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Squares of Q sets

by

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Abstract. A Q set is an uncountable separable metric space in which every subset is a G_δ. We show the following statement is consistent with ZFC: There is a Q set of cardinality ω_2 but no square of a space of cardinality ω_2 is a Q set.

A Q set is an uncountable separable metric space in which every subset is a G_δ. The existence of Q sets is consistent with and independent of ZFC. The existence of Q sets is equivalent to several propositions of set theory and topology and is central in a web of interesting implications — see [T], [P], [F]. One concept investigated recently is that of a *strong Q set*, defined to be a Q set all of whose finite powers are Q sets. The main results are:

If X is a strong Q set, then the Pixley–Roy space built from X is a normal nonmetrizable Moore space [PT].

Conversely, if X is a separable metric space whose Pixley–Roy space is a normal nonmetrizable Moore space, then X is a strong Q set [R].

If there is a Q set, then there is a strong Q set of cardinality ω_1, [P].

We complement this last result with

THEOREM. *It is consistent, relative to* ZFC, *that there be a Q set of cardinality ω_2, but no square of a space of cardinality ω_2 is a Q set.*

We sketch the proof of the theorem below. We start with a model, M, of GCH. We define a notion of forcing $\boldsymbol{P}$ which adds a set Y of ω_2 Cohen reals and makes Y into a Q set in the manner of [FM]. Let $Z = \{z_\beta\colon \beta < \omega_2\}$ be a set of reals of cardinality ω_2. We will show that $\Delta = \{(z_\beta, z_{\beta^*})\colon \beta < \beta^* < \omega_2\}$ is not a G_δ in $Z \times Z$. Let $\mathcal{U} = \{U_k\colon k \in \omega\}$ be a family of open sets containing Δ. Using counting arguments (Lemma 3), we find a large subset H of ω_2 such that z_β, $\beta \in H$, are independent over $\mathcal{U}$ (mutually Cohen generic over $\mathcal{U}$, in some sense). Choose β, $\beta^* \in H$. There must be $k \in \omega$ and $t \in \boldsymbol{P}$ such

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that $t \Vdash (z_{\beta^*}, z_\beta) \notin U_k$. Choose $\delta, \delta^* \in H$, $\delta < \delta^*$, about which t says nothing. We extend t to a condition t_1 (Lemma 4) which says about z_δ everything t says about z_{β^*}, says about z_{δ^*} everything t says about z_β, and says nothing new about z_β, z_{β^*}. Because $(z_\delta, z_{\delta^*}) \in \Delta$ there are basic open sets B_0, B_1 and conditions t_2, t_3, $\bar{t}$ such that t_1, t_2, t_3, $\bar{t}$ are compatible, t_2 forces $z_\delta \in B_0$, t_3 forces $z_{\delta^*} \in B_1$, and $\bar{t}$ forces $B_0 \times B_1 \subset U_k$. By the construction of H, we can find t_2^*, t_3^* such that t_2^* forces $z_{\beta^*} \in B_0$, t_3^* forces $z_\beta \in B_1$, and t_1, t_2^*, t_3^*, $\bar{t}$ are compatible (Lemma 5). Thus, $(z_{\beta^*}, z_\beta) \in U_k$. We conclude that $\bigcap \mathscr{U} \neq \Delta$, and Z^2 is not a Q set.

What needs to be done to turn the sketch above into a proof? First, we must define $\boldsymbol{P}$ and verify that forcing with $\boldsymbol{P}$ adds a set Y of ω_2 Cohen reals and then makes Y a Q set (Lemma 0).

Second, we were careless about talking about objects in the extension. To be precise, we should say z_β, a term for an element of C, the Cantor set; U_k, a term for an open subset of $C \times C$; and G, the canonical term for the generic filter. We leave it to the reader to make these and associated changes. Let us emphasize that "we work in the ground model". In particular, we apply the Δ-system lemma in the ground model, and the set H is in the ground model.

Third, we need to make precise the notion "t says about z_β what it says about z_δ". We will do this after Lemma 3. One difficulty of this proof lies in the fact that two slightly different notions are used.

Finally, we must, of course, prove the lemmas.

Let us establish some notation. ${}^XY$ is the set of functions from X to Y. We denote the set of finite subsets of a set A by $[A]^{<\omega}$, the set of subsets of A of cardinality $\varkappa$ by $[A]^\varkappa$. Since every Q set has cardinality less than 2^ω, and every separable metric space of cardinality less than 2^ω is homeomorphic to a subspace of C, the Cantor set, we lose no generality by considering only subsets of C. We do this because $\mathscr{B}$, the family of clopen subsets of C, is a countable base for C closed under intersection and difference.

We will use the following special case of the *Δ-system lemma* (see [J], Appendix 2, for example); assuming CH, whenever $\{S_\alpha\colon \alpha \in \omega_2\}$ is a family of countable sets, there are $J \in [\omega_2]^{\omega_2}$ and a set S_R such that for all $\{\alpha, \beta\} \in [J]^2$, $S_\alpha \cap S_\beta = S_R$. $\{S_\alpha\colon \alpha \in J\}$ is called a *Δ-system*; S_R is called the *root*.

Towards defining $\boldsymbol{P}$ we define a preliminary partial order. Let $\boldsymbol{Q}$ be the set of all triples (p, a, b) satisfying

1.1. $p \in {}^{\omega_2}\mathscr{B}$, $a \in {}^{\omega_3}([\omega_2]^{<\omega})$, $b \in {}^{\omega_3}\mathscr{B}$,

1.2. $|\{\beta \in \omega_2\colon p(\beta) \neq C\}| < \omega$, $|\{\gamma \in \omega_3\colon a(\gamma) \cup b(\gamma) \neq \emptyset\}| < \omega$,

1.3. for all $\beta \in \omega_2$, $p(\beta) \neq \emptyset$,

1.4. for all $\beta \in \omega_2$, for all $\gamma \in \omega_3$, $\beta \in a(\gamma)$ implies $p(\beta) \cap b(\gamma) = \emptyset$,

1.5. $(p, a, b) \leqslant (p^*, a^*, b^*)$ iff for all $\beta \in \omega_2$, $p(\beta) \subset p^*(\beta)$ and for all $\gamma \in \omega_3$ $a(\gamma) \supset a^*(\gamma)$ and $b(\gamma) \supset b^*(\gamma)$.

Let $(E_\alpha)_{\alpha < \omega_3}$, where E_α has the form $\big((q(\alpha, i, \beta))_{i \in \omega}\big)_{\beta \in \omega_2}$, count, ω_3 times each, every ω_2 sequence of ω sequences of pairwise incompatible elements of $\boldsymbol{Q}$. (The intended meaning is that $\{q(\alpha, i, \beta)\colon i \in \omega\}$ is a countable pairwise incompatible subset of conditions forcing $\beta \in A_\alpha$, where $(A_\alpha)_{\alpha \in \omega_3}$ lists all subsets of ω_2 in the extension.)

We define suborders $\boldsymbol{P}_\alpha$, $\alpha \leqslant \omega_3$ of $\boldsymbol{Q}$ by induction on α. $\boldsymbol{P}_0 = \{(p, a, b) \in \boldsymbol{Q}\colon$ for all $\gamma \in \omega_3$, $a(\gamma) \cup b(\gamma) = \emptyset\}$. $\boldsymbol{P}_\lambda = \bigcup_{\alpha < \lambda} \boldsymbol{P}_\alpha$, when λ is a limit ordinal. $\boldsymbol{P}_{\alpha+1} = \boldsymbol{P}_\alpha$ if $\{q(\alpha, i, \beta)\colon i \in \omega,\ \beta \in \omega_2\}$ is *not* a subset of $\boldsymbol{P}_\alpha$. If $\{q(\alpha, i, \beta)\colon i \in \omega,\ \beta \in \omega_2\}$ *is* a subset of $\boldsymbol{P}_\alpha$, $\boldsymbol{P}_{\alpha+1}$ is the set of all $(p, a, b) \in \boldsymbol{Q}$ such that

2.1. for all $\gamma > \alpha$ $a(\gamma) \cup b(\gamma) = \emptyset$,

2.2. $(p, a^*, b^*) \in \boldsymbol{P}_\alpha$ where a^*, b^* are defined by

$$a^*(\alpha) = \emptyset, \qquad b^*(\alpha) = \emptyset,$$
$$a^*(\gamma) = a(\gamma), \qquad b^*(\gamma) = b(\gamma) \text{ if } \gamma \neq \alpha,$$

2.3. if $\beta \in a(\alpha)$ then there is $i \in \omega$ such that

$$(p, a, b) \leqslant (p, a^*, b^*) \leqslant q(\alpha, i, \beta),$$

$\boldsymbol{P}_{\omega_3} = \boldsymbol{P}$ is the partial order we will use.

Our next task is to determine when two elements of $\boldsymbol{P}$, $t = (p, a, b)$ and $\bar{t} = (\bar{p}, \bar{a}, \bar{b})$, are compatible. In view of 1.5, the natural common extension seems to be $(p \cap \bar{p}, a \cup \bar{a}, b \cup \bar{b})$, where $(p \cap \bar{p})(\beta) = p(\beta) \cap \bar{p}(\beta)$, $(a \cup \bar{a})(\gamma) = a(\gamma) \cup \bar{a}(\gamma)$, and $(b \cup \bar{b})(\gamma) = b(\gamma) \cup \bar{b}(\gamma)$. However, we must consider also 1.4, and so we define

$$(p \cap \bar{p})_{t\bar{t}}(\beta) = \big(p(\beta) \cap \bar{p}(\beta)\big) - \bigcup \{b(\gamma) \cup \bar{b}(\gamma)\colon \beta \in a(\gamma) \cap \bar{a}(\gamma)\}.$$

(The subscript, $t\bar{t}$, indicates the dependence on all of t and $\bar{t}$, not only p and $\bar{p}$. Now, we ask when $\bar{\bar{t}} = \big((p \cap \bar{p})_{t\bar{t}}, (a \cup \bar{a}), (b \cup \bar{b})\big) \in \boldsymbol{P}$. It is straightforward to check 1.1, 1.2, and 1.4. We can verify 2.1, 2.2, and 2.3 by induction on $\gamma < \omega_3$. Further, 1.5 gives $\bar{\bar{t}} \leqslant t$ and $\bar{\bar{t}} \leqslant \bar{t}$. Hence the only way that t, $\bar{t}$ can fail to be compatible is for $\bar{\bar{t}}$ not to satisfy 1.3. The above analysis is summarized in the following lemma.

LEMMA 0. *Let t, $\bar{t}$ be elements of $\boldsymbol{P}$. Define $\bar{\bar{t}} = \big((p \cap \bar{p})_{t\bar{t}}, (a \cup \bar{a}), (b \cup \bar{b})\big)$ as in the paragraph above. Then the following are equivalent*

a) *t and $\bar{t}$ are compatible,*

b) *$\bar{\bar{t}} \in \boldsymbol{P}$,*

c) *for all $\beta < \omega_2$, $(p \cap \bar{p})_{t\bar{t}}(\beta) \neq \emptyset$.*

Next, we use Lemma 0 to show that $\boldsymbol{P}$ has CCC. Let $\{t_\alpha:\ \alpha\in\omega_1\}$ be an uncountable subset of $\boldsymbol{P}$, where $t_\alpha=(p_\alpha, a_\alpha, b_\alpha)$. For $\alpha\in\omega_1$, set $\mathrm{sp}(\alpha)=\{\beta\in\omega_1:\ p_\alpha(\beta)\neq C\}$; $\mathrm{sa}(\alpha)=\{\gamma\in\omega_2:\ a_\alpha(\gamma)\neq\emptyset\}$; $\mathrm{sb}(\alpha)=\{\gamma\in\omega_2:\ b_\alpha(\gamma)\neq\emptyset\}$; and $\mathrm{sr}(\alpha)=\mathrm{sp}(\alpha)\cup(\bigcup\mathrm{range}\,a_\alpha)$. Apply the Δ-system lemma for finite sets four times to get an uncountable $W\subset\omega_1$ such that $\{\mathrm{sp}(\alpha):\ \alpha\in W\}$, $\{\mathrm{sa}(\alpha):\ \alpha\in W\}$, $\{\mathrm{sb}(\alpha):\ \alpha\in W\}$, and $\{\mathrm{sr}(\alpha):\ \alpha\in W\}$ are all Δ-systems. Since there are only countably many possibilities for each α, we are further refine to an uncountable $W^1\subset W$ so that for all $\alpha\in W^1$ we have $\mathrm{card}(\mathrm{sa}(\alpha)=n_a$, $\mathrm{card}(\mathrm{sb}(\alpha))=n_b$, $\mathrm{card}(\mathrm{sp}(\alpha))=n_p$, and $\mathrm{card}(\mathrm{sr}(\alpha))=n_r$. Moreover, for $i<n_p$, $j<n_a$, and $k<n_b$, if $\beta(\alpha, i)$ is the ith element of $\mathrm{sp}(\alpha)$, $\gamma(\alpha, j)$ is the jth element of $\mathrm{sa}(\alpha)$, and $\delta(\alpha, k)$ the kth element of $\mathrm{sb}(\alpha)$, then the values of $p_\alpha(\beta(\alpha, i))$ and $b_\alpha(\delta(\alpha, k))$ in $\mathscr{B}$ and the truth of "$\beta(\alpha, i)\in a_\alpha(\gamma(\alpha, j))$" does not depend on α.

We claim that if $\alpha, \bar\alpha\in W^1$, then $t_\alpha=(p, a, b)=\bar t$ and $t_{\bar\alpha}=(\bar p, \bar a, \bar b)=\bar t$ are compatible because condition c) of Lemma 0 is satisfied. If β is in the root of $\{\mathrm{sr}(\gamma):\ \gamma\in W^1\}$, then $(p\cap\bar p)_{t\bar t}(\beta)=p(\beta)\neq\emptyset$. If $\beta\notin\mathrm{sr}(\alpha)$, then $(p\cap\bar p)_{t\bar t}(\beta)=\bar p(\beta)\neq\emptyset$; similarly if $\beta\notin\mathrm{sr}(\alpha)$.

LEMMA 1. *$\boldsymbol{P}$ adds a set, Y, of Cohen reals and makes Y a Q-set.*

Proof. For each $\beta\in\omega_2$, let y_β the unique element of $\bigcap\{p(\beta):\ (p, a, b)\in G\}$. Each y_β is a Cohen real because $\mathscr{B}-\{\emptyset\}$ is isomorphic to the usual Cohen poset. Let $Y=\{y_\beta:\ \beta\in\omega_2\}$.

Towards showing that Y is a Q-set, let $A^\#$ be an arbitrary subset of Y, and set $A=\{\beta\in\omega_2:\ y_\beta\in A^\#\}$. Let $E=\big(q(i, \beta)_{i\in\omega}\big)_{\beta\in\omega_2}$ be such that for all $\beta\in\omega_2$, $\{q(i, \beta):\ i\in\omega\}$ is a maximal pairwise incompatible subset of $\boldsymbol{P}$ of conditions forcing $\beta\in A$. There is $\alpha<\omega_3$ such that $\{q(i, \beta):\ i\in\omega,\ \beta\in\omega_2\}$ is a subset of $\boldsymbol{P}_\alpha$. Because E was listed ω_3 times, there are distinct α_n, $n\in\omega$, of ordinals greater than α such that $E_{\alpha_n}=E$. For $n\in\omega$ let

$$F_n=\{y_\beta\in Y:\ \exists(p, a, b)\in G\,[\beta\in a(\alpha_n)]\},$$

$$V_n=\bigcup\{b(\alpha_n):\ (p, a, b)\in G\}.$$

Our first claim is that $V_n\cap F_n=\emptyset$. Aiming for a contradiction, assume that $y_\beta\in V_n\cap F_n$. Then for some $(p, a, b)\in G$, $\beta\in a(\alpha_n)$ and for some $(\bar p, \bar a, \bar b)\in G$, $y_\beta\in b(\alpha_n)$. Since G is a filter, there is a common extension, $(\bar p, \bar a, \bar b)$. We have $y_\beta\in\bar p(\beta)$ (definition of y_β) and $y_\beta\in b(\alpha_n)\subset\bar b(\alpha_n)$, hence $y_\beta\in\bar p(\beta)\cap\bar b(\alpha_n)\neq\emptyset$. However, $\beta\in a(\alpha_n)\subset\bar a(\alpha_n)$, so by 1.4, $\bar p(\beta)\cap\bar b(\alpha_n)=\emptyset$ – contradiction.

Next, we claim that $F_n\cup V_n=Y$, and hence that F_n is closed in Y. It will suffice to show that for each $\beta<\omega_2$, the set

$$D_\beta=\{(p, a, b)\in\boldsymbol{P}:\ \beta\in b(\alpha_n)\cup a(\alpha_n)\}$$

is dense in $\boldsymbol{P}$. So let $t=(p, a, b)\in\boldsymbol{P}$ be arbitrary. If $\beta\in a(\alpha_n)$, then $t\in D_\beta$. If $\beta\notin a(\alpha_n)$, we define p^* from ω_2 to $\mathscr{B}$ and b^* from ω_3 to $\mathscr{B}$ so that

$$\begin{gathered}p^*(\delta)=p(\delta)\quad\text{if}\quad\delta\notin a(\alpha_n)\cup\{\beta\},\\ \emptyset\neq p^*(\delta)\leqslant p(\delta)\quad\text{if}\quad\delta\in a(\alpha_n)\cup\{\beta\},\\ \{p^*(\delta):\ \delta\in a(\alpha_n)\cup\{\beta\}\}\text{ is disjoint},\\ b^*(\gamma)=b(\gamma)\quad\text{if}\quad\gamma\neq\alpha_n,\\ b^*(\alpha_n)=b(\alpha_n)\cup p(\beta).\end{gathered}$$

Then (p^*, a, b^*) extends (p, a, b) and is in D_β.

Finally, we claim that $\bigcup\{F_n:\ n\in\omega\}=\{y_\beta:\ \beta\in A\}$. It is clear from the definition that $F_n\subset\{y_\beta:\ \beta\in A\}$. We will demonstrate the other inclusion by showing that for all $\beta\in A$, the set

$$D^\beta=\{(p, a, b)\in\boldsymbol{P}:\ \exists n\in\omega\,[\beta\in a(\alpha_n)]\}$$

is dense below any $t_0\in G$. Since $\beta\in A$, there is $i\in\omega$ such that $q(i, \beta)\in G$. Since G is a filter, there is $t=(p, a, b)$ extending t_0 and $q(i, \beta)$. By 1.2, there is $n\in\omega$ such that $a(\alpha_n)\cup b(\alpha_n)=\emptyset$. Define a^* by $a^*(\alpha_n)=\{\beta\}$, $a^*(\gamma)=a(\gamma)$ if $\gamma\in\omega_3-\{\alpha_n\}$. Because $t^*=(p, a^*, b)$ is so similar to $t\in\boldsymbol{P}$, to verify that $t^*\in\boldsymbol{P}$ only 1.4 and 2.3 for $\gamma=\alpha_n$ need to be checked. Since $b(\alpha_n)=\emptyset$, 1.4 checks; 2.3 checks because t extends $q(i, \beta)=q(\alpha_n, i, \beta)$. Hence t^* extends t_0 and is in D^β.

Since A was arbitrary, we may conclude that Y is a Q-set.

We will not use the following lemma, which is a simple version of Lemma 5. We have included it here to give the reader some idea of the direction of our argument.

LEMMA 2. *Suppose* 1. $t=(p, a, b)$, $t^*=(p^*, a^*, b^*)$ *and* $\bar t=(\bar p, \bar a, \bar b)$ *are elements of* P;

2. *t and $\bar t$ are compatible,*

3. *there are permutations of order 2,* $'\colon\omega_2\to\omega_2$ *and* $''\colon\omega_3\to\omega_3$ *such that*

a) $\beta\neq\beta'$ *implies* $\bar p(\beta)=C$,

b) $\gamma\neq\gamma''$ *implies* $\bar a(\gamma)\cup\bar b(\gamma)=\emptyset$,

c) $\bar p(\beta)=p^*(\beta')$,

d) $b(\gamma)=b^*(\gamma'')$,

e) $\beta\in a(\gamma)$ *iff* $\beta'\in a^*(\gamma'')$,

f) $\beta\in\bar a(\gamma)$ *implies* $\beta=\beta'$ *and* $\gamma=\gamma''$.

Then t^* and $\bar t$ are compatible.

Proof. For all $\beta\in\omega_2$,

$$\begin{aligned}(p\cap\bar p)_{t\bar t}(\beta)&=p^*(\beta)\cap\bar p(\beta)-\bigcup\{b^*(\gamma)\cup\bar b(\gamma):\ \beta\in a^*(\gamma)\cup\bar a(\gamma)\}\\&=p(\beta')\cap\bar p(\beta')-\bigcup\{b(\gamma'')\cup\bar b(\gamma''):\ \beta'\in a(\gamma'')\cup\bar a(\gamma'')\}\\&=(p^*\cap\bar p)_{t^*\bar t}(\beta)\neq\emptyset.\ \blacksquare\end{aligned}$$

Let $Z = \{z_\nu: \nu < \omega_2\} \in [C]^{\omega_2}$. Let $\mathscr{U} = \{U_k: k \in \omega\}$ be a set of open subsets of $C \times C$. Let Δ be $\{(z_\nu, z_{\nu*}): \nu < \nu^* < \omega_2\}$. We will show that $\bigcap \mathscr{U} \cap (Z \times Z) \neq \Delta$. Since we are working in the ground model, what we will really show is that the set of conditions forcing this statement is dense.

For $k \in \omega$ and $B_0, B_1 \in \mathscr{B}$, let $\Gamma(k, B_0, B_1)$ be a maximal pairwise incompatible set of conditions forcing $B_0 \times B_1 \subset U_k$. For $\nu \in \omega_2$, $B \in \mathscr{B}$, let $\Gamma(\nu, B) = \{r(\nu, B, i): i \in \omega\}$ be a maximal pairwise incompatible set of conditions forcing $z_\nu \in B$. Set, for $t = (p, a, b) \in \boldsymbol{P}$ and for $\beta, \nu \in \omega_2$,

$$\begin{aligned}
K_t &= \{\beta \in \omega_2: p(\beta) \neq C\} \cup \operatorname{range}(a),\\
L_t &= \{\gamma \in \omega_3: a(\gamma) \cup b(\gamma) \neq \emptyset\},\\
K_U &= \bigcup \{K_t: t \in \Gamma(k, B_0, B_1),\ k \in \omega,\ B_0, B_1 \in \mathscr{B}\},\\
L_U &= \bigcup \{L_t: t \in \Gamma(k, B_0, B_1),\ k \in \omega,\ B_0, B_1 \in \mathscr{B}\},\\
K^\beta &= \bigcup \{K_{q(\alpha, i, \beta)}: \alpha \in K_U,\ i \in \omega\},\\
L^\beta &= \bigcup \{K_{q(\alpha, i, \beta)}: \alpha \in K_U,\ i \in \omega\},\\
K'_\nu &= \bigcup \{K_t: t \in \Gamma(\nu, B),\ B \in \mathscr{B}\},\\
L'_\nu &= \bigcup \{L_t: t \in \Gamma(\nu, B),\ B \in \mathscr{B}\},\\
K_\nu &= \bigcup \{K^\beta: \beta \in L'_\nu\} \cup K'_\nu \cup K_U,\\
L_\nu &= \bigcup \{L^\beta: \beta \in L'_\nu\} \cup L'_\nu \cup L_U.
\end{aligned}$$

All the above sets are countable.

For each $\delta \in \omega_2$, let $(\gamma(\delta, \alpha))_{\alpha < \varkappa(\delta)}$ list K_δ in the order preserving way. For all $\delta, \delta^* \in \omega_2$ with $\varkappa(\delta) = \varkappa(\delta^*)$, let $\tau_{\delta\delta*}: \omega_3 \to \omega_3$ be the permutation of order two which exchanges $\gamma(\delta, \alpha)$ and $\gamma(\delta^*, \alpha)$ for all $\alpha < \varkappa(\delta)$, and fixes everything else. Similarly list L_δ as $(\beta(\delta, \alpha))_{\alpha < \lambda(\delta)}$ and if $\lambda(\delta) = \lambda(\delta^*)$, define $\theta_{\delta\delta*}: \omega_2 \to \omega_2$, a permutation of order two.

Definition. For $\delta, \delta^* \in \omega_2$ and $t = (p, a, b)$, $t^* = (p^*, a^*, b^*) \in P$, let $(\delta, t \leftrightarrow \delta^*, t^*)$ mean

$$\theta_{\delta\delta*} \text{ and } \tau_{\delta\delta*} \text{ are defined,}$$

and, abbreviating them by ′,″, respectively,

1. for all $\beta \in \omega_2$, $p(\beta) = p^*(\beta')$,
2. for all $\gamma \in \omega_3$, $b(\gamma) = b^*(\gamma'')$,
3. for all $\beta \in \omega_2$, $\gamma \in \omega_3$, $\beta \in a(\gamma)$ iff $\beta' \in a^*(\gamma'')$.

The proof of the following lemma is a straightforward counting argument, very similar to the proof given above that $\boldsymbol{P}$ has CCC. The fact that CH holds in the ground model allows us to apply the Δ-system lemma to families of ω_2 – many countable sets.

Lemma 3. *There is $H \in [\omega_w]^{\omega_2}$ such that*

1. *$\{K_\delta: \delta \in H\}$ is a Δ-system with root K_R,*
2. *$\{L_\delta: \delta \in H\}$ is a Δ-system with root L_R,*

and for each $\delta, \delta^* \in H$,

3. *for all $\beta \in L_R$, $\theta_{\delta\delta*}(\beta) = \beta$,*
4. *for all $\gamma \in K_R$, $\tau_{\delta\delta*}(\gamma) = \gamma$,*
5. *for all $\alpha \in K_U$, $i \in \omega$, $(\delta, q(\alpha, i, \delta) \leftrightarrow \delta^*, q(\alpha, i, \delta^*))$,*
6. *for all $i \in \omega$, $B \in \mathscr{B}$, $(\delta, r(\delta, B, i) \leftrightarrow \delta^*, r(\delta^*, B, i))$.*

Definition. Let $t = (p, a, b) \in P$ and $\eta, \eta^* \in H$. Abbreviate $\theta_{\eta\eta*}$ and $\tau_{\eta\eta*}$ by ′ and ″, respectively. Let $(t; \eta \leftrightarrow \eta^*)$ mean

1. for all $\beta \in \omega_2$, $p(\beta) = p(\beta')$,
2. for all $\gamma \in \omega_3$, $b(\gamma) = b(\gamma'')$,
3. if $\beta \in L_\eta \cup L_{\eta*}$ and $\gamma \in K_\eta \cup K_{\eta*}$, then $\beta \in a(\gamma)$ iff $\beta' \in a(\gamma'')$,
4. $L_t \cap (L_\eta \cup L_{\eta*}) \subset L_R$.

Remark. $(t; \eta \leftrightarrow \eta^*)$ is weaker than $(\eta, t \leftrightarrow \eta^*, t)$ – compare condition 3 in each definition. The use of these two similar but slightly different notions seems necessary. Lemma 4 is false if the conclusion is strengthened to $(\delta_0, t^* \leftrightarrow \delta_0^*, t^*)$. Lemma 5 is false if we weaken $(\eta_2, t_2 \leftrightarrow \eta_1, t_2^*)$ or $(\eta_3, t_3 \leftrightarrow n_0, t_3^*)$.

Lemma 4. *Let $\delta_0, \ldots, \delta_n \in H$, $t = (p, a, b) \in P$. The set $J = \{\delta_0, \ldots, \delta_n\} \cup \cup \{\delta^* \in H: L_{\delta^*} \cap L_t - L_R \neq \emptyset\} \cup \{\delta^* \in H: K_{\delta^*} \cap K_t - K_R\}$ is countable. If $\delta_0^*, \ldots, \delta_n^*$ are distinct elements of $H - J$, then there is $t^* \in P$, $t^* < t$ such that for all $i \leqslant n(t^*, \delta_i \leftrightarrow \delta_i^*)$.*

Proof. The countability of J is routine. For the second assertion of the lemma, we do the special case $n = 0$. The general case is only notionally more complex. Let ′,″ be as in Lemma 3. We define $t^* = (p^*, a^*, b^*)$ by

$$\begin{aligned}
p^*(\beta) &= p(\beta) \cap p(\beta'),\\
a^*(\gamma) &= a(\gamma) \cup \{\beta: \beta' \in L_\delta,\ \gamma'' \in K_\delta \text{ and } \beta' \in a(\gamma'')\},\\
b^*(\gamma) &= b(\gamma) \cup b(\gamma'').
\end{aligned}$$

It is straightforward to check that for all $i \leqslant n$, $(t^* \delta_i \leftrightarrow \delta_i^*)$. Most of the verification that $t^* \in \boldsymbol{P}$ is routine; below we verify 1.3, 2.3 and 1.4.

1.3. Because $L_{\delta*} \cap L_t \subset L_R$, either $\beta \in \beta'$ or one of $p(\beta)$, $p(\beta') = C$. In either case $p^*(\beta) \neq \emptyset$.

2.3. Suppose $\beta \in a^*(\gamma)$. If $\beta \in a(\gamma)$ it suffices to note that $t \in \boldsymbol{P}$. Otherwise $\gamma \in K_{\delta*}$, $\beta \in L_{\delta*}$, and $\beta' \in a(\gamma'')$. Because of this last fact, there is $i \in \omega$ such that $t \leqslant q(\gamma'', i, \beta') = (p_1, a_1, b_1)$. We claim that $t^* \leqslant q(\gamma, i, \beta) = (p_1^*, a_1^*, b_1^*)$. For all $\eta \in \omega_2$, $p_1^*(\eta) = p_1(\eta') \supset p(\eta') \supset p^*(\eta)$. For all $\eta \in \omega_2$, $\alpha \in \omega_3$, $\eta \in a_1^*(\alpha) \leftrightarrow \eta' \in a_1(\alpha'') \to \eta' \in a(\alpha'') \to \eta \in a^*(\alpha)$. (The last implication uses the fact that $\eta' \in a_1(\alpha'')$ implies $\eta' \in L_\delta$ and $\alpha'' \in K_\delta$.) For all $\alpha \in \omega_3$, $b_1^*(\alpha) = b_1(\alpha'') \subset b(\alpha'') \subset b^*(\alpha)$.

1.4. Suppose $\beta \in a^*(\gamma)$; we must show that $p^*(\beta) \cap b^*(\gamma) = \emptyset$. If $\beta \in a(\gamma)$, then $b^*(\gamma) = b(\gamma)$ and $p^*(\beta) \subset p(\beta)$ so $p^*(\beta) \cap b^*(\gamma) \subset p(\beta) \cap b(\gamma) = \emptyset$. Otherwise, $\gamma \in K_{\delta*}$, $\beta \in L_{\delta*}$, and $\beta' \in a(\gamma'')$. Then $p^*(\beta) \cap b^*(\gamma) = p(\beta') \cap \cap b(\gamma'') = \emptyset$. ∎

LEMMA 5. *Suppose $t_1, t_2, t_3, \bar{t}, t_2^*, t_3^*$ are elements of P such that for some $\eta_0, \eta_1, \eta_2, \eta_3 \in H$, $k, i, j \in \omega$, $B_0, B_1 \in \mathscr{B}$ we have*

$$\begin{gathered} t_2 = r(\eta_2, B_0, i), \quad t_2^* = r(\eta_1, B_0, i), \\ t_3 = r(\eta_3, B_1, j), \quad t_3^* = r(\eta_0, B_1, j), \\ \bar{t} \in \Gamma(k, B_0, B_1), \\ (t_1, \eta_0 \leftrightarrow \eta_3), \qquad (t_1, \eta_1 \leftrightarrow \eta_2). \end{gathered}$$

If $t_1, t_2, t_3, \bar{t}$, are compatible, then $t_1, t_2^*, t_3^*, \bar{t}$ are compatible.

Proof. The proof is an elaboration of that of Lemma 2. Abbreviate the composition $\theta_{\eta_2\eta_1} \circ \theta_{\eta_3\eta_0}$ by $'$ and the composition $\tau_{\eta_2\eta_1} \circ \tau_{\eta_3\eta_0}$ by $''$. Note that by hypothesis and part 6 of Lemma 3, we have $(\eta_2, t_2 \leftrightarrow \eta_1, t_2^*)$ and $(\eta_3, t_3 \leftrightarrow \eta_0, t_3^*)$. We will refer to this fact as $(\leftrightarrow)$. The following definitions allow us to define and work with the given common extension of $t_1, t_2, t_3, \bar{t}$ and the claimed common extension of $t_1, t_2^*, t_3^*, \bar{t}$. We simultaneously make the same definitions with $t_2, t_3, a, b, c_1, c_2, d$, and p (but not t_1 or $\bar{t}$) starred. For all $\gamma < \omega_3$ and $\beta < \omega_2$, let

$$\begin{aligned} a(\gamma) &= a_1(\gamma) \cup a_2(\gamma) \cup a_3(\gamma) \cup \bar{a}(\gamma), \\ b(\gamma) &= a_1(\gamma) \cup a_2(\gamma) \cup a_3(\gamma) \cup \bar{a}(\gamma), \\ d(\beta) &= p_1(\beta) \cap p_2(\beta) \cap p_3(\beta) \cap \bar{p}(\beta), \\ c_0(\beta) &= \bigcup \{b(\gamma)\colon\ \beta \in a(\gamma) \,\&\, \gamma = \gamma''\}, \\ c_i(\beta) &= \bigcup \{b_i(\gamma)\, \beta \in a_i(\gamma) \,\&\, \gamma \neq \gamma''\} \quad \text{for} \quad i = 1, 2, 3, \\ p(\beta) &= d(\beta) - (c_0(\beta) \cup c_1(\beta) \cup c_2(\beta) \cup c_3(\beta)). \end{aligned}$$

By 4 of $(t, \eta \leftrightarrow \eta^*)$ and $L_{\bar{t}} \subset L_R$, we have that

$$\bigcup \{b(\gamma)\colon\ \beta \in a(\gamma)\} = c_0(\beta) \cup c_1(\beta) \cup c_2(\beta) \cup c_3(\beta).$$

From Lemma 0, we see that for all $\beta < \omega_2$, $p(\beta) \neq \emptyset$, and that to prove Lemma 5, it will suffice to show that for all $\beta < \omega_2$, $p^*(\beta) \neq \emptyset$.

Case 1. $\beta = \beta'$. First,

$$\begin{aligned} d^*(\beta) &= p_1(\beta) \cap p_2^*(\beta) \cap p_3^*(\beta) \cap \bar{p}(\beta) \\ &= p_1(\beta) \cap p_2(\beta') \cap p_3(\beta') \cap \bar{p}(\beta) \\ &= p_1(\beta) \cap p_2(\beta) \cap p_3(\beta) \cap \bar{p}(\beta) = d(\beta). \end{aligned}$$

The first and fourth equalities are definitions; the second is by $(\leftrightarrow)$, and the third is by $\beta = \beta'$. Next,

$$\begin{aligned} c_0^*(\beta) &= \bigcup \{b^*(\gamma)\colon\ \beta \in a^*(\gamma) \,\&\, \gamma = \gamma''\} \\ &= \bigcup \{b(\gamma)\colon\ \beta \in a(\gamma) \,\&\, \gamma = \gamma''\} = c_0(\beta). \end{aligned}$$

Here the middle equality is from $(\leftrightarrow)$, $\beta = \beta'$ and $\gamma = \gamma''$. Because only t_1 is used in the definition, $c_1^*(\beta) = c_1(\beta)$. Next,

$$\begin{aligned} c_2^*(\beta) &= \bigcup \{b_2^*(\gamma)\colon\ \beta \in a_2^*(\gamma) \,\&\, \gamma \neq \gamma''\} \\ &= \bigcup \{b_2(\gamma'')\colon\ \beta' \in a_2(\gamma'') \,\&\, \gamma \neq \gamma''\} \\ &= \bigcup \{b_2(\gamma)\colon\ \beta \in a_2(\gamma) \,\&\, \gamma \neq \gamma''\} = c_2(\beta). \end{aligned}$$

The first and fourth equalities are definitions, the second uses $(\leftrightarrow)$, and the third uses $\beta = \beta'$, $(\gamma'')'' = \gamma$, and the symmetry of $\neq$. Finally, $c_3^*(\beta) = c_3(\beta)$ by exactly the same argument. Hence if $\beta = \beta'$, then $p^*(\beta) = p(\beta) \neq \emptyset$.

Case 2. $\beta \neq \beta'$. The key is to note that for all $\gamma < \omega_2$, $\beta \notin a_1(\gamma) \cup \bar{a}(\gamma)$. By the same arguments as in Case 1, we show that $d^*(\beta) = d(\beta')$, $c_2^*(\beta) = c_2(\beta')$, and $c_2^*(\beta) = c_2(\beta')$. Next, $c_1^*(\beta) = \emptyset = c_1(\beta')$. Finally, we show that $c_0^*(\beta) = \emptyset = c_0^*(\beta)$. If $\beta \in a_2(\gamma)$, then $\gamma \in K_{\eta_2}$. (Similarly, for a_3, a_2^*, and a_3^*). If further $\gamma = \gamma''$, then $\gamma \in K_R$, $\beta \in L_R$, and $\beta = \beta'$. But Case 2 is $\beta \neq \beta'$. Hence $p^*(\beta) = p(\beta') \neq \emptyset$. ■

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