Big(\bigcup_{i\in\omega}A_i\Big)^{>p}=\bigcup_{i\in\omega}\Big(\bigcup_{j<i}A_j\Big)^{>p},$$
$$(\mathscr{X}-A)^{>p}=\bigcup_{q<1-p}\big(\mathscr{M}(\mathscr{X})-A^{>q}\big).$$

These formulas show that $A^{>p}$ is Borel whenever A is. Now the Measure Formula expresses $(\mathscr{A}(A))^{>p}$ as obtainable by $\mathscr{G}$ from sets $(V\cap A(t))^{>r}$. Hence it shows that $A^{>p}$ is a C-set whenever A is. In [16], where this last result first appeared, it was used to derive the following:

(b) MEASURE DUALITY THEOREM (Shreve). *If f: $[0,1]\to[0,1]$ is C-measurable, then so is f^*: $([0,1])\to[0,1]$ where $f^*(\mu)=\int f d\mu$.*

§ 7. Category. In discussing category we use modern terminology: rare, meager, nonmeager, comeager, almost open. (Rather than: nowhere dense, 1st category, 2nd category, residual, Baire property.) The work of Vaught and of Kechris has recently culminated in a proof of a Category Formula (corresponding to 6 (a) above) valid for arbitrary topological spaces and including the classical [9, 11 VII] result that $\mathscr{A}$ preserves almost openness in arbitrary spaces. (And hence, by [13, chapter 22], that $\mathscr{A}$ preserves measurability in much more general circumstances than were considered in the preceding section.) We do not intend to reproduce this proof here in complete generality; nor do we intend even in the special cases we do consider to present in detail what can already be found in [18] and [7]. Our aim is merely to outline the treatment of category, sketching the proofs, and presenting only those technical details that will be needed in the next section.

So let $\mathscr{X}$ be a topological space satisfying the Baire Category Theorem and admitting a countable π-*basis* $\mathscr{U}$, i.e. a countable family of open sets containing $\mathscr{X}$ and excluding $\varnothing$, such that every nonempty open set has some element of $\mathscr{U}$ as a subset. Then $A\subseteq\mathscr{X}$ is almost open iff for every $U\in\mathscr{U}$, either $A\cap U$ is meager, or for some $V\in\mathscr{U}$ with $V\subseteq U$, $A\cap V$ is comeager in the relative topology on V. We reserve the letters U, V, W, X, Y for elements of $\mathscr{U}$. Let A be a nested $Q(\omega)$-indexed system of almost open sets.

(a) CATEGOR FORMULA (Vaught). *$\mathscr{A}(A)$ is almost open, and for any $W\in\mathscr{U}$, $\mathscr{A}(A)\cap W$ is nonmeager iff*:

(*) $\exists k_0\in\omega\ \exists U_0\subseteq W\ \forall V_0\subseteq U_0\ \exists k_1\in\omega\ \exists U_1\subseteq V_0\ \forall V_1\subseteq U_1\ \ldots$ $\forall n[A((k_0, k_1, \ldots)|n)\cap V_{n-1}$ *is nonmeager*].

Proof. In view of the determinateness of the game (*), it will suffice to show that if CON (resp. PRO) has a winning strategy, then $\mathscr{A}(A)$ is meager in W (resp. comeager in some $U_0\subseteq W$). As the two cases are virtually symmetrical, we consider the case where CON has a winning strategy ψ. ψ remains a winning strategy if the $A(s)$ are replaced by open $A'(s)$ with $A(s)\triangle A'(s)$ meager, so we may assume the $A(s)$ are already open. By the Baire Category Theorem, nonmeager and nonempty coincide for open sets, so ψ amounts to a winning strategy for PRO in the game:

(**) $\forall k_0\ \forall U_0\subseteq W\ \exists V_0\subseteq U_0\ \forall k_1\ \forall U_1\subseteq V_0\ \exists V_1\subseteq U_1\ \ldots\ \exists n[A((k_0, k_1, \ldots)|n)\cap$ $\cap V_{n-1}=\varnothing]$.

The first step in the proof is to trade ψ for a winning strategy φ for PRO in the game:

(***) $\forall X_0\subseteq W\ \exists Y_0\subseteq X_0\ \forall X_1\subseteq Y_0\ \exists Y_1\subseteq X_1\ \ldots\ \bigcap_{n\in\omega}Y_n\cap\mathscr{A}(A)=\varnothing$.

As we define φ we associate to each partial play agreeing with φ a partial play agreeing with ψ. Suppose that φ has been partially defined and that $s=(X_0, \ldots, Y_{n-1})$ agrees with φ as defined so far, and that $X\subseteq Y_{n-1}$, and finally that $\varphi(X_0, \ldots, X_{n-1}, X)$ has not yet been defined. Let the sequence with code number n be $(k_0, \ldots, k_{m-1}, k)$. (We assume $m>0$, the opposite case requiring slight verbal modifications.) Suppose as induction hypothesis that the partial play t agreeing with ψ that has been associated to $s|2n^*$, where n^* is the code number of $(k_0, \ldots, k_{m-1})$, has the form

$$(k_0, U_0, V_0, \ldots, k_{m-1}, U_{m-1}, V_{m-1})$$

with $V_{m-1}=Y_{n^*}$, so that $X\subseteq V_{m-1}$. As $\varphi(X_0, \ldots, X_{n-1}, X)$ choose

$$Y=\psi(k_0, U_0, \ldots, k_{m-1}, U_{m-1}, k, X).$$

To the partial play $s\oplus X\oplus Y$ associate $t\oplus k\oplus X\oplus Y$, which is of the form required to keep the induction going. We suppress the details of the proof that φ is indeed a winning strategy. (Briefly, to each complete play $\sigma=(X_0, Y_0, \ldots)$ agreeing with φ and each $\xi=(k_0, \ldots)\in\omega^\omega$ corresponds a complete play $\tau=(k_0, U_0, V_0, \ldots)$ agreeing with ψ and having $\bigcap_n V_n=\bigcap_n Y_n$. Viz. $\tau|3m$ is the partial play associated as in the above construction to $\sigma|2\#(\xi|m)$.)

The second step is to use φ to build a set S of finite sequences. Fix an enumeration $(W(i):\ i\in\omega)$ of $\mathscr{U}$. Let $S_0=\{(\,)\}$. Let $s\oplus W(i)\oplus Y\in S_{n+1}$ iff $s\in S_n$ and (i) for all $j<i$ and $s\oplus W(j)\oplus Z\in S_{n+1}$, $X\cap Z=\varnothing$, and (ii) $s\oplus W(i)\oplus Y$ agrees with φ. Let S be the union of the S_n for $n>0$. We remark for future reference that the passage from ψ to S was highly constructive, and could be effected by a Borel function from the space of strategies to the power set of the relevant set of finite sequences.

The third step is to introduce a G_δ set C. For $n>0$, let C_n be the union of the Y for $(X, \ldots, Y)\in S_n$; let C be the intersection of the C_n. We suppress the details of the proof that C is comeager in W and disjoint from $\mathscr{A}(A)$ as required to complete the proof. (Briefly, for comeagerness, it is readily verified that for each $(X, \ldots, Y) \in S_n$, the complement of the union of the Y' for $(X, \ldots, Y, X', Y') \in S_{n+1}$ is rare in Y. As for the other property, the disjointness condition (i) in the definition of S_{n+1} guarantees that for each $x \in C$ there is a *unique* sequence $(X_0, Y_0, \ldots)$ such that for each n $x \in Y_n$ and the initial segment of this sequence of length $2n$ belongs to S_n. Then the agreement condition (ii) implies this sequence is a complete play agreeing with φ.)

We make the set of nonempty closed subspaces of $\mathscr{X}$ into a topological space $\mathscr{K}(\mathscr{X})$ by taking as a subbasis the sets $A^+ = \{K\colon A \cap K \neq \emptyset\}$ for $A \subseteq \mathscr{X}$ open. Even when $\mathscr{X}$ is standard $\mathscr{K}(\mathscr{X})$ need not be, but it at least has what is known as a *standard Borel structure*, as is shown in [3], where several other topologies leading to the same Borel structure are considered. $\mathscr{X}$ can be embedded in $\mathscr{K}(\mathscr{X})$ by identifying x and $\{x\}$. Call $A \subseteq \mathscr{X}$ *strongly* almost open iff $A \cap K$ is almost open in the relative topology on K for every $K \in \mathscr{K}(\mathscr{X})$. For such A and for $U \in \mathscr{U}$ define $A^{+U} = \{K \in \mathscr{K}(\mathscr{X})\colon (A \cap U) \cap K$ is nonmeager in the relative topology on $K\}$. For $U = \mathscr{X}$ we simply write A^+, and if A is open this agrees with the earlier definition of A^+ by the Baire Category Theorem. Much as in the preceding section it can be argued that A^{+U} is Borel whenever A is. For applications see [18] and [1]. We also have:

(b) CATEGORY DUALITY THEOREM (Vaught). *A^{+U} is a C-set in $\mathscr{K}(\mathscr{X})$ whenever A is a C-set in $\mathscr{X}$.*

§ 8. Selection theorems. In this section we turn at last to selection theory and prove the result promised in the Abstract. Let $\mathscr{X}$ and $\mathscr{Y}$ be standard spaces. Fix a basis $(W(i)\colon i \in \omega)$ for $\mathscr{X}$ containing $\mathscr{X}$ and excluding $\emptyset$, and fix a bounded complete metric ν for $\mathscr{X}$. Let $\mathscr{H}$ be a σ-field of subsets of $\mathscr{Y}$.

A multifunction from $\mathscr{Y}$ to $\mathscr{X}$ is simply a function F from $\mathscr{Y}$ to $\mathscr{P}(\mathscr{X})$. Let $\operatorname{Dom} F = \{y\colon F(y) \neq \emptyset\}$. A *selector* for F is a function $f\colon \operatorname{Dom} F \to \mathscr{X}$ satisfying $f(y) \in F(y)$ for all y. A *section* is a selector satisfying $f(y) = f(z)$ whenever $F(y) = F(z)$. Of course selectors and sections always exist by the Axiom of Choice; but to obtain measurable ones, we need hypotheses on F.

Values: Sometimes we assume the values $F(y)$ of F are compact or closed or $(F_\sigma \cap G_\delta)$ of G_δ. Measurability: For open $U \subseteq \mathscr{X}$ let

$$F^{-1}[U] = \{y\colon F(y) \cap U \neq \emptyset\}.$$

Call F *$\mathscr{H}$-measurable* if $F^{-1}[U] \in \mathscr{H}$ for all open U. Graphs: Sometimes we impose conditions on $\operatorname{Graph} F = \{(y, x)\colon x \in F(y)\}$. It always makes sense to ask whether this set is Borel or analytic or a C-set. If $\mathscr{H}$ is a uniform class it also makes sense to ask whether it is an $\mathscr{H}$-set. This is always the case when F is closed-valued and $\mathscr{H}$-measurable, for then:

$$\operatorname{Graph} F = \bigcap_{i \in \omega} \{[F^{-1}[W(i)] \times \mathscr{X}] \cup [\mathscr{Y} \times (\mathscr{X} - W(i))]\}.$$

We briefly review some known selection results that apply to C-sets. (a) below has already been alluded to. (b) admits of a somewhat sharper formulation than that quoted here. (c) deserves to be better known.

(a) ANALYTIC UNIFORMIZATION THEOREM (Yankov/von Neumann, cf. [20, § 5]). *A multifunction G whose graph is analytic admits an analytically measurable selector g.*

(b) FUNDAMENTAL SELECTION THEOREM (Kuratowski & Ryll-Nardzewski/Castaing, cf. [20, § 4]). *A closed-valued $\mathscr{H}$-measurable multifunction admits an $\mathscr{H}$-measurable section.*

(c) KEY SELECTION THEOREM (Dellachérie). *There exists a Borel $h\colon \mathscr{K}(\mathscr{X}) \to \mathscr{X}$ satisfying $h(K) \in K$ for all K.*

Proofs. (c) To each K associate the $\eta \in \omega^\omega$ such that $W(\eta(0)) = \mathscr{X}$ and for all n $\eta(n+1)$ is the *least* i such that (i) $W(i) \cap K \neq \emptyset$, (ii) closure $W(i) \subseteq W(\eta(n))$, (iii) ν-diameter $W(i) \leqslant \frac{1}{2}\nu$-diameter $W(\eta(n))$. Define h by $\bigcap_n W(\eta(n)) = \{h(K)\}$.

(b) To any multifunction F associate the *derived* function $\partial F\colon \operatorname{Dom} F \to \mathscr{K}(\mathscr{X})$ sending y to closure $F(y)$. F is $\mathscr{H}$-measurable iff ∂F is. If F is closed-valued and $\mathscr{H}$-measurable, we obtain an $\mathscr{H}$-measurable section f by composing ∂F and the h of (c).

(a) Represent Graph G as $\mathscr{A}(A)$ with each $A(s)$ closed. Apply (b) to $F(y) = \{(x, \xi) \in \mathscr{X} \times \omega^\omega\colon \forall n \in \omega\ x \in A(\xi|n)\}$, $\mathscr{H} = \sigma$-field generated by analytic sets, obtaining a section f. Set $g(y) = x$ where $f(y) = (x, \xi)$. ■

The following is due to Kallman & Mauldin for the case $\mathscr{H}$ = Borel, cf. [4].

(d) "SOFT" SELECTION THEOREM. *Assume $\mathscr{H}$ is a uniform family. An $(F_\sigma \cap G_\delta)$-valued $\mathscr{H}$-measurable multifunction F whose graph is an $\mathscr{H}$-set admits an $\mathscr{H}$-measurable section f.*

Proof. Let F_i be the multifunction from $\mathscr{Y}$ to the space $W(i)$ defined by $F_i(y) = F(y) \cap W(i)$, and $\partial F_i\colon \operatorname{Dom} F_i = F^{-1}[W(i)] \to \mathscr{K}(W(i))$ the derived function as in (b). Let $h_i\colon \mathscr{K}(W(i)) \to W(i)$ be as in (c), f_i the composition $h_i \partial F_i$. Let $E_i = \{y \in F^{-1}[W(i)]\colon f_i(y) \in F(y)\}$, $D_i = E_i - \bigcup_{j<i} E_j$.

Then $D_i \in \mathscr{H}$ since $E_i = (\text{identity} \times f_i)^{-1}[\operatorname{Graph} F]$. By elementary topology, for any nonempty Z which is both F_σ and G_δ, there is an i such that $Z \cap W(i)$ is nonempty and relatively closed in $W(i)$. It follows $\operatorname{Dom} F = \bigcup_i D_i$. So set $f(y) = f_i(y)$ for the i with $y \in D_i$. ■

(e) CATEGORY SELECTION THEOREM. *For each C-set $A \subseteq \mathscr{X}$ there exists a C-measurable $h\colon A^+ \to \mathscr{X}$ satisfying $h(K) \in A \cap K$ for all $K \in A^+$.*

Proof. Recall that the Category duality Theorem says that $A^+ = \{K \in \mathscr{K}(\mathscr{X})\colon A \cap K$ is nonmeager in the relative topology on $K\}$ is a C-set.

We give the proof only in the case A co-analytic, the general case calling for extensions of the machinery of §§ 4 and 7 above that, though conceptually straightforward, are notationally forbidding. So let $A = \mathscr{X} - \mathscr{A}(A)$ where A is a nested indexed system of open sets. Reserve U, V, W to range over our chosen basis for $\mathscr{X}$.

Let $K \in A^+$ be given. Then there is a W meeting K such that $A \cap W \cap K$ is comeager in $W \cap K$. By the Category Formula we then have the following, where the U's and V's are required to meet K:

$$(*)\quad \exists W\ \forall k_0\ \forall U_0 \subseteq W\ \exists V_0 \subseteq U_0\ \forall k_1\ \forall U_1 \subseteq V_0\ \exists V_1 \subseteq U_1 \ldots \quad \exists n [A((k_0, k_1, \ldots)|n) \cap V_{n-1} = \emptyset].$$

By the Canonical strategy Theorem there is a C-measurable h_0 such that $h_0(K)$ is a winning strategy ψ for PRO in $(*)$.

In the proof of the Category Formula we used such a ψ to construct first another strategy φ, then a set S of finite sequences, then a G_δ set C contained in $K \cap A$; and we remarked that there was a Borel function h_1 that would take us from ψ to S.

An argument like the proof of (c) above produces a Borel function h_2 which applied to S yields an element of C. Hence it suffices to set $h = h_2 h_1 h_0$. ■

The following is due to S. M. Srivastava for the case of Borel sets. Neither his proof [17] nor that of Miller [10] extends to C-sets. Our proof extends (clumsily and with difficulty) to Borel sets.

(f) "HARD" SELECTION THEOREM. *A G_δ-valued C-measurable multifunction F whose graph is a C-set admits a C-measurable selector f.*

Proof. Define a C-measurable $F'\colon \mathscr{Y} \to \mathscr{K}(\mathscr{Y} \times \mathscr{X})$ by $F'(y) = \{y\} \times$ closure $F(y)$. Let h be as in (e) for $A = \operatorname{Graph} F$. By elementary topology, a G_δ set is comeager in its closure, hence $g = hF'$ is defined on *all* y, and it suffices to let $f(y) = x$ where $g(y) = (y, x)$. ■

Counterexamples exist to show each of the three hypotheses of (f) is indispensible. We do not know whether one could hope for a *section* in (f).

We close by citing a result from [2]. A *section* for an equivalence relation E on $\mathscr{X}$ is a section f for the multifunction $F(x) =$ equivalence class of x. The fixed points of f form a set containing exactly one representative of each class. E is countably *$\mathscr{H}$-generated* if there is a countable $\mathscr{H}_0 \subseteq \mathscr{H}$ such that two points are equivalent just when they belong to all the same elements of $\mathscr{H}_0$. Srivastava and Miller have each observed that an analytic equivalence whose classes are G_δ sets is countably C-generated. We have:

(g) PARTITION SELECTION THEOREM. *A countably C-generated equivalence relation admits a C-measurable section.*

The proof requires special facts about C-sets. (g) is false for Borel sets.

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