Products of hereditarily indecomposable continua are λ-connected

by

Charles L. Hagopian * (Sacramento, Ca.)

Dedicated to the memory of
Professor Kazimierz Kuratowski

Abstract. Suppose X and Y are continua. The product $X\times Y$ is arcwise connected only when both X and Y are arcwise connected. However $X\times Y$ may be λ-connected while X and Y are not λ-connected. We prove that $X\times Y$ is λ-connected when X and Y are hereditarily indecomposable.

A *continuum* is a nondegenerate compact connected metric space. A continuum is *decomposable* if it is the union of two proper subcontinua; otherwise, it is *indecomposable*. A continuum is *hereditarily indecomposable* if each of its subcontinua is indecomposable.

Kuratowski [6, p. 262] defined a continuum M to be of *type* λ if M is irreducible between two points p and q is of type λ if and only if there is a condensation. Gordh [1, Theorem 2.7, p. 650] proved that a continuum M irreducible between two points p and q is of type λ if and only if there is a decomposition $\mathscr{D}$ of M into continua of condensation such that each element of $\mathscr{D}$ not containing p or q separates M. Furthermore $\mathscr{D}$ is upper semi-continuous and $M/\mathscr{D}$ is an arc [1, Corollary 2.1, p. 648]. Additional information about continua of type λ is given in [7] and [8].

In [5] Knaster and Mazurkiewicz used this concept to generalize arcwise connectivity. They defined a continuum M to be λ-connected if for each pair p, q of points of M, there exists a continuum of type λ in M that is irreducible between p and q. The author [2, Theorem 2] proved that if M is a λ-connected plane continuum, then each pair of points of M can be joined by a hereditarily decomposable subcontinuum of M. Hence every planar continuous image of a λ-connected plane continuum is λ-connected [2, Theorem 5]. However, unlike arcwise connectivity, λ-connectivity is not a continuous invariant for nonplanar continua. The continuous image of a λ-connected

* The author wishes to thank G. R. Gordh, Jr. and E. E. Grace for comments that led to the improvement of this paper. This research was partially supported by the National Science Foundation Grant MCS 79-16811.

continuum may even be hereditarily indecomposable. Knaster and Mazurkiewicz [5, Example 2, p. 89] established this fact by showing that the product of a pseudo-arc and a circle is λ-connected. They used the natural projection function to map this product onto a pseudo-arc.

In a recent conversation Howard Cook raised the question of whether the product of two pseudo-arcs is λ-connected. The following theorem answers Cook's question in the affirmative.

THEOREM. *If X and Y are hereditarily indecomposable continua, then $X \times Y$ is λ-connected.*

Proof. Let $(x(0), y(0))$ and $(x(1), y(1))$ be distinct points of $X \times Y$. We shall define a continuum of type λ in $X \times Y$ that is irreducible between $(x(0), y(0))$ and $(x(1), y(1))$. According to Gordh's characterization (above) it suffices to construct a disjoint collection $\mathscr{D}$ of subcontinua of $X \times Y$ whose union M is a continuum irreducible between $(x(0), y(0))$ and $(x(1), y(1))$ such that each element of $\mathscr{D}$ has void interior relative to M and each element of $\mathscr{D}$ that misses $(x(0), y(0))$ and $(x(1), y(1))$ separates M.

To accomplish this let C be the Cantor ternary set in the unit interval $[0, 1]$.

Let D be the set of numbers in $C\setminus\{0, 1\}$ that are of the form $i/3^n$ with integers i and n. Note that D consists of the points of C that are accessible from $[0, 1]\setminus C$ [7, p. 175].

Let E be the set of all numbers in D that are of the form $i/3^n$ with i odd.

For each number r in E having simplest form $i/3^n$, let r^* be the number $(i+1)/3^n$ in $D\setminus E$. Note that r^* is the successor to r in C. We shall refer to r and r^* as opposite accessible points of C.

Let F be the set of all numbers in E having simplest form $i/3^n$ with n odd.

We assume without loss of generality that $x(0) \neq x(1)$ and X is irreducible between $x(0)$ and $x(1)$.

Let $\mathscr{A}$ be the unique arc from $\{x(0)\}$ to X in $\mathscr{C}(X)$, the hyperspace of closed connected subsets of X [4, Theorem 8.4, p. 34]. The elements of $\mathscr{A}$ are ordered by inclusion. Let $d_{\mathscr{A}}$ be the Hausdorff metric on $\mathscr{A}$.

Let d_X be the metric on X. Assume without loss of generality that $d_X(x(0), x(1)) > 2$.

Let h be an order-preserving homeomorphism of C into $\mathscr{A}$ such that $d_{\mathscr{A}}(h(0), \{x(0)\}) = 1$, $h(1) = X$, and for each number r in E that has simplest form $i/3^n$

$$d_{\mathscr{A}}\left(h(r), h\left(\frac{i-1}{3^n}\right)\right) < \frac{1}{4^n} \tag{1}$$

and

$$d_{\mathscr{A}}\left(h(r^*), h\left(\frac{i+2}{3^n}\right)\right) < \frac{1}{4^n}. \tag{2}$$

Since X is hereditarily indecomposable, it follows from (1) and (2) that there exist points $x(\frac{1}{3})$ of $h(\frac{1}{3})\setminus\bigcup\{h(r)\colon r < \frac{1}{3} \text{ and } r \in C\}$ and $x(\frac{2}{3})$ of $h(\frac{2}{3})\setminus\bigcup\{h(r)\colon r < \frac{2}{3} \text{ and } r \in C\}$ such that $d_X(x(0), x(\frac{1}{3})) = d_X(x(\frac{2}{3}), x(1)) = \frac{1}{4}$ [7, Theorem 2, p. 209 and Theorem 5, p. 212].

We proceed inductively. Assume that for each positive integer m less than a given integer n, if $r \in E$ and r has simplest form $i/3^m$, then points $x(r)$ and $x(r^*)$ have been defined such that

$$x(r) \in h(r)\setminus\bigcup\{h(s)\colon s < r \text{ and } s \in C\}, \tag{3m}$$

$$x(r^*) \in h(r^*)\setminus\bigcup\{h(s)\colon s < r^* \text{ and } s \in C\}, \tag{4m}$$

and

$$d_X\left(x(r), x\left(\frac{i-1}{3^m}\right)\right) = d_X\left(x(r^*), x\left(\frac{i+2}{3^m}\right)\right) = \frac{1}{4^m}. \tag{5m}$$

For each number r in E that has simplest form $i/3^n$, let $x(r)$ and $x(r^*)$ be points satisfying (3n)–(5n). Note that since $(i-1)/3^n$ and $(i+2)/3^n$ are not in simplest form, $x((i-1)/3^n)$ and $x((i+2)/3^n)$ in (5n) have been defined. By induction, for each number r in E, there exist such points $x(r)$ and $x(r^*)$.

Let $\mathscr{B}$ be the unique arc from $\{y(0)\}$ to Y in $\mathscr{C}(Y)$. The elements of $\mathscr{B}$ are ordered by inclusion. Let $d_{\mathscr{B}}$ be the Hausdorff metric on $\mathscr{B}$.

Let $I(0, 1)$ and $J(0, 1)$ be nonempty disjoint open subsets of Y.

Let $k(0)$ be an element of $\mathscr{B}\setminus\{Y\}$ that intersects $I(0, 1)$ and $J(0, 1)$.

Let $p(0)$, $p(1)$, $q(0)$, and $q(1)$ be points of Y such that $p(0) \in I(0, 1) \cap k(0)$, $q(0) \in J(0, 1) \cap k(0)$, $p(1) \in I(0, 1) \cap Y$, $q(1) \in J(0, 1) \cap Y$, and $\{p(1), q(1)\}$ misses $\bigcup\{B\colon B \in \mathscr{B}\setminus\{Y\}\}$.

Let $I(0, \frac{1}{3})$ and $I(\frac{2}{3}, 1)$ be open subsets of Y whoses closures are disjoint sets in $I(0, 1)$ with diameters less than $\frac{1}{3}$ such that $p(0) \in I(0, \frac{1}{3})$, $p(1) \in I(\frac{2}{3}, 1)$, and $k(0) \cap I(\frac{2}{3}, 1) = \emptyset$.

Define elements $k(\frac{1}{3})$ and $k(\frac{2}{3})$ of $\mathscr{B}\setminus\{Y\}$ and points $p(\frac{1}{3})$, $p(\frac{2}{3})$, and $q(\frac{1}{3})$ of Y such that $k(0) \subset k(\frac{1}{3}) \subset k(\frac{2}{3})$, $d_{\mathscr{B}}(k(0), k(\frac{1}{3})) < \frac{1}{3}$, $d_{\mathscr{B}}(k(\frac{2}{3}), Y) < \frac{1}{3}$, $k(\frac{1}{3}) \cap I(\frac{2}{3}, 1) = \emptyset$, $p(\frac{1}{3}) \in k(\frac{1}{3}) \cap I(0, \frac{1}{3})$, $p(\frac{2}{3}) \in k(\frac{2}{3}) \cap I(\frac{2}{3}, 1)$, $q(\frac{1}{3}) \in k(\frac{1}{3}) \cap J(0, 1)$, $\{p(\frac{1}{3}), q(\frac{1}{3})\}$ misses $\bigcup\{B\colon B < k(\frac{1}{3}) \text{ and } B \in \mathscr{B}\}$, and $k(\frac{2}{3})$ is irreducible between $p(\frac{2}{3})$ and $k(\frac{1}{3})$. Let $q(\frac{2}{3}) = q(\frac{1}{3})$.

Let $J(0, \frac{1}{9})$, $J(\frac{2}{9}, \frac{1}{3})$, and $J(\frac{8}{9}, 1)$ be open subsets of Y whose closures are disjoint sets in $J(0, 1)$ with diameters less than $\frac{1}{9}$ such that $q(0) \in J(0, \frac{1}{9})$, $q(\frac{1}{3}) \in J(\frac{2}{9}, \frac{1}{3})$, $q(1) \in J(\frac{8}{9}, 1)$, $k(0) \cap J(\frac{2}{9}, \frac{1}{3}) = \emptyset$, and $k(\frac{2}{3}) \cap J(\frac{8}{9}, 1) = \emptyset$. Let $J(\frac{2}{3}, \frac{7}{9}) = J(\frac{2}{9}, \frac{1}{3})$.

Define elements $k(\frac{1}{9})$, $k(\frac{2}{9})$, $k(\frac{7}{9})$, and $k(\frac{8}{9})$ of $\mathscr{B}\setminus\{Y\}$ and points $p(\frac{1}{9})$, $q(\frac{1}{9})$, $q(\frac{2}{9})$, $p(\frac{7}{9})$, $q(\frac{7}{9})$, and $q(\frac{8}{9})$ of Y such that

(i) $k(0) \subset k(\frac{1}{9}) \subset k(\frac{2}{9}) \subset k(\frac{1}{3}) \subset k(\frac{2}{3}) \subset k(\frac{7}{9}) \subset k(\frac{8}{9})$,

(ii) for $r = [, \frac{2}{9}, \frac{2}{3}$, and $\frac{8}{9}$, $d_{\mathscr{B}}(k(r), k(r+\frac{1}{0})) < \frac{1}{9}$ $(k(1) = Y)$.

(iii) for $r=\frac{1}{9}$ and $\frac{7}{9}$, $k(r)\cap J(r+\frac{1}{9}, r+\frac{2}{9})=\emptyset$,

$$p(r)\in k(r)\cap I(r-\tfrac{1}{9}, r+\tfrac{2}{9}),\qquad q(r)\in k(r)\cap J(r-\tfrac{1}{9}, r),$$

and $\{p(r), q(r)\}$ misses $\bigcup\{B\colon B<k(r) \text{ and } B\in\mathscr{B}\}$, and

(iv) for $r=\frac{2}{9}$ and $\frac{8}{9}$, $q(r)\in k(r)\cap J(r, r+\frac{1}{9})$.

Let $p(\frac{2}{9})=p(\frac{1}{9})$ and $p(\frac{8}{9})=p(\frac{7}{9})$.

Let $\mathscr{I}_1=\{I(0,\frac{1}{3}), I(\frac{2}{3},1)\}$ and $\mathscr{J}_1=\{J(0,\frac{1}{9}), J(\frac{2}{9},\frac{1}{3}), J(\frac{2}{3},\frac{7}{9}), J(\frac{8}{9},1)\}$. Let $S=\{0,\frac{1}{9},\frac{2}{9},\frac{1}{3},\frac{2}{3},\frac{7}{9},\frac{8}{9},1\}$. Let $K_1=\{k(r)\colon r\in S\}$, $P_1=\{p(r)\colon r\in S\}$, and $Q_1=\{q(r)\colon r\in S\}$.

Proceeding inductively, assume that for each positive integer m less than a given integer n we have defined two collections

$$\mathscr{I}_m=\left\{I\left(\frac{2i}{3^{2m-1}},\frac{2i+1}{3^{2m-1}}\right)\colon \frac{2i}{3^{2m-1}}\in C\right\}$$

and

$$\mathscr{J}_m=\left\{J\left(\frac{2i}{3^{2m}},\frac{2i+1}{3^{2m}}\right)\colon \frac{2i}{3^{2m}}\in C\right\}$$

of open sets in Y, a set

$$K_m=\left\{k\left(\frac{i}{3^{2m}}\right)\colon \frac{i}{3^{2m}}\in C\right\}$$

of elements of $\mathscr{B}$, and two subsets

$$P_m=\left\{p\left(\frac{i}{3^{2m}}\right)\colon \frac{i}{3^{2m}}\in C\right\}\quad\text{and}\quad Q_m=\left\{q\left(\frac{i}{3^{2m}}\right)\colon \frac{i}{3^{2m}}\in C\right\}$$

of Y such that

(6m) each element of $\mathscr{I}_m\cup\mathscr{J}_m$ has diameter less than $1/3^{2m-1}$,

(7m) the closures of distinct elements of $\mathscr{I}_m\cup\mathscr{J}_m$ are disjoint,

(8m) each element $I(r,s)$ of $\mathscr{I}_{m-1}$ contains the closure of each element $I(t,u)$ of $\mathscr{I}_m$ with $t\geqslant r$ and $u\leqslant s$,

(9m) each element $J(r,s)$ of $\mathscr{J}_{m-1}$ contains the closure of each element $J(t,u)$ of $\mathscr{J}_m$ with $t\geqslant r$ and $u\leqslant s$,

(10m) for each number $\frac{i}{3^{2m}}$ in $C\setminus\{1\}$, $k\left(\frac{i}{3^{2m}}\right)$ is a proper subcontinuum of $k\left(\frac{i+1}{3^{2m}}\right)$,

and

(11m) for each number $\frac{2i}{3^{2m}}$ in C, $d_{\mathscr{H}}\left(k\left(\frac{2i}{3^{2m}}\right), k\left(\frac{2i+1}{3^{2m}}\right)\right)<\frac{1}{3^{2m}}$.

Also for each number r in E of the form $i/3^{2m}$

(12m) $\{p(r), q(r)\}$ misses $\bigcup\{B\colon B<k(r) \text{ and } B\in\mathscr{B}\}$.

And for each number r in C of the form $i/3^{2m}$

(13m) $p(r)\in k(r)\cap I(s,t)$ $(s\leqslant r\leqslant t$ and $I(s,t)\in\mathscr{I}_m)$.

(14m) $q(r)\in k(r)\cap J(s,t)$ $(s\leqslant r\leqslant t$ and $J(s,t)\in\mathscr{J}_m)$.

Furthermore for each number r in F of the form $i/3^{2m-1}$

(15m) $k(r)$ misses each element $I(s,t)$ of $\mathscr{I}_m$ with $s>r$,

(16m) $k(r^*)$ is irreducible between $p(r^*)$ and $k(r)$,

(17m) $$q(r^*)=q(r),$$

and

(18m) if $\frac{i+5}{3^{2m-1}}\in C$, then $I\left(r^*,\frac{i+2}{3^{2m-1}}\right)=I\left(\frac{i+5}{3^{2m-1}},\frac{i+6}{3^{2m-1}}\right)$.

Moreover for each number r in $E\setminus F$ of the form $i/3^{2m}$

(19m) $k(r)$ misses each element $J(s,t)$ of $\mathscr{J}_m$ with $s>r$,

(20m) $$p(r^*)=p(r),$$

and

(21m) if $\frac{i+5}{3^{2m}}\in C$, then $J\left(r^*,\frac{i+2}{3^{2m}}\right)=J\left(\frac{i+5}{3^{2m}},\frac{i+6}{3^{2m}}\right)$.

Define two collections $\mathscr{I}_n$ and $\mathscr{J}_n$ of open subsets of Y, a set K_n of elements of $\mathscr{B}$, and two subsets P_n and Q_n of Y satisfying (6n)–(21n). By induction, for each positive integer n there exist such sets $\mathscr{I}_n$, $\mathscr{J}_n$, K_n, P_n, and Q_n.

Suppose $r_1, r_2, \ldots$ and $s_1, s_2, \ldots$ are sequences in D that converge to the same number in $C\setminus D$. Then by (5m), $x(r_1), x(r_2), \ldots$ and $x(s_1), x(s_2), \ldots$ converge to the same point of X. By (10m) and (11m), $k(r_1), k(r_2), \ldots$ and $k(s_1), k(s_2), \ldots$ converge to the same continuum in Y. By (6m), (8m), (13m), (18m), and (20m), $p(r_1), p(r_2), \ldots$ and $p(s_1), p(s_2), \ldots$ converge to the same point of Y. By (6m), (9m), (14m), (17m), and (21m), $q(r_1), q(r_2), \ldots$ and $q(s_1)$, $q(s_2), \ldots$ converge to the same point of Y.

For each number r in $C\setminus(D\cup\{0,1\})$, let $r_1, r_2, \ldots$ be a sequence in D converging to r, let $x(r)$ be the limit of $x(r_1), x(r_2), \ldots$, let $k(r)$ be the limit of $k(r_1), k(r_2), \ldots$, let $p(r)$ be the limit of $p(r_1), p(r_2), \ldots$, and let $q(r)$ be the limit of $q(r_1), q(r_2), \ldots$

By (5m),

(22) $\{x(r)\colon r\in C\}$ is a Cantor set in X.

By (10m) and (11m), k is an order preserving homeomorphism of C into $\mathscr{B}$.

It follows from (6m)–(9m), (13m), (14m), (17m), (18m), (20m), and (21m) that

(23) $\{p(r)\colon r\in C\}$ and $\{q(r)\colon r\in C\}$ are disjoint Cantor sets in Y.

By (8m)–(10m), (13m)–(15m), and (17m)–(21m),

(24) for each number r in $C\setminus D$, $\bigcup\{k(s)\colon s<r \text{ and } s\in C\}$ misses $p(r)$ and $q(r)$.

For each number r in F, let $M(r)$ be the continuum

$$\big(h(r)\times\{p(r)\}\big)\cup\big(\{x(r)\}\times k(r)\big)\cup\big(h(r^*)\times\{q(r)\}\big)\cup \\ \cup\big(\{x(r^*)\}\times k(r^*)\big)\cup\big(h(r^*)\times\{p(r^*)\}\big).$$

For each number r in $E\setminus F$, let $M(r)$ be the continuum

$$\big(h(r)\times\{q(r)\}\big)\cup\big(\{x(r)\}\times k(r)\big)\cup\big(h(r^*)\times\{p(r)\}\big)\cup \\ \cup\big(\{x(r^*)\}\times k(r^*)\big)\cup\big(h(r^*)\times\{q(r^*)\}\big).$$

Let

(25) $$M(r^*)=M(r)\quad\text{for each number } r^* \text{ in } D\setminus E.$$

For each number r in $C\setminus D$, let $M(r)$ be the continuum

$$\big(h(r)\times\{p(r)\}\big)\cup\big(\{x(r)\}\times k(r)\big)\cup\big(h(r)\times\{q(r)\}\big).$$

Suppose $r_1, r_2, \ldots$ is a convergent sequence in C. It follows from (5m)–(9m), (13m), (14m), (17m), (18m), (20m), (21m), and the definitions of h and k that

(26) if $r_1, r_2, \ldots$ converges to a number r in $C\setminus D$, then $M(r)$ is the limit of $M(r_1), M(r_2), \ldots$,

(27) if $r_1, r_2, \ldots$ converges to a number r in E, then the subcontinuum $\big(h(r)\times\{p(r)\}\big)\cup\big(\{x(r)\}\times k(r)\big)\cup\big(h(r)\times\{q(r)\}\big)$ of $M(r)$ is the limit of $M(r_1), M(r_2), \ldots$,

and

(28) if $r_1, r_2, \ldots$ converges to a number r^* in $D\setminus E$, then the subcontinuum $\big(h(r^*)\times\{p(r^*)\}\big)\cup\big(\{x(r^*)\}\times k(r^*)\big)\cup\big(h(r^*)\times\{q(r^*)\}\big)$ of $M(r)$ is the limit of $M(r_1), M(r_2), \ldots$

By (3m)–(5m), (7m)–(9m), (12m)–(15m), (17m)–(21m), (25), and the definitions of h and k

(29) if r and s are distinct numbers in C and $M(r)\cap M(s)\neq\emptyset$, then r and s are opposite accessible points of C and $M(r)=M(s)$.

Let $M=\bigcup\{M(r)\colon r\in C\}$. By (26)–(28),

(30) M is compact.

Note that

(31) M is connected.

To see assume the contrary. Then, by (30), M is the union of two disjoint nonempty closed sets V and W. Since each $M(r)$ is connected, each $M(r)$ is a subset of V or W. Assume without loss of generality that $M(1)\subset W$. The existence of the least upper bound of $\{r\colon M(r)\subset V$ and $r\in C\}$ contradicts (25)–(28). Hence (31) is true.

Next we show that

(32) if G is a nonempty open subset of M missing $M(0)$ and $M(1)$, then G separates M between $M(0)$ and $M(1)$.

To accomplish this let T and U be open sets in X and Y, respectively, such that $M\cap(T\times U)$ is a nonempty subset of G. Since $I(0,1)\cap J(0,1)=\emptyset$, we can assume without loss of generality that U misses $I(0,1)$ or $J(0,1)$.

Let $P=\{(x(r),p(r))\colon r\in C\}$ and let $Q=\{(x(r),q(r))\colon r\in C\}$. By (22) and (23), P and Q are Cantor sets. Hence we can also assume without loss of generality that $T\times U$ misses $P\cup Q$.

Since $\bigcup\{M(r)\colon r\in D\}$ is dense in M, one of the following three cases holds.

Case 1. Suppose there is a number r_1 in D such that $(T\times U)\cap\cap\big(h(r_1)\times\{q(r_1)\}\big)\neq\emptyset$. By (6m), (9m), and (14m), there is a number r_2 in F such that $(T\times U)\cap\big(h(r_2)\times\{q(r_2)\}\big)\neq\emptyset$.

The $x(r_2^*)$-component L of $X\setminus T$ misses $h(r_2)$; for otherwise, $L\cup h(r_2)$ is a decomposable continuum and this contradicts the assumption that X is hereditarily indecomposable. Hence there exist disjoint closed sets V and W such that $V\cup W=X\setminus T$, $h(r_2)\setminus T\subset V$, and $x(r_2^*)\in W$ [7, Theorem 3, p. 170]. By (5m), (6m), (9m), (14m), and (17m), there is a number r_3 in $E\setminus F$ such that $r_2<r_3$, $\{x(r)\colon r_2<r\leqslant r_3$ and $r\in C\}\subset W$, and $\{q(r)\colon r_2\leqslant r\leqslant r_3$ and $r\in C\}\subset U$.

Let H be the union of $\big(\bigcup\{M(r)\colon r<r_2$ and $r\in C\}\big)\setminus(T\times U)$, $h(r_2)\times\times\{p(r_2)\}$, $\{x(r_2)\}\times k(r_2)$, and $\bigcup\{(h(r)\cap V)\times\{q(r)\}\colon r_2\leqslant r\leqslant r_3$ and $r\in C\}$.

Since U misses $\{p(r)\colon r\in C\}$, H is a closed open subset of $M\setminus(T\times U)$. Furthermore H contains $M(0)$ and misses $M(1)$. Hence (32) is true.

Case 2. Suppose there is a number r_1 in D such that $(T\times U)\cap\cap\big(h(r_1)\times\{p(r_1)\}\big)\neq\emptyset$. Let r_2 be a number in $E\setminus F$ such that $(T\times U)\cap\cap\big(h(r_2)\times\{p(r_2)\}\big)\neq\emptyset$ and follow the argument given in Case 1.

Case 3. Suppose there is a number r_1 in D such that $(T\times U)\cap\cap\big(\{x(r_1)\}\times k(r_1)\big)\neq\emptyset$. By (5m), there is a number r_2 in F such that

$(T\times U)\cap(\{x(r_2^*)\}\times k(r_2^*))\neq\emptyset$. By (16m), $k(r_2^*)$ is irreducible between $p(r_2^*)$ and $k(r_2)$.

The $p(r_2^*)$-component L of $Y\backslash U$ misses $k(r_2)$; for otherwise, $L\cup k(r_2^*)$ is a decomposable continuum and this contradicts the assumption that Y is hereditarily indecomposable. Hence there exist disjoint closed sets V and W such that $V\cup W=Y\backslash U$, $k(r_2)\backslash U\subset V$, and $p(r_2^*)\in W$. By (5m), (6m), (8m), (9m), (13m), (14m), (17m) and (20m), there is a number r_3 in $E\backslash F$ such that $r_2<r_3$, $\{p(r)\colon r_2<r\leqslant r_3 \text{ and } r\in C\}\subset W$, $\{q(r)\colon r_2\leqslant r\leqslant r_3 \text{ and } r\in C\}\subset V$, and $\{x(r)\colon r_2<r\leqslant r_3 \text{ and } r\in C\}\subset T$.

Let H be the union of

$$(\bigcup\{M(r)\colon r<r_2 \text{ and } r\in C\})\backslash(T\times U),\qquad h(r_2)\times\{p(r_2)\},$$

$$\bigcup\{\{x(r)\}\times(k(r)\cap V)\colon r_2\leqslant r\leqslant r_3 \text{ and } r\in C\},$$

and

$$\bigcup\{h(r)\times\{q(r)\}\colon r_2\leqslant r\leqslant r_3 \text{ and } r\in C\}.$$

Since H is a closed open subset of $M\backslash(T\times U)$ that contains $M(0)$ and misses $M(1)$, (32) is true.

By (30) and (31), M is a continuum. Since $(x(0),y(0))\in M(0)$ and $(x(1),y(1))\in M(1)$, it follows from (26) and (32) that M is irreducible between these points. By (29), $\mathscr{D}=\{M(r)\colon r\in C\backslash(D\backslash E)\}$ is a disjoint collection of continua whose union is M. By (26)-(28), each element of $\mathscr{D}$ is a continuum of condensation in M and each element of $\mathscr{D}\backslash\{M(0),M(1)\}$ separates M. Thus M is a continuum of type λ [1, Theorem 2.7, p. 650]. Hence $X\times Y$ is λ-connected.

Question. Is every product of continua λ-connected?

A continuum M is *aposyndetic at a point* p if for each point q of $M\backslash\{p\}$ there is a continuum neighborhood of p in M that misses q. A continuum is *aposyndetic* if it is aposyndetic at each of its points.

Jones [3, Theorem 7, p. 406] proved that the product of any two continua is aposyndetic. Hence one might hope to answer the question above by showing that every aposyndetic continuum is λ-connected. This approach fails.

Example. There is an aposyndetic continuum M in Euclidean 3-space that is not λ-connected. Let K be Knaster's simplest indecomposable plane continuum [7, Example 1, p. 204]. Let L be the set of all points p in K such that p is an endpoint of a semicircle in K. Let M be the continuum $(K\times\{0,1\})\cup(L\times[0,1])$ in $K\times[0,1]$.

There is a base $\mathscr{E}$ of M with the property that for each element E of $\mathscr{E}$, $M\backslash E$ is connected. Hence M is aposyndetic.

To see that M is not λ-connected, let p and q be points of distinct composants of K. Let V be a continuum in M that contains $\{(p,0),(q,0)\}$. It suffices to show that V has an indecomposable subcontinuum that contains a nonempty open subset of M.

Let π be the projection function of $K\times[0,1]$ onto K. Note that $\pi(V)=K$. Let W be a subcontinuum of V that is irreducible with respect to being mapped onto K by π. It follows that W is indecomposable [7, Theorem 4, p. 208].

Let G be an open set in K whose closure $\mathrm{Cl}\,G$ misses L. Let

$$A=W\cap\pi^{-1}(\mathrm{Cl}\,G)\cap(K\times\{1\})\quad\text{and}\quad B=W\cap\pi^{-1}(\mathrm{Cl}\,G)\cap(K\times\{0\}).$$

Since $\pi(A)$ and $\pi(B)$ are closed sets whose union is $\mathrm{Cl}\,G$, either $\pi(A)$ or $\pi(B)$ contains a nonempty open subset of K. Assume without loss of generality that $\pi(A)$ contains a nonempty open subset S of K. Since $L\cap\mathrm{Cl}\,G=\emptyset$, $A\cap\pi^{-1}(S)$ is a nonempty open subset of M. Hence the indecomposable subcontinuum W of V contains a nonempty open subset of M. Therefore M is not λ-connected.

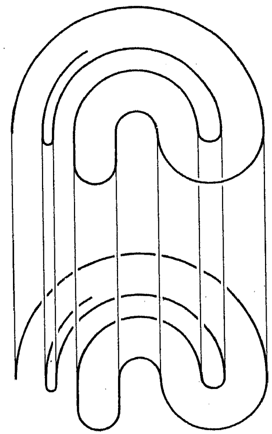

Fig. 1

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DEPARTMENT OF MATHEMATICS
CALIFORNIA STATE UNIVERSITY
Sacramento, California 95819

Accepté par la Rédaction le 17. 8. 1981

Completely distributive lattices

by

M. S. Lambrou (Iraklion — Crete)

Abstract. If a complete lattice with 0 and 1 satisfies the infinite distributivity laws it is called completely distributive. In this paper we give simple proofs of known characterizations of complete distributivity as well as new characterizations in terms of maps from the lattice to itself satisfying the condition $a = \bigvee\{b/a \nleq p(b)\}$ for all a in the lattice, where $p\colon L \to L$ is the map.

1. Introduction. Although the motivation for the results of this paper, whose purpose is to study complete distributivity of lattices, arise from Functional Analysis, we shall keep the theorems and their proofs lattice theoretic. In Functional Analysis, and more specifically in the study of invariant subspaces of operators on a normed vector space H, one examines conditions on a set L of subspaces of H to be *reflexive* in the sense that it coincides with the family of subspaces that are invariant under each operator that leaves invariant the elements of the set (see Radjavi and Rosenthal [13] for the relavant definitions). A necessary, but far from sufficient, condition for the reflexivity of L is that L is a complete lattice (under the usual lattice operations on subspaces). There are several sufficient conditions known. For instance Ringrose in [17] has shown that every complete totally ordered lattice of subspaces of a Hilbert space (*complete nest* in his terminology) is reflexive. Halmos in [7] has shown that complete atomic Boolean lattices of subspaces are also reflexive. Both these examples are examples of completely distributive lattices. Longstaff in [12] has shown that in fact complete and completely distributive lattices of subspaces of Hilbert spaces are reflexive, and so he extended the previous two cases. A necessary and sufficient condition for a complete and completely distributive lattice to be a complete atomic Boolean lattice is given in [10]. Another equivalent condition, but this time Functional Analytic, is given in [9]. It is easy to see that if the underlying Hilbert space is finite dimensional then a lattice is complete and completely distributive if and only if it is distributive. In the finite dimensional Hilbert space case R. Johnson in [8] has shown that a necessary and sufficient condition for a finite lattice to be reflexive is that it is distributive. In general Hilbert spaces neither of these two conditions implies the other. Indeed, Halmos [7] constructed a reflexive lattice which is (lattice) isomorphic to the non-modular pentagon M_5. An example in the opposite direction is due to Conway ([6]) who constructed a non-reflexive complete lattice