Classical hierarchies from a modern standpoint

Part II. R-sets

by

John P. Burgess * (Princeton, N. J.)

Abstract. The Borel sets and the C-sets form but the bottom two levels of the hierarchy of R-sets. It is shown how the properties established for these bottom levels in Part I can be lifted to all levels of the hierarchy. It is further shown that the Borel sets, C-sets, and R-sets coincide with the bottom three levels of Vaught's hierarchy of Borel-game sets.

Chapter D. Survey of R-sets

§ 9. Basics

(a) Introductory. From time to time analysts, topologists, logicians, and probabilists have found it convenient or illuminating to introduce certain families of Lebesgue measurable sets of reals properly including the Borel sets. In Part I of this series we surveyed the useful properties of one such family, the C-sets. In the present Part II we turn to a larger family, the R-sets of Kolmogorov and the Russian school [22], [23], [24], [25], [26], [27], showing how the game-theoretic methods applied to C-sets in Part I lift with a minimum of fuss to the R-sets. Here and in Part III to come we will offer two new characterizations of the R-sets, quite different in appearance from the original definition; the existence of such diverse characterizations suggest that the R-sets form a very natural class.

Notation and terminology of Part I will be retained. The author remains indebted to the persons thanked in Part I, especially Professor Vaught.

(b) R-transform, R-operations, R-hierarchy. We begin with a quick review of material mostly to be found in the works cited above. First some generalities about *set operations* in the sense of § 2 (c). The *composition* $\Gamma \circ \Gamma'$ of two operations Γ and Γ' with index sets I and I' is the operation

$$\Gamma''(A) = \Gamma\big((\Gamma'(A(i, i'): i' \in I'): i \in I)\big).$$

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* Research partially supported by U.S. National Science Foundations grant MCS-77-03590.

The *conjunction* $\bigwedge_{k\in K}\Gamma_k$ of a family $(\Gamma_k\colon k\in K)$ of operations with index sets I_k and truth tables B_k is the operation Γ^* with index set I^* and truth table B^*, where:

$$I^* = \bigcup_{k\in K}\{k\}\times I_k\,,$$
$$B^* = \bigcap_{k\in K}\{J\subseteq I^*\colon \{i\in I_k\colon (k,i)\in J\}\in B_k\}\,.$$

The operation Γ with index set I and truth table B is *nontrivial* iff B is a nonempty proper subset of $\mathscr{P}(I)$, and *positive* iff $J\in B$ and $J\subseteq K$ always imply $K\in B$. (All the specific operations we have considered or will be considering are nontrivial and positive.) For nontrivial positive Γ we define the *R-transform* $R\Gamma$ of Γ as the operation with index set $Q(I)$ (= finite sequences from I) and truth table:

(*) $$\{S\subseteq Q(I)\colon \exists T\subseteq s[(\ )\in T\ \&\ \forall s\in T\{i\colon s\oplus i\in T\}\in B]\}\,.$$

E.g. $R(\text{join}) = \mathscr{A}$, an example motivating the general concept.

$R\Gamma$ has a useful game analysis: We claim $x\in R\Gamma(A)$ iff:

(**) $$\exists J_0\in B\ \forall i_0\in J_0\ \exists J_1\in B\ \forall i_1\in J_1\ \ldots\ \forall n[x\in A((i_0, i_1, \ldots)|n)]\,.$$

Indeed, let $S = \{s\in Q(I)\colon x\in A(s)\}$. If $T\subseteq S$ is as required by (*) for x to belong to $R\Gamma(A)$, then we get a winning strategy φ for the $\exists$-player PRO in the game (**) by setting $\varphi(s) = \{i\colon s\oplus i\in T\}$. Conversely, if φ is a winning strategy, the set of $(i_0, \ldots, i_n)$ for which there exists a partial play $(J_0, i_0, \ldots, J_n, i_n)$ agreeing with φ constitutes a suitable T.

$R\Gamma$ has a useful inductive analysis:

$$A^0(s) = \bigcap_{t\lhd s} A(t)\,,$$
$$A^{\beta+1}(s) = \bigcup_{J\in B}\ \bigcap_{i\in J} A^\beta(s\oplus i) = \Gamma\big((A^\beta(s\oplus i)\colon i\in I)\big)\,,$$
$$A^\alpha(s) = \bigcap_{\beta<\alpha} A^\beta(s)\ \text{at limits}\ \alpha\leqslant\Omega\,.$$

The inductive analysis of games in § 4 applies *mutatis mutandis* to (**) to show that the game associated with (**) is determined and that $R\Gamma(A) = A^\Omega((\ ))$.

R has some useful reducibility properties: It is a pleasant exercise to verify the following (or they can be looked up in [23]):

(i) if Γ is reducible to Γ', then $R\Gamma$ is reducible to $R\Gamma'$,
(ii) Γ is reducible to $R\Gamma$,
(iii) meet is reducible to $R\Gamma$,
(iv) $RR\Gamma$ is reducible to $R\Gamma$,
(v) $R\Gamma\circ R\Gamma$ is reducible to $R\Gamma$.

It is amusing to give a "soft" proof of the reducibility of $\mathscr{G}$ to $\mathscr{A}$ using these facts: First note that $\mathscr{G} = R(\text{join}\circ\text{meet})$. Now join and meet are reducible to $\mathscr{A}$ by (ii) and (iii). Hence their composition is reducible to $\mathscr{A}\circ\mathscr{A}$ and so to $\mathscr{A}$ by (v). Hence $\mathscr{G}$ is reducible to $R\mathscr{A}$ by (i), and so to $\mathscr{A}$ by (iv).

We now define the *R-operations* Φ_α for $\alpha<\Omega$:

$$\Phi_0 = \text{meet}\,,$$
$$\Phi_{\beta+1} = R(\text{co-}\Phi_\beta)\,,$$
$$\Phi_\alpha = \bigwedge_{\beta<\alpha}\Phi_\beta\ \text{at limits}\,.$$

E.g. $\Phi_1 = \mathscr{A}$. The game analysis of Φ_2 boils down to the statement that $x\in\Phi_2(A)$ iff:

$$\forall\xi_0\in\omega^\omega\ \exists n_0\in\omega\ \forall\xi_1\ \exists n_1\ \ldots\ \forall k[x\in A((\xi_0|n_0, \xi_1|n_1, \ldots)|k)]\,.$$

Then by determinateness, $x\in\text{co-}\Phi_2(A)$ iff:

$$\exists\xi_0\ \forall n_0\ \exists\xi_1\ \forall n_1\ \ldots\ \exists k[x\in A((\xi_0|n_0, \xi_1|n_1, \ldots)|k)]\,.$$

And then $x\in\Phi_3(A)$ iff:

$$\exists\xi_{00}\ \forall n_{00}\ \exists\xi_{01}\ \ldots\ \forall n_{01}\ \ldots\ \exists k_0\ \exists\xi_{10}\ \forall n_{10}\ \exists\xi_{11}\ \forall n_{11}\ \ldots\ \exists k_1\ \ldots\ \exists k_2\ \ldots$$
$$\forall r\big[x\in A\big(((\xi_{00}|n_{00}, \xi_{01}|n_{01}, \ldots)|k_0, (\xi_{10}|n_{10}, \xi_{11}|n_{11}, \ldots)|k_1, \ldots)|r\big)\big]$$

where for the first time we are considering a game with a sequence of moves of length $>\omega$. And so on for all the Φ_i, $i<\omega$. The game associated with Φ_ω begins with CON choosing an $i<\omega$, after which the players go through a game of the sort associated with Φ_i. The games for $\alpha>\omega$ are more easily imagined than described. Inductively, Φ_β is reducible to Φ_α whenever $\beta<\alpha$.

We finally introduce the hierarchy of *R-sets*:

$$\mathscr{R}^0 = \text{Borel sets}\,,$$
$$\mathscr{B}_0^{\delta+1} = \mathscr{R}^\delta\,,$$
$$\mathscr{A}_{\beta+1}^{\delta+1} = \text{sets obtainable by } \Phi_{\delta+1} \text{ from sets in } \mathscr{B}_\beta^{\delta+1}\,,$$
$$\mathscr{C}_{\beta+1}^{\delta+1} = \text{complements of sets in } \mathscr{A}_{\beta+1}^{\delta+1}\,,$$
$$\mathscr{D}_{\beta+1}^{\delta+1} = \mathscr{A}_{\beta+1}^{\delta+1}\cap\mathscr{C}_{\beta+1}^{\delta+1}\,,$$
$$\mathscr{B}_{\beta+1}^{\delta+1} = \text{smallest } \sigma\text{-field containing } \mathscr{A}_{\beta+1}^{\delta+1} \text{ and stable under } \Phi_\delta\,,$$
$$\mathscr{B}_\alpha^{\delta+1} = \text{smallest } \sigma\text{-field containing all } \mathscr{B}_\beta^{\delta+1} \text{ for } \beta<\alpha, \text{ and stable under } \Phi_\delta, \text{ at limits } \alpha<\Omega\,,$$
$$\mathscr{R}^{\delta+1} = \bigcup_{\alpha<\Omega}\mathscr{B}_\alpha^{\delta+1}\,,$$
$$\mathscr{R}^\gamma = \text{smallest uniform family containing all } \mathscr{R}^\delta \text{ for } \delta<\gamma \text{ at limits } \gamma<\Omega\,,$$
$$R\text{-sets} = \bigcup_{\gamma<\Omega}\mathscr{R}^\gamma\,.$$

E.g. $\mathscr{R}^1 = C$-sets (and for superscript 1 the above definitions agree with those in § 3). The families $\mathscr{A}_1^2$, $\mathscr{C}_1^2$, $\mathscr{D}_1^2$ have no accepted names, but we will call them *primitive*, *co-primitive*, and *bi-primitive* R-sets.

Much, of course, was known about R-sets to the Russian authors cited above.

Noteworthy are the fact that for standard spaces the $\mathscr{R}^\gamma$ are all distinct, and the fact that for arbitrary spaces the R-transform preserves the property of preserving the property of almost openness (= Baire property); hence the R-operations preserve, and the R-sets enjoy, this property. We would like to extend such modern results as the Duality Theorems of §§ 6, 7 and the "Hard" Selection Theorem of § 8 to R-sets.

§ 10. From C-sets to R-sets. Our proofs in Part I were chosen so as to generalize as easily as possible from $\mathscr{R}^1 = C$-sets to all $\mathscr{R}^\gamma$. We indicate in this section the generalization to $\mathscr{R}^2$, which is typical.

In addition to $\Phi = \Phi_2$ we make use of three auxiliary operations,

$$\Psi = R((\text{co-}\mathscr{A})\circ\mathscr{A}), \qquad \Theta = R(\text{co-}\mathscr{G}), \qquad \Xi = R((\text{co-}\mathscr{G})\circ\mathscr{G}).$$

These may be proved interreducible with Φ using the reducibility properties (i)–(v) of R enumerated in the preceding section. They act on systems indexed by $Q^*(Q(\omega))$, $Q(Q^*(\omega))$, and $Q^*(Q^*(\omega))$ respectively:

(i) $x \in \Psi(A)$ iff:

$$\forall \xi_0 \in \omega^\omega\, \exists n_0 \in \omega\, \exists \zeta_0 \in \omega^\omega\, \forall m_0 \in \omega\, \forall \xi_1\, \exists n_1\, \exists \zeta_1\, \forall m_1 \ldots$$
$$\forall k\, [x \in A((\xi_0|n_0, \zeta_0|m_0, \ldots)|2k)],$$

(ii) $x \in \Theta(A)$ iff:

$$\forall a_{00} \in \omega\, \exists b_{00} \in \omega\, \forall a_{01}\, \exists b_{01} \ldots \exists n_0\, \forall a_{10}\, \exists b_{10}\, \exists a_{11}\, \exists b_{11} \ldots \exists n_1 \ldots \exists n_2 \ldots$$
$$\forall k \big[x \in A\big(((a_{00}, b_{00}, \ldots)|2n_0, (a_{10}, b_{10} \ldots)|2n_1, \ldots)|k\big)\big],$$

(iii) $x \in \Xi(A)$ iff:

$$\forall a_{00}\, \exists b_{00}\, \forall a_{01}\, \exists b_{01} \ldots \exists n_0\, \exists c_{00}\, \forall d_{00}\, \exists c_{01}\, \forall d_{01} \ldots \forall m_0$$
$$\forall a_{10}\, \exists b_{10}\, \forall a_{11}\, \exists b_{11} \ldots \exists n_1\, \exists c_{10}\, \forall d_{10}\, \exists c_{11}\, \forall d_{11} \ldots \forall m_1 \ldots$$
$$\exists n_2 \ldots \forall m_2 \ldots \forall k \big[x \in A\big(((a_{00}, b_{00}, \ldots)|2n_0, (c_{00}, d_{00}, \ldots)|2m_0, \ldots)|2k\big)\big].$$

It is a pleasant task to go through Part I carrying over all the results there from $\mathscr{R}^1$ to $\mathscr{R}^2$ with the aid of these auxiliaries. Let us sketch how this is to be done:

Hierarchy Theorems. The proof of § 5(a) (resp. § 5(b)) carries over virtually unchanged with Φ (resp. Ψ) in place of $\mathscr{A}$ (resp. $\mathscr{G}$).

Regularity Properties. In § 7, to prove the Category Formula and Duality Theorem we expressed the set there called $(\mathscr{A}(A))^+$ in terms of $\mathscr{G}$ applied to simpler sets. Similarly, one can express $(\Phi(A))^+$ in terms of Θ applied to simpler sets. A version of the resulting Category Formula has been written out in [21, § 2], and the proof there gives a hint also how to proceed in the measure case.

Selection Theorems. The "soft" result of § 8 is valid as it stands for $\mathscr{R}^2$. For the "hard" result we need a Canonical Strategy Theorem for the game connected with Θ. To get this, consider the inductive analysis of $R\Gamma$ of the preceding section,

applied to $\Gamma = \text{co-}\mathscr{G}$. Introduce an associated notion of *rank* as in § 4. Unpacking the definitions, x-rank$(\sigma) = \Omega$ iff:

$$(*) \qquad \forall a_0 \exists b_0 \forall a_1 \exists b_1 \ldots \exists n\, [x\text{-rank}(\sigma \oplus (a_0, b_0, \ldots)|2n) \geqslant x\text{-rank}(\sigma)].$$

This is the sort of game considered in § 4, and the canonical strategy for it as there defined will here be called the *canonical σ-substrategy*. Now in the game (ii), if x-rank$(( )) = \Omega$, PRO's *canonical strategy* is defined thus: If play so far has produced $\sigma = (s_0, \ldots, s_{k-1})$ with x-rank$(\sigma) = \Omega$, then choose $b_{k0}, b_{k1}, b_{k2}, \ldots$ according to the canonical σ-substrategy, and as n_k choose the least n with

$$x\text{-rank}(\sigma \oplus (a_{k0}, b_{k0}, \ldots)|2n) = \Omega.$$

PRO by playing thus keeps the ranks up and wins the game (ii). PRO's canonical strategy can be identified with, say, the set of pairs (σ, φ) such that x-rank$(\sigma) = \Omega$ and φ is the canonical σ-substrategy, and hence with a point in a standard space. Canonical strategies for CON are similarly defined. To show that the function associating to x the canonical strategy for PRO (or CON as the case may be) is $\mathscr{R}^2$-measurable, we need a Comparison Lemma. About this we will only say that $\{x\colon x\text{-rank}(\sigma) \leqslant x\text{-rank}(\tau)\}$ can be obtained by Ξ applied to sets in the field generated by the $A(\sigma)$. Cf. in this connection [27].

This rough outline could hardly pretend to be a fully rigorous proof. To execute in detail the project of extending each result in Part I to all levels of the R-hierarchy would take dozens of pages. We hope we have said enough to equip our hardier readers to carry out this task for themselves if they desire.

Chapter E. Borel-game characterization

§ 11. Preliminaries. In recent years Vaught and his students have developed a general theory of operations defined by games of length ω. (Cf. item [19] of the Bibliography of Part I.) These ideas lead to a new characterization of the R-sets.

The following format, though slightly artificial, proves convenient: Let $\mathscr{X}$ be any space, $\mathscr{Y} = \{\xi \in \omega^\omega\colon \forall n\, \xi(3n) < 2\}$. For A a $Q^*(\omega)$-indexed system of subsets of $\mathscr{X}$, define the *characterizer* $\chi_A\colon \mathscr{X} \times \omega^\omega \to \mathscr{Y}$ by $\chi_A(x, \xi) = \zeta$ where $\zeta(3n)$ is 1 or 0 according as $x \in A(\xi|2n)$ or not, and $\zeta(3n+i+1) = \xi(2n+i)$ for $i = 0$ or 1. For $B \subseteq \mathscr{Y}$ the *game operation* Γ with *target* B is defined by letting $x \in \Gamma(A)$ iff:

$$\exists c_0 \in \omega\; \forall d_0 \in \omega\, \exists c_1 \forall d_1 \ldots \mathscr{X}_A(x, (c_0, d_0, \ldots)) \in B.$$

Intuitively, membership in $\Gamma(A)$ corresponds to the existence of a winning strategy for PRO in the game where the players alternately choose the terms of a sequence $\xi = (c_0, d_0, \ldots)$ and PRO wins iff a certain condition involving ξ and $\{n\colon x \in A(\xi|2n)\}$ is fulfilled. E.g. $\mathscr{G}$ is the game operation with target $\{\xi\colon \forall n\, \xi(3n) = 1\}$.

If $\mathscr{H}$ is a class of subsets of $\mathscr{Y}$, the *$\mathscr{H}$-game operations* are those with targets in $\mathscr{H}$, and the *$\mathscr{H}$-game subsets* of $\mathscr{X}$ those obtainable by $\mathscr{H}$-game operations applied to indexed systems of open sets. (Vaught and Schilling have shown that Borel-game operations preserve and Borel-game sets enjoy the property of almost openness.) In general, for $\mathscr{X} = \mathscr{Y}$, passing from $\mathscr{H}$ to the $\mathscr{H}$-game sets produces a considerable enlargement. The magnification process $\mathscr{H} \to \mathscr{H}$-game applied to the lowest levels of the Borel hierarchy produces some familiar classes.

Characterization Theorem.

(a) *clopen-game = Borel,*

(b) *closed-game = analytic, open-game = co-analytic,*

(c) $(F_\sigma \cap G_\delta)$*-game = C-sets,*

(d) G_δ*-game = primitive R-sets,* F_σ*-game = co-primitive R-sets,*

(e) $(G_{\delta\sigma} \cap F_{\sigma\delta})$*-game = R-sets.*

Proofs. (b): $\mathscr{A}$ is reducible to $\mathscr{G}$ which is clearly a closed-game operation, so analytic $\subseteq$ closed-game. The proof that $\mathscr{G}$ is reducible to $\mathscr{A}$ generalizes to handle any closed-game operation, proving the opposite inclusion.

(a): The inclusion clopen-game $\subseteq$ Borel follows from (b). To get the opposite inclusion it suffices to note that if each E_{ij} is obtainable from open sets by the clopen-game operation with target B_{ij}, then $\bigcup_i \bigcap_j E_{ij}$ is obtainable by the clopen-game operation with target $B = \{i \oplus j \oplus k \oplus \xi \colon \xi \in B_{ij} \;\&\; k < 2\}$. There is in fact a correspondence between the levels of the Borel hierarchy and the levels of Kalmar's hierarchy on clopen sets.

(c): Will be omitted.

(e) and (d): There is in fact a level-by-level correspondence between the R-hierarchy and the classical difference hierarchy on $(G_{\delta\sigma} \cap G_{\sigma\delta})$. (d) states the correspondence at the bottom level; at the next we have:

(f) *(difference of two* G_δ*'s)-game* $= \mathscr{C}_1^3$.

We will offer a proof of (f), illustrating the general principles involved in (e) while avoiding notational nightmares. A proof of (d) can be obtained as a direct simplification.

Towards proving (f), note that the sets in $\mathscr{C}_1^3$ are obtainable from open sets by application of co-$(R(\text{co-}(R(\text{co-}\mathscr{G}))))$, and that the *general form* of the game associated with this operation is:

$$(*) \qquad \begin{aligned} &\forall a_{000} \exists b_{000} \forall a_{001} \exists b_{001} \forall a_{002} \exists b_{002} \ldots \exists k_{00} \\ &\forall a_{010} \exists b_{010} \forall a_{011} \exists b_{011} \forall a_{012} \exists b_{012} \ldots \exists k_{01} \ldots \exists k_{02} \ldots \forall j_0 \\ &\forall a_{100} \exists b_{100} \forall a_{101} \exists b_{101} \forall a_{102} \exists b_{102} \ldots \exists k_{10} \\ &\forall a_{110} \exists b_{110} \forall a_{111} \exists b_{111} \forall a_{112} \exists b_{112} \ldots \exists k_{11} \ldots \exists k_{12} \ldots \forall j_1 \ldots \forall j_2 \ldots \\ &\exists i (\sigma_0, \sigma_1, \ldots) | i \in W \text{ where} \\ &\sigma_i = (s_{10}, s_{i1}, \ldots) | j_i \text{ and } s_{ij} = (a_{ij0}, b_{ij0}, \ldots) | 2k_{ij}. \end{aligned}$$

We have already observed that games associated with R are determined. In $(*)$ we refer to the (ω^2+1)-sequence of moves from CON's choice of a_{i00} to CON's choice of j_i as the *i-subgame*; we refer to the $(\omega+1)$-sequence from CON's choice of a_{ij0} to PRO's choice of k_{ij} as the *i, j-subsubgame*.

Note also that a G_δ subset of ω^ω can be represented as $\{\xi \colon \forall j \exists k\; \xi | k \in M(j, k)\}$ where the $M(j, k) \subseteq Q(\omega)$ are such that $s \in M(j, k)$ and $s \lhd t$ and $k \leqslant k'$ imply $t \in M(j, k')$. Thus the game associated with a (difference of two G_δ's)-game operation has the *general form*:

$$(**) \qquad \exists c_0 \forall d_0 \exists c_1 \forall d_0 \ldots \text{ for } \xi = (c_0, d_0, \ldots) \text{ we have:}$$
$$\forall j \exists k\; \xi | k \in M(j, k) \;\&\; \neg \forall i \exists j\; \xi | j \in N(i, j)$$

where the $M(j, k)$ have the property just mentioned, and similarly for the $N(i, j)$.

To prove (f) we will show more generally that to the game G^* given by $(*)$ we can associate a game G' of the same form as $(**)$, and to the game G^{**} given by $(**)$ a game G'' of the same form as $(*)$, in such a way that whichever player has a winning strategy in G^* (resp. G'') has one in G' (resp. G^{**}). This incidentally proves the determinateness of games of form $(**)$ (a result originally due to Morton Davis).

§ 12. The proof.

(a) Auxiliary games. The G'' promised at the end of the preceding section is formally defined thus:

$$\begin{aligned} &\exists c_{000} \forall d_{000} \exists c_{001} \forall d_{001} \ldots \exists k_{00} \\ &\exists c_{010} \forall d_{010} \exists c_{011} \forall d_{011} \ldots \exists k_{01} \ldots \exists k_{02} \ldots \forall j_0 \\ &\exists c_{100} \forall d_{100} \exists c_{101} \forall d_{101} \ldots \exists k_{10} \\ &\exists c_{110} \forall d_{110} \exists c_{111} \forall d_{111} \ldots \exists k_{11} \ldots \exists k_{12} \ldots \forall j_1 \ldots \forall j_2 \\ &\exists i [\forall i' \leqslant i \forall j' \leqslant j_i,\; t_{i'j'} \in M(j', k_{i'j'}) \;\&\; \tau_i \notin N(i, j_i)] \text{ where} \\ &s_{ij} = (c_{ij0}, d_{ij0}, \ldots, c_{ijk}, d_{ijk}) \text{ for } k = k_{ij} \text{ and} \\ &\tau_{-1} = (\;) \text{ and } t_{ij} = \tau_{i-1} \oplus s_{i0} \oplus \ldots \oplus s_{ij} \text{ and } \tau_i = t_{ij} \text{ for } j = j_i. \end{aligned}$$

Less formally, the i, j-subsubgame of G'' consists in the players alternately choosing the terms of a sequence $\xi_{ij} = t \oplus (c_{ij0}, d_{ij0}, \ldots)$ where $t = t_{i,j-1}$ if $j > 0$ and $= \tau_{i-1}$ otherwise, and then PRO choosing a k_{ij} determining an initial segment t_{ij}. If $t_{ij} \notin M(j, k_{ij})$ PRO in effect has lost the game at this point (unless CON lost at some earlier point). The i-subgame consists in the players going through the i, j-subsubgames producing all the t_{ij}, whose union we call ξ_i, and then CON choosing a j_i determining an initial segment τ_i. If $\tau_i \notin N(i, j_i)$ CON in effect has lost the game at this point (unless PRO lost at some earlier point). If the players get through all the subgames without either one winning, this counts in the end as a win for CON.

The game G' promised at the end of the preceding section will only be described informally. Though its total length is ω we think of it as consisting of a *potentially* infinite sequence of subgames, each consisting of a *potentially* infinite sequence of subsubgames, each consisting of a *potentially* inifinite sequence of rounds. If in any play of G' the i,j-subsubgame *actually* goes through infinitely many rounds, then the $(i,j+1)$-subsubgame never gets started, nor does the $(i+1)$-subgame; while if the i-subgame *actually* goes through infinitely many subsubgames, then the $(i+1)$-subgame never gets started. This keeps the total length in bounds.

The i,j-subsubgame opens by CON signalling (by a choice of a 0 or a 1) either a *challenge* or a *pase*. In the former case, the whole i-subgame ends at once and the players proceed to the $(i+1)$-subgame; in this case we record as σ_i the sequence of s_{ij}, for $j'<j$. In the latter case, the players proceed to the rounds of the i,j-subsubgame. The kth such round opens with PRO either challenging or passing. In the former case, the whole i,j-subsubgame ends at once and the players proceed to the $(i,j+1)$-subgame; in this case we record as s_{ij} the sequence of $c_{ijk'}$, $d_{ijk'}$ for $k'<k$. In the latter case, PRO proceeds to choose a $c_{ijk}\in\omega$, then CON chooses a $d_{ijk}\in\omega$, then the players proceed to the $(k+1)$st round.

Who wins is decide as follows: First, if some i,j-subsubgame goes on forever because PRO fails to challenge on any round, PRO forfeits the game. Second, if this first provision does not apply, but some i-subgame goes on forever because CON fails to challenge at the beginning of any subsubgame, CON forfeits the game. Third, if neither of the first two provisions applies, then a sequence $(\sigma_0, \sigma_1, \ldots)$ will have been generated. PRO wins iff some initial segment belongs to the set W of $(*)$.

A little thought shows that the set of plays *not* constituting a forfeit for PRO is a G_δ set M, while the set of plays not constituting a forfeit for either player is a G_δ subset $M'\subseteq M$, and finally the set of plays in which neither player forfeits but PRO wins constitutes a relatively open $M''\subseteq M'$. Thus the winning set $M-(M'-M'')$ for PRO is a difference of two G_δ's, and G' is of the required form.

(b) Who wins? We claim:

If PRO has a winning strategy for G^*, then he has one for G'.

If CON has a winning strategy for G^*, then she has one for G'.

If PRO has a winning strategy for G'', then he has one for G^{**}.

If CON has a winning strategy for G'', then she has one for G^{**}.

As the proofs are much of a muchness, we consider only the first of these claims. Let φ be a winning strategy for PRO in G^*. He should play thus in G':

In the $0,0$-subsubgame of G', PRO pretends he is playing the $0,0$-subsubgame of G^* and produces his a_{00k} according to φ *until* a k is reached with the following property: In addition to the $b_{00k'}$ for $k'<k$ that CON has *actually* played, PRO can imagine $b_{00k'}$ for $k'\geqslant k$ such that, if CON had played $(b_{00k'}\colon k'\in\omega)$ in the $0,0$-subsubgame of G^*, φ would have dictated that PRO choose k as his k_{00}. When such a k is reached, PRO *challenges*. (Play cannot go on forever without such a k being reached, else an infinite sequence $(b_{00k}\colon k\in\omega)$ would actually be produced by CON, to which we could apply φ!) Then in the $0,1$-subsubgame of G', PRO pretends he is playing the $0,1$-subsubgame of G^*, and so on through all the $0,j$-subsubgames *until* (if this ever happens) CON *challenges* at some j. At such a point $b_{0j'k}$ will have been played by CON or imagined by PRO for $j'<j$ and $k\in\omega$. PRO supplements

these by imagining $b_{0j'k}=0$ for $j'\geqslant j$. Then in the $1,0$-subsubgame of G', PRO imagines that he is playing the $1,0$-subsubgame of G^*, CON having played the $b_{0j'k}$ for j' and $k\in\omega$ as her first ω^2-sequence of moves, and j as her j_0. Etc.

Playing thus, PRO does not forfeit G'. And if CON does not forfeit, then to the actual play of G' corresponds a complete imaginary play of G^* agreeing with φ. Since φ was a winning strategy for G^*, it follows that PRO wins the play of G'. Thus the strategy described is a winning one for PRO in G'. ■

References

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DEPARTMENT OF PHILOSOPHY
PRINCETON UNIVERSITY
Princeton, New Jersey

Accepté par la Rédaction le 3. 3. 1980