$\mathscr{C}_p$-Movably regular convergences

by

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Abstract. This paper introduces the notions of a weakly $\mathscr{C}_p$-movably and a $\mathscr{C}_p$-movably regular convergence of compacta lying in a metric space X, where $\mathscr{C}_p$ is an arbitrary class of metrizable pairs. The definitions are motivated by the desire to get conditions under which will the limit in Borsuk's fundamental metric of a sequence of compacta in X be (weakly) $\mathscr{C}_p$-movable, a shape invariant property generalizing the notion of the (FANR) pointed FANR.

We prove a number of results which describe properties of (weakly) $\mathscr{C}_p$-movably regularly converging sequences of compacta and present several examples of situations where these types of convergences appear.

The main theorems show that it is possible to define a metric on the collection of all (weakly) $\mathscr{C}_p$-movable compacta in X which preserves (weakly) $\mathscr{C}_p$-movably regular convergence.

1. Introduction. This is the third paper in a series in which we study various types of globally regular convergences of compacta. In the previous two we considered $\mathscr{C}$-movably regular convergence [Č3] and $\mathscr{C}$-calmly regular convergence [Č4].

The present paper first introduces, motivated by [Č5], shape invariant properties weak $\mathscr{C}_p$-movability and $\mathscr{C}_p$-movability for compacta, where $\mathscr{C}_p$ is a class of metrizable pairs. They represent generalizations of the notions of the fundamental absolute neighborhood retract (FANR) and of the pointed FANR, respectively. On the other hand, they include the notions of $\mathscr{C}$-movability [ČŠ] and $\mathscr{C}$-calmness [Č2].

Then we define the concepts of a weakly $\mathscr{C}_p$-movably regular (or mo $(\mathscr{C}_p)$-regular) convergence and of a $\mathscr{C}_p$-movably regular (or mo $(\mathscr{C}_p^*)$-regular) convergence of compacta in a metric space X. The idea is to require that the sequence $\{A_1, A_2, \ldots\}$ of compacta in X approximate the limit A_0 more and more closely in the sense of (weak) $\mathscr{C}_p$-movability.

The definitions of the mo $(\mathscr{C}_p)$-regular and mo $(\mathscr{C}_p^*)$-regular convergence, besides providing generalizations of many theorems about (weak) $\mathscr{C}_p$-movability (see §§ 3 and 4), is justified by the new information that these notions give about the collections mo $(\mathscr{C}_p, X)$ and mo $(\mathscr{C}_p^*, X)$ of all weakly $\mathscr{C}_p$-movable and $\mathscr{C}_p$-movable compacta in X, respectively. The main results (6.6) and (6.7) of this paper show that one can define metrics d_1 on mo $(\mathscr{C}_p, X)$ and d_2 on mo $(\mathscr{C}_p^*, X)$ such that $A_n \to A_0$ with respect to d_1 iff $A_n \to A_0$ mo $(\mathscr{C}_p)$-regularly and $\lim d_Z(A_n, A_0) = 0$, and $A_n \to A_0$ with respect to d_2 iff $A_n \to A_0$ mo $(\mathscr{C}_p^*)$-regularly and $\lim d_Z(A_n, A_0) = 0$. Here d_Z denotes the metric on the hyperspace 2^X of all non-empty compacta in X closely related to Borsuk's fundamental metric d_F (see § 3).

Our proof that metrics d_1 and d_2 exist is based on the idea from [Č3] and [Č4] and thus also relies heavily on Begle's method from [Be].

In §§ 3-5 we prove many interesting properties of sequences $\{A_n\}$ of compacta in X that converge (weakly) $\mathscr{C}_p$-movably regularly to a compactum A_0 in X. We present several examples of situations where such sequences appear naturally. In doing this, we improve some results in [Č3], [Č4], and [Č5] and also introduce new concepts (metrics d_Z, d_{Z^n}, Z-domination, and quasi-domination for metrizable pairs) which generalize corresponding notions defined recently by Borsuk.

The paper is self-contained but it is preferable if the reader is familiar with author's papers [Č3], [Č4], and [Č5].

2. Preliminaries. Throughout the paper $\mathscr{C}$ and $\mathscr{D}$ will be arbitrary (non-empty) classes of topological spaces. By $\mathscr{P}$ $(\mathscr{P}^n)$ we denote the class of all compact ANR's for the class $\mathscr{M}$ of all metrizable spaces (of dimension $\leqslant n$), and $\mathscr{S}^n$ denotes the class of spheres $\{S^0, S^1, \ldots, S^n\}$. A map will be called a *$\mathscr{C}$-map* provided its domain is a member of a class $\mathscr{C}$.

We reserve $\mathscr{C}_p$ and $\mathscr{D}_p$ for arbitrary (non-empty) classes of metrizable pairs (i.e., pairs (K, K_0) where K is a metrizable space and K_0 is a closed subset of K). We restricted ourselves to metrizable pairs only because we need in our arguments the homotopy extension theorem (HET) [H, p. 117]. Hence, as long as this theorem is true for members of a class $\mathscr{C}_p$ of pairs of topological spaces our results remain true for these more general classes $\mathscr{C}_p$. The special classes of pairs used in the paper are $\mathscr{P}_p$ (the class of all pairs of compact ANR's), $\mathscr{P}_p^n$ (the class of all pairs of compact ANR's of dimension $\leqslant n$), $\mathscr{K}_p^n$ (the class of all pairs (K, K_0) where K is a finite simplicial complex of dimension $\leqslant n$ and K_0 is a subcomplex of K), and $\mathscr{B}_p^n$ (the class $\{(B^1, S^0), (B^2, S^1), \ldots, (B^n, S^{n-1})\}$, where B^k is the k-dimensional unit solid ball and S^{k-1} is its boundary $(k-1)$-sphere).

Let $\mathscr{C}_p$ be a class of pairs. We associate to it two classes of spaces as follows. $\mathscr{C}_p' = \{K \mid \exists K_0$ such that $(K, K_0) \in \mathscr{C}_p\}$ and $\mathscr{C}_p'' = \{K_0 \mid \exists K$ such that $(K, K_0) \in \mathscr{C}_p\}$. Conversely, if $\mathscr{C}$ is a class of spaces, let $\mathscr{C}_H$ denote the class of pairs $\{(K \times I, K \times \{0\} \cup K \times \{1\}) \mid K \in \mathscr{C}\}$, where $I = [0, 1]$ is the unit interval.

A map of pairs is a *$\mathscr{C}_p$-map* provided its domain is a member of a class $\mathscr{C}_p$.

Let U and V be subsets of a space X, $V \subset U$. Then $i_{V,U}$ denotes the inclusion of V into U. Two maps of pairs $f, g\colon (K, K_0) \to (U, V)$ are *homotopic in* (U, V) if there is a map of pairs $H\colon (K \times I, K_0 \times I) \to (U, V)$ such that $H_0 = f$ and $H_1 = g$.

We shall say that maps f and g of a space Z into a metric space (Y, d) are *ε-close* provided $d(f(z), g(z)) < \varepsilon$ for every $z \in Z$. If Z is a subset of Y and f is ε-close to the inclusion $i_{Z,Y}$, we call f an *ε-map*. We shall repeatedly use the following property of a compact ANR Y [H, p. 111]. For every $\varepsilon > 0$ there is an $\eta > 0$ such that every two η-close maps f and g of a space Z into Y are ε-homotopic in Y, i.e., there is a homotopy $H\colon Z \times I \to Y$ (called an *ε-homotopy*) between f and g with H_0 and H_t being ε-close for every $t \in I$. In particular, there is an $\varepsilon > 0$ such that every two ε-close maps into Y are homotopic in Y.

If not stated otherwise, we reserve X for an arbitrary metric space with a fixed metric d; $A_0, A_1, A_2, \ldots$ are compact subsets of X; d_H is the Hausdorff metric on the hyperspace 2^X of all non-empty compacta in X; M is an ANR for the class of all metrizable spaces which contains X metrically; a neighborhood means an open neighborhood; and $N(\varepsilon, A_0)$ denotes the ε-neighborhood of A_0 in M.

3. $\mathscr{C}_p$-movably regular convergences. Let B be a subset of a space M and let U and V, $V \subset U$, be neighborhoods of B in M. We denote by $\mathscr{C}(U, V; B)$, $\mathscr{C}_h(U, V; B)$, and $\mathscr{C}_p(U, V; B)$ the following statements.

$\mathscr{C}(U, V; B)$ For every neighborhood W of B in M and a $\mathscr{C}$-map $f\colon K \to V$ there is a homotopy $f_t\colon K \to U$, $0 \leqslant t \leqslant 1$, with $f_0 = f$ and $f_1(K) \subset W$.

$\mathscr{C}_h(U, V; B)$ For every neighborhood W of B in M, there is a neighborhood W_0 of B in M, $W_0 \subset V \cap W$, such that $\mathscr{C}$-maps into W_0 which are homotopic in V are also homotopic in W.

$\mathscr{C}_p(U, V; B)$ For every neighborhood W of B in M, there is a neighborhood W_0 of B in M, $W_0 \subset V \cap W$, such that for every $\mathscr{C}_p$-map $f\colon (K, K_0) \to (V, W_0)$ there is a homotopy $f_t\colon K \to U$, $0 \leqslant t \leqslant 1$, with $f_0 = f$, $f_1(K) \subset W$, and $f_1 | K_0 = f | K_0$.

We shall denote by $\mathscr{C}_p^*(U, V; B)$ a stronger statement which differs from the statement $\mathscr{C}_p(U, V; B)$ only in the fact that the homotopy f_t maps K_0 into W, i.e. f_t is a homotopy of pairs $f_t\colon (K, K_0) \to (U, W)$.

A compactum A is (*weakly*) *$\mathscr{C}_p$-movable* if for some, and hence for every, embedding of A into an ANR M the following holds. For each neighborhood U of A in M there is a neighborhood V of A in M, $V \subset U$, such that $\mathscr{C}_p^*(U, V; A)$ $(\mathscr{C}_p(U, V; A))$ is true.

Observe that a Z-set A in the Hilbert cube Q is (weakly) $\mathscr{C}_p$-movable iff $Q - A$ is (weakly) $\mathscr{C}_p$-movable at ∞ [Č5]. Hence, a compactum A is an FANR (a pointed FANR) iff A is weakly $\mathscr{P}_p$-movable ($\mathscr{P}_p$-movable) [Č5, Theorem (3.3)]. Also, note that every connected compactum is weakly $\{(I, \{0\} \cup \{1\})\}$-movable and that every $\{(I, \{0\} \cup \{1\})\}$-movable compactum must be $\{S^1\}$-movable [ČŠ]. Hence, the dyadic solenoid is an example of a weakly $\{(I, \{0\} \cup \{1\})\}$-movable space which is not $\{(I, \{0\} \cup \{1\})\}$-movable.

(3.1) DEFINITION. A sequence $A_1, A_2, \ldots$ of compacta in a metric space X which lies in an ANR M converges *(weakly) $\mathscr{C}_p$-movably regularly* (or mo $(\mathscr{C}_p^*)$*-regularly* (mo $(\mathscr{C}_p)$*-regularly*)) *in* M to a compactum $A_0 \subset X$ provided for every neighborhood U of A_0 in M there is a neighborhood V of A_0 in M, $V \subset U$, such that $\mathscr{C}_p^*(U, V; A_n)$ $(\mathscr{C}_p(U, V; A_n))$ holds for almost all indices n.

It can be routinely proved that the definition (3.1) is shape theoretic in the sense that (weakly) $\mathscr{C}_p$-movably regular convergence is independent of the choice of M and the embedding of X into M. We shall write $A_n - \text{mo}\,(\mathscr{C}_p^*) \to A_0$ $(A_n - \text{mo}\,(\mathscr{C}_p) \to A_0)$ to indicate that the sequence $\{A_n\}$ of compacta in X converges (weakly) $\mathscr{C}_p$-movably regularly to a compactum A_0 in X in some, and hence in every, ANR containing X.

A careful examination of the proofs of (2.9) in [Č3], (2.4) in [Č4], and (4.3), (4.4) in [Č4] reveals that they remain true if we replace the condition $\lim d_F(A_n, A_0) = 0$ with weaker conditions $\lim Z_{A_0}(A_n) = 0$ and $\lim d_Z(A_n, A_0) = 0$, respectively, which we define next.

Let A and B be compacta in a metric space X which lies in an ANR M. Let $Z_{A,M}(B)$ denote the infimum of those $\varepsilon > 0$ such that for every neighborhood U of B in M there is an ε-map of A into U. Put $d_{Z,M}(A, B) = \max\{Z_{A,M}(B), Z_{B,M}(A)\}$. One easily proves that the value $Z_{A,M}(B)$ (and therefore also the value $d_{Z,M}(A, B)$) does not depend on the choice of the space M. Hence, we can drop M from our notation. It is clear that d_Z is a metric on 2^X which is stronger than d_H and weaker than the fumdamental metric d_F [B2].

(3.2) THEOREM. *Let $\{A_k\}_{k=0}^{\infty}$ be a sequence in 2^X. If $\lim Z_{A_0}(A_n) = 0$ and* $(A_n - \text{mo}\,(\mathscr{C}_p) \to A_0)$ $A_n - \text{mo}\,(\mathscr{C}_p^*) \to A_0$, *then A_0 is (weakly) $\mathscr{C}_p$-movable.*

Proof. We shall consider only a slightly more complicated case of the mo $(\mathscr{C}_p)$-regular convergence.

As in [Č4], we can assume that $A = \bigcup_{n=0}^{\infty} A_n$ lies in the Hilbert cube Q. Let U be a neighborhood of A_0 in Q. Pick a neighborhood V' of A_0 in Q and an index $k_{V'}$ such that $\mathscr{C}_p(U, V'; A_k)$ holds for all $k \geqslant k_{V'}$. Inside V' take compact ANR neighborhoods V_1 and V_2 such that $V_2 \subset \text{int}\, V_1 \subset V_1 \subset V'$ and let $\varepsilon_1 = d(Q - \text{int}\, V_1, V_2)$. Put $V = \text{int}\, V_2$. We claim that $\mathscr{C}_p(U, V; A_0)$ is true.

Indeed, let W be an arbitrary open neighborhood of A_0. Let W', $W' \subset V$, be a compact ANR neighborhood of A_0, and let $\varepsilon_2 = d(Q - W, W')$. Choose an $\eta > 0$ such that η-close maps into V_1 are $\min\{\varepsilon_1, \varepsilon_2\}$-homotopic in V_1 and then pick $k \geqslant k_{V'}$ so that $Z_{A_0}(A_k) < \eta$ and $A_k \subset W'$. Let W_0' be a neighborhood of A_k, $W_0' \subset W \cap V$, chosen with respect to W using $\mathscr{C}_p(U, V'; A_k)$. Pick an η-map $f\colon A_0 \to W_0'$ and let W_0 be a neighborhood of A_0, $W_0 \subset W \cap V$, such that f extends to an η-map $\tilde{f}\colon W_0 \to W_0'$.

Suppose $g\colon (K, K_0) \to (V, W_0)$ is a $\mathscr{C}_p$-map. Since $\tilde{f} \circ g|K_0$ and $g|K_0$ are η-close maps into V_1, they are $\min\{\varepsilon_1, \varepsilon_2\}$-homotopic in V_1 and hence homotopic in W. By the HET, there is a homotopy $g_t\colon (K, K_0) \to (V_1, W)$, $0 \leqslant t \leqslant 1/3$, such that $g_0 = g$ and $g_{1/3}|K_0 = \tilde{f} \circ g|K_0$. Let $f_t\colon K \to U$, $1/3 \leqslant t \leqslant 2/3$, be a homotopy satisfying $f_{1/3} = g_{1/3}$, $f_{2/3}(K) \subset W$, and $f_{2/3}|K_0 = g_{1/3}|K_0$. Applying the HET again, this time to the map $f_{2/3}$ and the partial homotopy $g_{1-t}|K_0$, $2/3 \leqslant t \leqslant 1$, we shall get a homotopy $h_t\colon K \to W$, $2/3 \leqslant t \leqslant 1$, with $h_{2/3} = f_{2/3}$ and $h_1|K_0 = g|K_0$. The join of homotopies g_t, f_t, and h_t shows that $\mathscr{C}_p(U, V; A_0)$ holds. Hence, A_0 is weakly $\mathscr{C}_p$-movable.

(3.3) EXAMPLES. (a) A constant sequence $\{A\}$ converges (weakly) $\mathscr{C}_p$-movably regularly to A iff A is (weakly) $\mathscr{C}_p$-movable.

(b) In the interval $X = [-1, 1]$ consider a sequence $\{A_n\}$ where $A_n = \{-1/n\} \cup \{1/n\}$ $(n = 1, 2, \ldots)$. Put $A_0 = \{0\}$. Then $\lim d_Z(A_n, A_0) = 0$ and A_0 is $\mathscr{C}_p$-movable for every class $\mathscr{C}_p$, but $\{A_n\}$ does not converge mo $(\mathscr{C}_p)$-regularly to A_0. Hence, the converse of (3.2) is not true.

(c) In the interval $X = [-1, 1]$ consider compacta $A = \{0\} \cup \{1/n \mid n = 1, 2, \ldots\}$ and $B = \{0\}$. The constant sequence $\{B\}$ converges $\mathscr{C}_p$-movably regularly to A for any class $\mathscr{C}_p$ such that $\mathscr{C}_p'$ consists of connected spaces. Since A is not weakly $\mathscr{C}_p$-movable for any such class $\mathscr{C}_p$ satisfying $\mathscr{C}_p'' \neq \varnothing$, this example shows that the condition $\lim Z_{A_0}(A_n) = 0$ in (3.2) is necessary.

Now we discuss the role of a class $\mathscr{C}_p$ in Definition (3.1). Theorems (2.8) in [Č4] and (2.6) in [Č3] suggest that only quantitative shape properties of pairs in $\mathscr{C}_p$ effect (weakly) $\mathscr{C}_p$-movably regular convergence. Indeed, using Fox's mutations between pairs of metrizable spaces instead of fundamental sequences in Borsuk's definitions in [B1], one can introduce a notion of quasi-domination for classes of pairs of metrizable spaces and prove easily the following theorem.

(3.4) THEOREM. *Let a sequence $\{A_n\}$ of compacta in a metric space X converge (weakly) $\mathscr{C}_p$-movably regularly to a compactum A_0 in X and let $\mathscr{C}_p$ quasi-dominate a class $\mathscr{D}_p$. Then $\{A_n\}$ also converges (weakly) $\mathscr{D}_p$-movably regularly to A_0.*

Recall now definitions of $\mathscr{C}$-movably and $\mathscr{C}$-calmly regular (or mo $(\mathscr{C})$-regular and ca $(\mathscr{C})$-regular) convergence ([Č3] and [Č4]). They differ from Definition (3.1) only in the fact that $\mathscr{C}(U, V; A_n)$ and $\mathscr{C}_h(U, V; A_n)$, respectively, replace $\mathscr{C}_p^*(U, V; A_n)$.

(3.5) THEOREM. (a) *If $A_n - \text{mo}\,(\mathscr{C}_p) \to A_0$, then $A_n - \text{ca}\,(\mathscr{D}) \to A_0$, where $\mathscr{D}$ is any class for which $\mathscr{D}_H \subset \mathscr{C}_p$.*

(b) *If $A_n - \text{mo}\,(\mathscr{C}_p') \to A_0$ and $A_n - \text{ca}\,(\mathscr{C}_p'') \to A_0$, then $A_n - \text{mo}\,(\mathscr{C}_p) \to A_0$.*

Proof. The proof of (a) is obvious while (b) can be proved by the method of the proof of Theorem (4.5) in [ČŠ].

(3.6) Remark. Example (3.3) (c) shows that $A_n - \text{mo}\,(\mathscr{C}_p^*) \to A_0$ does not imply $A_n - \mathrm{mo}(\mathscr{C}'_p) \to A_0$ in general. This implication will be true, for example, if $A_n - \mathrm{mo}(\mathscr{C}''_p) \to A_0$, or if the class $\mathscr{C}_p$ has the property that for every $K \in \mathscr{C}'_p$ there exists a space $K_0 \in \mathscr{C}''_p$ such that $K \cap K_0 = \emptyset$ and $(K \cup K_0, K_0) \in \mathscr{C}_p$.

(3.7) COROLLARY. *$A_n - \mathrm{mo}(\mathscr{P}_p) \to A_0$ iff $A_n - \mathrm{mo}(\mathscr{P}) \to A_0$ and $A_n - \mathrm{ca}(\mathscr{P}) \to A_0$.*

We shall give now several examples of situations in which $\mathrm{mo}(\mathscr{C}_p)$-regular and $\mathrm{mo}(\mathscr{C}^*_p)$-regular convergences appear naturally.

A compactum A is *$\mathscr{C}$-trivial* [ČŠ] if, for some (and hence for every) embedding of A into an ANR M, for each neighborhood U of A in M there is a smaller neighborhood V of A in M such that every $\mathscr{C}$-map into V is null-homotopic in U. Observe that A has trivial shape iff A is $\mathscr{P}$-trivial [Č1].

We write $\lim N_{A_0}(A_n) = 0$ if for every neighborhood U of A_0 in X almost all A_n are contained in U.

(3.8) EXAMPLE. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of connected $\mathscr{C}''_p$-trivial compacta in X and let $A_0 \subset X$ be a $\mathscr{C}'_p$-trivial compactum. If $\lim N_{A_0}(A_n) = 0$, then $A_n - \mathrm{mo}(\mathscr{C}_p) \to A_0$.

Proof. The assumptions imply $A_n - \mathrm{mo}(\mathscr{C}'_p) \to A_0$ and $A_n - \mathrm{ca}(\mathscr{C}''_p) \to A_0$. Hence, we can apply (3.5) (b).

(3.9) EXAMPLE. Let $\{A_n\}_{n=0}^{\infty}$ be a sequence of compacta of trivial shape in X and assume that $\lim N_{A_0}(A_n) = 0$. Then $A_n - \mathrm{mo}(\mathscr{C}^*_p) \to A_0$ for every class $\mathscr{C}_p$.

Proof. Assume that a compactum $\bigcup_{n=0}^{\infty} A_n$ lies in Q and let U be a neighborhood of A_0 in Q. Pick a neighborhood V of A_0 inside U which is homeomorphic to Q [Ch] and let n_0 be such that $n \geqslant n_0$ implies $A_n \subset \mathrm{int}\, V$. We claim that $\mathscr{C}^*_p(U, V; A_n)$ holds for all $n \geqslant n_0$.

Indeed, let $n \geqslant n_0$ and let W be a neighborhood of A_n in Q. Select a Hilbert cube neighborhood W_0 of A_n in $V \cap W$. Clearly, every $\mathscr{C}_p$-map $f\colon (K, K_0) \to (V, W_0)$ deforms inside (V, W_0) (keeping $f|K_0$ fixed) to a map into W_0.

(3.10) EXAMPLE. If $\{A_n\}_{n=0}^{\infty}$ is a sequence of $\mathscr{S}^k$-trivial compacta in X and $\lim N_{A_0}(A_n) = 0$, then $A_n - \mathrm{mo}((\mathscr{P}^k_p)^*) \to A_0$.

Proof. Since the class $\mathscr{K}^k_p$ of all pairs of finite simplicial complexes of dimension $\leqslant k$ quasi-dominates the class $\mathscr{P}^k_p$, by (3.4), it suffices to prove that $A_n - \mathrm{mo}((\mathscr{K}^k_p)^*) \to A_0$. This will follow, by induction, if we prove that $A_n - \mathrm{mo}((\mathscr{B}^k_p)^*) \to A_0$.

Let U be a neighborhood of A_0 in M. Pick a neighborhood V of A_0 in U such that every map $S^m \to V$, $0 \leqslant m \leqslant k$, is null-homotopic in U and an index n_0 so that $A_n \subset V$ for all $n \geqslant n_0$. Using the method of the proof of Example (3.5) in [Č5], it can be checked that $(\mathscr{B}^k_p)^*(U, V; A_n)$ holds for all $n \geqslant n_0$.

Our next example improves (7.3) (a) in [Č5] and (2.8) in [Č3].

(3.11) THEOREM. *Let a sequence $\{A_k\}_{k=1}^{\infty}$ of LC^{n-1} compacta in X converge homotopy $(n-1)$-regularly* [Cu] *to an LC^{n-1} compactum A_0 in X. Then $A_k - \mathrm{mo}(\mathscr{P}^n_p) \to A_0$.*

Proof. Assume that $\bigcup_{i=0}^{\infty} A_i \subset Q$. By (3.4), it suffices to prove that $A_k - \mathrm{mo}(\mathscr{K}^n_p) \to A_0$ in Q.

For a compact ANR neighborhood U of A_0 in Q, take a compact neighborhood U' of A_0 in the int U and put $\alpha = \{\mathrm{int}\, U, Q - U'\}$. Pick a refinement β of α with the property described in [H, p. 112]. Since A_0 is an LC^{n-1} compactum, there is a refinement γ of β for which the assertion $E(\gamma, \beta, n)$ holds [K, Lemma 1]. Let $V^* = \bigcup\{V' \in \gamma \mid V' \cap A_0 \neq \emptyset\}$ and let V, $V \subset V^*$, be a compact ANR neighborhood of A_0 in Q. We claim that $\mathscr{K}^n_p(U, V; A_k)$ holds for almost all k.

Let $\varepsilon > 0$ has the property that ε-close maps into U and V are homotopic in U and V, respectively. Using Lemma (4.1) in [Cu], select an i_0 and δ, $0 < \delta < 3\varepsilon/4$, such that partial realization of mesh $< \delta$ in A_i ($i = 0$ or $i \geqslant i_0$) of an at most n-dimensional finite complex can be extended to a full realization of mesh $< \varepsilon/2$ in A_i. Let $i_1 \geqslant i_0$ be such that $d_H(A_i, A_0) < \varepsilon/3$ and $A_i \subset \mathrm{int}\, V$ for all $i \geqslant i_1$.

Fix an index $i \geqslant i_1$ and let W be a neighborhood of A_i in Q. Pick a compact ANR neighborhood W' of A_i inside $V \cap W$ and then choose a neighborhood W_0 of A_i with respect to W' in the same way as V was chosen with respect to U.

Consider a $\mathscr{K}^n_p$-map $f\colon (K, K_0) \to (V, W_0)$. The choice of W_0 gives us a homotopy $f_t\colon (K, K_0) \to (V, W)$, $0 \leqslant t \leqslant 1/5$, with $f_0 = f$ and $f_{1/5}(K_0) \subset A_i$. Pick a fine triangulation T of K_0 so that for every simplex $\sigma \in T$, $f_{1/5}(\sigma)$ has diameter $< \delta/3$. For every vertex v of σ select a point y_v in A_0 such that $d(f_{1/5}(v), y_v) < \delta/3$. A map which associates the point y_v to the vertex v of T is a partial realization of mesh $< \delta$ in A_0 of T. Hence, it extends to a full realization $g\colon K_0 \to A_0$ of mesh $< \varepsilon/2$. Since g and $f_{1/5}|K_0$ are ε-close maps into V, there is a homotopy $g_t\colon K \to V$, $1/5 \leqslant t \leqslant 2/5$, with $g_0 = f_{1/5}$ and $g_{2/5}|K_0 = g$. The choice of V provides us with a homotopy $h_t\colon K \to U$, $2/5 \leqslant t \leqslant 3/5$, with $h_{2/5} = g_{2/5}$, $h_{3/5}(K) \subset A_0$, and $h_t|K_0 = g$ for all t. If we repreat the argument that we applied to the map $f_{1/5}|K_0$ to get g, we see that there is a homotopy $k_t\colon K \to V$, $3/5 \leqslant t \leqslant 4/5$, such that $k_{3/5} = h_{3/5}$, $k_{4/5}(K) \subset A_i$, and $k_{4/5}|K_0 = f_{1/5}|K_0$. The last fact implies that there is a homotopy $m_t\colon K \to W$, $4/5 \leqslant t \leqslant 1$, with $m_{4/5} = k_{4/5}$ and $m_1|K_0 = f|K_0$. The join of homotopies f_t, g_t, h_t, k_t, and m_t shows that $\mathscr{K}^n_p(U, V; A_i)$ holds.

We observed in Example (3.3) (b) that the convergence with respect to the metric d_Z to a $\mathscr{C}_p$-movable compactum is not sufficient for the $\mathrm{mo}(\mathscr{C}_p)$-regular convergence. We shall now introduce a stronger metric d_{Z^*} on a certain subset of 2^X and prove that if A_0 is weakly $\mathscr{C}_p$-movable and $A_n \to A_0$ in the metric d_{Z^*}, then $A_n - \mathrm{mo}\,(\mathscr{C}_p) \to A_0$.

A compactum A *Z-dominates* a compactum B if embedded into ANR's M and N, respectively, they satisfy the following condition.

(3.12) For every neighborhood W of B in N there is a neighborhood W' of A in M such that for every neighborhood W_0' of A in W' there is a neighborhood W_0 of B in N and maps $g\colon W_0 \to W_0'$ and $f\colon W' \to W$ with $f\circ g$ homotopic in W to the inclusion $i_{W_0,W}$ of W_0 into W.

One can prove that the choice of spaces M and N is immaterial in the above definition and that A will Z-dominate B if A quasi-dominates B [B1].

Compacta A and B are called *Z-equivalent* if they Z-dominate each other. For example, this will be the case provided A and B are quasi-affinite [B1]. A class of all compacta in X which are Z-equivalent to a compactum C will be denoted by $X[C]_Z$.

Let C be a compactum, let $A, B \in X[C]_Z$, and assume that X lies in an ANR M. Let $Z^*_{A,M}(B)$ denote the infimum of those $\varepsilon > 0$ such that (3.12) for $M = N$ holds with f and g required to be ε-maps. Put $d_{Z^*,M}(A, B) = \max\{Z^*_{A,M}(B), Z^*_{B,M}(A)\}$. It can be proved that the value $Z^*_{A,M}(B)$ (and therefore also the value $d_{Z^*,M}(A, B)$) does not depend on the choice of a space M. Hence, we can omit M from our notation. Clearly, d_{Z^*} is a metric on $X[C]_Z$ which is stronger than the metric d_Z and is weaker than the strong fundamental metric d_{SF} (when restricted to $X[C]$, the class of all compacta in X shape equivalent to C) [Č7].

(3.13) THEOREM. *Let A_0 be a weakly $\mathscr{C}_p$-movable compactum in X and let $\{A_n\}_{n=1}^{\infty}$ be a sequence in $X[A_0]_Z$. If $\lim Z^*_{A_0}(A_n) = 0$, then $A_n - \mathrm{mo}\,(\mathscr{C}_p) \to A_0$.*

Proof. We can assume that X is a subset of Q. Let U be an arbitrary neighborhood of A_0 in Q. Pick a compact ANR neighborhood V of A_0 in Q such that $\mathscr{C}_p(U, V; A_0)$ holds. Let $\varepsilon > 0$ has the property that ε-close maps into V are homotopic in V. Choose an index n_0 such that $n \geqslant n_0$ implies $A_n \subset \mathrm{int}\, V$ and $Z^*_{A_0}(A_n) < \varepsilon$.

Consider an index $n \geqslant n_0$ and a neighborhood W of A_n in Q. Select a neighborhood W' of A_0 in Q as in (3.12). Then choose a neighborhood W_0' of A_0, $W_0' \subset W' \cap V$, using $\mathscr{C}_p(U, V; A_0)$. Finally, pick a neighborhood W_0 of A_n in Q, $W_0 \subset W \cap V$, and ε-maps $g\colon W_0 \to W_0'$ and $f\colon W' \to W$ such that $f\circ g \simeq i_{W_0,W}$ in W.

Suppose $h\colon (K, K_0) \to (V, W_0)$ is a $\mathscr{C}_p$-map. Observe that maps $h' = h|K_0$ and $g\circ h'$ are homotopic in V. By the HET, there is a homotopy $h_t\colon K \to V$, $0 \leqslant t \leqslant 1/4$, with $h_0 = h$ and $h_{1/4}|K_0 = g\circ h'$. Hence, $h_{1/4}$ is a $\mathscr{C}_p$-map into (V, W_0'). The choice of W_0' gives us a homotopy $k_t\colon K \to U$, $1/4 \leqslant t \leqslant 1/2$, such that $k_{1/4} = h_{1/4}$, $k_{1/2}(K) \subset W'$, and $k_{1/2}|K_0 = h_{1/4}|K_0$. Note that $f\circ k_{1/2}$ and $k_{1/2}$ are ε-close maps into V. Hence, there is a homotopy $m_t\colon K \to V$, $1/2 \leqslant t \leqslant 3/4$, with $m_{1/2} = k_{1/2}$ and $m_{3/4} = f\circ k_{1/2}$. The map $m_{3/4}$ maps K into W and on K_0 it agrees with $f\circ g\circ h'$. Since $f\circ g\circ h' \simeq h'$ in W, we can apply the HET once more and get a homotopy $p_t\colon K \to W$, $3/4 \leqslant t \leqslant 1$, such that $p_{3/4} = m_{3/4}$ and $p_1|K_0 = h' = h|K_0$. The join of homotopies h_t, k_t, m_t, and p_t shows that $\mathscr{C}_p(U, V; A_n)$ holds.

(3.14) Remark. With the notation from definitions preceding (3.13), let $Z^{**}_A(B)$ denote the infimum of those $\varepsilon > 0$ such that (3.12) for $M = N$ holds with f and g required to be ε-maps and the homotopy between $f\circ g$ and $i_{W_0,W}$ is an ε-homotopy in W. By an argument considerably more complicated than the one used in the proof of (3.13) one can prove the following.

Let A_0 be a $\mathscr{C}_p$-movable compactum in X and let $\{A_n\}_{n=1}^{\infty}$ be a sequence in $X[A_0]_Z$. If $\lim Z^{**}_{A_0}(A_n) = 0$, then $A_n - \mathrm{mo}\,(\mathscr{C}_p^*) \to A_0$.

The last example uses the notion of the approximate fibration [C1].

(3.15) THEOREM. *Let $p\colon E \to B$ be an approximate fibration. If A_0, A_1, $A_2, \ldots$ is a sequence of compacta in B such that $A_n - \mathrm{mo}\,(\mathscr{C}_p^*) \to A_0$ ($A_n - \mathrm{mo}\,(\mathscr{C}_p) \to A_0$ and $(\mathscr{C}_p'')_H \subset \mathscr{C}_p$), then $p^{-1}(A_n) - \mathrm{mo}\,(\mathscr{C}_p^*) \to p^{-1}(A_0)$ ($p^{-1}(A_n) - \mathrm{mo}\,(\mathscr{C}_p) \to p^{-1}(A_0)$).*

Proof. (a) Assume $A_n - \mathrm{mo}\,(\mathscr{C}_p^*) \to A_0$. Let $\tilde{U}$ be an arbitrary neighborhood of $p^{-1}(A_0)$ in E. Pick a neighborhood U of A_0 in B such that $p^{-1}(U) \subset \tilde{U}$. Inside U select a neighborhood V of A_0 in B so that $\mathscr{C}_p^*(U, V; A_n)$ holds for almost all n and let $\tilde{V} = p^{-1}(V)$. We claim that $\mathscr{C}_p^*(\tilde{U}, \tilde{V}; p^{-1}(A_n))$ is true for almost all n.

Indeed, suppose $\mathscr{C}_p^*(U, V; A_n)$ holds and let $\tilde{W}$ be a neighborhood of $p^{-1}(A_n)$ in E. Choose a neighborhood W' of A_n in V such that $p^{-1}(W') \subset \tilde{W}$. Inside W' take a neighborhood W of A_n in B with $\mathrm{Cl}\, W \subset W'$. Finally, pick W_0, $W_0 \subset W \cap V$, using $\mathscr{C}_p^*(U, V; A_n)$ and put $\tilde{W}_0 = p^{-1}(W_0)$.

Consider a $\mathscr{C}_p$-map $f\colon (K, K_0) \to (\tilde{V}, \tilde{W}_0)$. Let ε denote the cover $\{\tilde{W}', U - \mathrm{cl}\, W\}$ of U. By the choice of n and [Č5, Theorem (5.1)], there is a homotopy $f_t\colon K \to U$, $0 \leqslant t \leqslant 1$, with $f_0 = p\circ f$, $f_1(K) \subset W$, and $f_t|K_0 = p\circ f|K_0$ for all t. Since $p|p^{-1}(U)\colon p^{-1}(U) \to U$ has the approximate homotopy lifting property with respect to the class $\mathscr{C}_p'$, it also has the regular approximate homotopy lifting property with respect to the class $\mathscr{C}_p'$ [C1, Proposition 1.5]. Hence, there is an ε-lift $\tilde{f}_t\colon K \to p^{-1}(U)$ of the homotopy f_t such that $\tilde{f}_0 = f$ and $\tilde{f}_t|K_0 = f|K_0$ for all t. But, because $f_1(K) \subset W$, clearly, $\tilde{f}_1(K) \subset \tilde{W}$.

(b) Assume $A_n - \mathrm{mo}\,(\mathscr{C}_p) \to A_0$ and $(\mathscr{C}_p'')_H \subset \mathscr{C}_p$. By [Č3, Example (2.10)], $p^{-1}(A_n) - \mathrm{mo}\,(\mathscr{C}_p') \to p^{-1}(A_0)$. Hence, by (3.5) (b), it remains to see that $p^{-1}(A_n) - \mathrm{ca}\,(\mathscr{C}_p'') \to p^{-1}(A_0)$.

For a neighborhood $\tilde{U}$ of $p^{-1}(A_0)$ in E, pick neighborhoods U and V of A_0 in B and $\tilde{V}$ of $p^{-1}(A_0)$ in E as we did in (a) so that $(\mathscr{C}_p'')_h(U, V; A_n)$ holds for almost all n.

Suppose $(\mathscr{C}_p'')_h(U, V; A_n)$ is true and let $\tilde{W}$ be a neighborhood of

$p^{-1}(A_n)$ in E. Now select neighborhoods W', W, and W_0 of A_n in B and $\tilde{W}_0$ of $p^{-1}(A_n)$ in E as in (a) but this time use $(\mathscr{C}''_p)_h(U, V; A_n)$ instead of $\mathscr{C}^*_p(U, V; A_n)$.

Consider $\mathscr{C}''_p$-maps $f, g\colon K \to \tilde{W}_0$ and assume that they are homotopic in $\tilde{V}$ via a homotopy $H\colon K \times I \to \tilde{V}$. Then $\mathscr{C}''_p$-maps $p \circ f$ and $p \circ g$ into W_0 are homotopic in V via $p \circ H$. By the choice of V and W_0, there is a homotopy $h\colon K \times I \to W$ between $p \circ f$ and $p \circ g$. On the subset $T = K \times I \times \{0\} \cup K \times I \times \{1\} \cup K \times \{0\} \times I$ of $K \times I \times I$ define a map m into V by $m|K \times I \times \{0\} = h$, $m|K \times I \times \{1\} = p \circ H$, and $m|K \times \{0\} \times I = p \circ f$. By the HET, m extends to a map $\mu'\colon K \times I \times I \to V$. We can reparametrize μ' and thus get a map $\mu\colon K \times I \times I \to V$ such that $\mu|T = m$ and $\mu|K \times \{1\} \times I = p \circ g$. If we apply the regular approximate homotopy lifting property of $p|p^{-1}(V)$ with respect to a cover $\varepsilon = \{W', V - \mathrm{Cl}\, W\}$, we shall get an ε-lift $\tilde{\mu}\colon K \times I \times I \to \tilde{V}$ of μ such that $\tilde{\mu}|K \times \{0\} \times I = f$ and $\tilde{\mu}|K \times \{1\} \times I = g$. It is clear that $\tilde{\mu}|K \times I \times \{0\}$ is a homotopy in $\tilde{W}$ between f and g.

(3.16) Remark. The characterization of approximate fibrations as completely movable maps [C2, Proposition 3.6] and (3.15) immediately imply the following. A surjective proper map $p\colon E \to B$ between locally compact, separable metric ANR's is an approximate fibration iff p is a mo $(\mathscr{P}^*_p)$-regular map (i.e., iff for every sequence $\{b_i\}$ of points in B converging to a point $b_0 \in B$, $p^{-1}(b_i) - \mathrm{mo}\,(\mathscr{P}^*_p) \to p^{-1}(b_0)$).

4. Operations preserving convergence. In this short section we shall state three theorems which describe ways of producing mo $(\mathscr{C}_p)$-regularly and mo $(\mathscr{C}^*_p)$-regularly converging sequences of compacta. We leave the proofs to the reader because they are similar to the proofs in § 3 in [Č3] and [Č4]. For the case of mo$(\mathscr{C}_p)$-regular convergence, in view of (3.5) (b), they follow from the corresponding statements in [Č3] and [Č4].

(4.1) Theorem. *Let, for each $i = 1, 2, \ldots, m$ $(m < \infty)$, $\{A^i_n\}$ be a sequence of compacta in a metric space X_i converging* $\mathrm{mo}(\mathscr{C}^*_p)$*-regularly* $(\mathrm{mo}(\mathscr{C}_p)$*-regularly*) *to a compactum A^i_0 in X_i. Let $A_n = \prod_{i=1}^{m} A^i_n$ $(n = 0, 1, 2, \ldots)$. Then $A_n - \mathrm{mo}(\mathscr{C}^*_p) \to A_0$ $(A_n - \mathrm{mo}(\mathscr{C}_p) \to A_0)$.*

(4.2) Theorem. *If $A_n - \mathrm{mo}(\mathscr{C}^*_p) \to A_0$ $(A_n - \mathrm{mo}(\mathscr{C}_p) \to A_0)$, where $\{A_n\}_{n=0}^{\infty}$ are compacta in a compact metric space X, then the sequence $\{SA_n\}_{n=1}^{\infty}$ of the* (unreduced) *suspensions of A_n is a sequence of compacta in SX converging* $\mathrm{mo}(\mathscr{C}^*_p)$*-regularly* $(\mathrm{mo}(\mathscr{C}_p)$*-regularly*) *to SA_0.*

(4.3) Theorem. *If $A_n - \mathrm{mo}(\mathscr{C}^*_p) \to A_0$ $(A_n - \mathrm{mo}(\mathscr{C}_p) \to A_0)$, then for every component C_0 of A_0 there is a component C_n of A_n such that $C_n - \mathrm{mo}(\mathscr{C}^*_p) \to C_0$ $(C_n - \mathrm{mo}(\mathscr{C}_p) \to C_0)$. Conversely, let every compactum A_n $(n = 0, 1, 2, \ldots)$ has precisely k $(k < \infty)$ components $C^1_n, \ldots, C^k_n$ and assume that $C^i_n - \mathrm{mo}(\mathscr{C}^*_p) \to C^i_0$ $(C^i_n - \mathrm{mo}(\mathscr{C}_p) \to C^i_0)$, $1 \leqslant i \leqslant k$. Then $A_n - \mathrm{mo}(C^*_p) \to A_0$ $(A_n - \mathrm{mo}(\mathscr{C}_p) \to A_0)$.*

5. Consequences of $\mathrm{mo}(\mathscr{C}_p)$-regular convergence. The results in this section show that if $A_n - \mathrm{mo}(\mathscr{P}_p) \to A_0$ and A_0 has a certain hereditary shape property, then almost all A_n also have that property. The following properties will be considered: $\mathscr{C}$-triviality, $\mathscr{C}$-movability [ČŠ], near 1-movability [M], $(\mathscr{C}, \mathscr{D})$-tameness, $(\mathscr{C}, \mathscr{D})$-smoothness [Č2], to have the shape $\mathscr{C}$-category $\mathrm{cat}_{\mathscr{C}} A \leqslant k$ [Č6], to be Z-dominated by a compactum B, and to have the kth (cohomology) Betti number $p_k(A; G) \leqslant m$ [Č3].

(5.1) Theorem. *If $A_n - \mathrm{ca}(\mathscr{C}) \to A_0$ and A_0 is $\mathscr{C}$-trivial, then almost all A_n are also $\mathscr{C}$-trivial.*

Proof. Pick a neighborhood U of A_0 in M and an index n_0 such that $\mathscr{C}_h(M, U; A_n)$ holds for all $n \geqslant n_0$. Then select a neighborhood V of A_0 in U using $\mathscr{C}$-triviality of A_0 and an $n_1 \geqslant n_0$ such that $A_n \subset V$ whenever $n \geqslant n_1$.

Let $n \geqslant n_1$ and let W be an arbitrary neighborhood of A_n in M. Pick a smaller neighborhood W_0 of A_n, $W_0 \subset W \cap V$, using $\mathscr{C}_h(M, U; A_n)$.

Consider a $\mathscr{C}$-map $f\colon K \to W_0$. Since $W_0 \subset V$, there is a homotopy $F\colon K \times I \to U$ with $F_0 = f$ and $F_1(K) = p \in U$. We can assume that $p \in W_0$ because the set $F(K \times I)$ is connected. The choice of U and W_0 implies that there is a homotopy $G\colon K \times I \to W$ satisfying $G_0 = F_0$ and $G_1 = F_1$. Hence, f is null-homotopic in W.

Recall [ČŠ] that a compactum A is *$\mathscr{C}$-movable* provided, for some (and hence for every) embedding of A into an ANR M, for each neighborhood U of A in M there is a smaller neighborhood V of A in M such that $\mathscr{C}(U, V; A)$ holds.

(5.2) Theorem. *If $A_n - \mathrm{ca}(\mathscr{C}) \to A_0$ and $A_n - \mathrm{mo}(\mathscr{C}) \to A_0$, then A_0 and almost all A_n are $\mathscr{C}$-movable.*

Proof. A_0 is $\mathscr{C}$-movable by [Č3, (2.3)]. In order to prove that almost all A_n are $\mathscr{C}$-movable, select neighborhoods U and V, $V \subset U$, of A_0 in M and an index n_0 such that $\mathscr{C}_h(M, U; A_n)$ and $\mathscr{C}(U, V; A_n)$ hold for all $n \geqslant n_0$.

Let $n \geqslant n_0$ and let W be an arbitrary neighborhood of A_n in M. Pick a neighborhood W_0 of A_n, $W_0 \subset W \cap V$, using $\mathscr{C}_h(M, U; A_n)$.

Consider a $\mathscr{C}$-map $f\colon K \to W_0$ and a neighborhood Z of A_n in M. Since $W_0 \subset V$ and $\mathscr{C}(U, V; A_n)$ holds, there is a homotopy $F\colon K \times I \to U$ with $F_0 = f$ and $F_1(K) \subset Z \cap W_0$. But, the choice of U and W_0 gives us a homotopy $G\colon K \times I \to W$ satisfying $G_0 = F_0$ and $G_1 = F_1$. Hence, A_n is $\mathscr{C}$-movable.

Let $\mathscr{P}d_p$ denote all pairs of the form $(D - \bigcup_{i=1}^{n} \mathrm{int}\, D_i, \partial D \cup \bigcup_{i=1}^{n} \partial D_i)$, where D is a 2-disc and $\{D_1, \ldots, D_n\}$ is a finite, disjoint collection of discs in int D.

(5.3) Theorem. *If $A_n - \mathrm{mo}(\mathscr{P}d_p) \to A_0$, $\lim Z_{A_0}(A_n) = 0$, and A_0 is nearly 1-movable* [M], *then almost all A_n are also nearly 1-movable.*

Proof. Pick neighborhoods U and V, $V \subset U$, of A_0 in $M (= Q)$ and an index n_0 such that U is a compact ANR, V "nearly 1-moves toward A_0 in U" [M], and $\mathscr{P}d_p(M, U; A_n)$ holds for all $n \geqslant n_0$. Let $\varepsilon > 0$ has the property that ε-close maps into U are homotopic in U. Take an $n \geqslant n_0$ such that $Z_{A_0}(A_n) < \varepsilon$ and $A_n \subset V$.

Let W be an arbitrary neighborhood of A_n in M. Select a neighborhood W_0 of A_n in W, $W_0 \subset V$, using $\mathscr{P}d_p(M, U; A_n)$. We claim that W_0 "nearly 1-moves toward A_n in W".

Indeed, let Z be a neighborhood of A_n in M and consider a map $f\colon \partial D \to W_0$. Let T be a neigborhood of A_0 in M for which there is an ε-map $g\colon T \to Z \cap W_0$. Since $W_0 \subset V$, there is $(K, K_0) \in \mathscr{P}d_p$ and an extension $F\colon K \to U$ of f such that $F(K_0 - \partial D) \subset T \cap U$. But, $F|K_0 - \partial D$ and $g \circ F|K_0 - \partial D$ are ε-close maps into U. Hence, we can assume that $F(K_0 - \partial D) \subset W_0 \cap Z$. Finally, the way U and W_0 were chosen gives us a homotopy $F_t\colon K \to M$, $0 \leqslant t \leqslant 1$, with $F_0 = F$, $F_1(K) \subset W$, and $F_1|K_0 = F|K_0$. Then F_1 is a homotopy which "nearly 1-moves f into Z in W".

A compactum A is *$(\mathscr{C}, \mathscr{D})$-tame* if, for some (and hence for every) embedding of A into an ANR M, for each neighborhood U of A in M there is a smaller neighborhood V of A in M such that for every $\mathscr{C}$-map $f\colon K \to V$ there is $L \in \mathscr{D}$ and maps $h\colon K \to L$ and $g\colon L \to U$ with f homotopic to $g \circ h$ in U. Observe that a compactum A is $(\mathscr{P}, \mathscr{C})$-tame iff A is $\mathscr{C}$-tame [ČŠ] and that A is $(\mathscr{C}, \mathscr{M}^k)$-tame iff $d_{\mathscr{C}}(A) \leqslant k$, where $\mathscr{M}^k$ denotes the collection of all at most k-dimensional metric spaces and $d_{\mathscr{C}}(A)$ is Nowak's coefficient of deformability of A with respect to a class $\mathscr{C}$ (see [N] and [ČŠ]). Hence, the fundamental dimension $\mathrm{Fd}(A)$ of A is $\leqslant k$ iff A is $(\mathscr{P}, \mathscr{P}^k)$-tame.

(5.4) Theorem. *If A_n $-\mathrm{ca}(\mathscr{C})\to$ A_0, A_n $-\mathrm{mo}(\mathscr{D})\to$ A_0, and A_0 is $(\mathscr{C}, \mathscr{D})$-tame, then almost all A_n are also $(\mathscr{C}, \mathscr{D})$-tame.*

Proof. Pick neighborhoods U, V_1, and V of A_0 in M, $V \subset V_1 \subset U$, and an index n_0 such that every $\mathscr{C}$-map into V factors up to homotopy in V_1 by a $\mathscr{D}$-map and $\mathscr{C}_h(M, U; A_n)$ and $\mathscr{D}(U, V_1; A_n)$ hold for all $n \geqslant n_0$.

Let $n \geqslant n_0$ and let W be an arbitrary neighborhood of A_n in M. Select a neighborhood W_0 of A_0 in W, $W_0 \subset V$, using the choice of U.

Consider a $\mathscr{C}$-map $f\colon K \to W_0$. Since $W_0 \subset V$, there is $L \in \mathscr{D}$, maps $h\colon K \to L$ and $g\colon L \to V_1$, and a homotopy $k_t\colon K \to V_1$, $0 \leqslant t \leqslant 1/2$, such that $k_0 = f$ and $k_{1/2} = g \circ h$. But, by the choice of V_1, for a $\mathscr{D}$-map $g\colon L \to V_1$ there is a homotopy $d_t\colon L \to U$, $1/2 \leqslant t \leqslant 1$, with $d_{1/2} = g$ and $d_1(L) \subset W_0$. The join of homotopies g_t and $d_t \circ h$ shows that f and $d_1 \circ h\colon K \to W_0$ are homotopic in U. The way U was chosen implies that they are also homotopic in W. Hence, A_n is $(\mathscr{C}, \mathscr{D})$-tame.

By a similar method one can also prove the following two theorems.

(5.5) Theorem. *If A_n $-\mathrm{ca}(\mathscr{C})\to$ A_0 and A_0 is $(\mathscr{C}, \mathscr{D})$-smooth [Č2], then almost all A_n are $(\mathscr{C}, \mathscr{D})$-smooth.*

(5.6) *If A_n $-\mathrm{ca}(\mathscr{C})\to$ A_0 and $\mathrm{cat}_{\mathscr{C}}(A_0) \leqslant k$ [Č6], then $\mathrm{cat}_{\mathscr{C}}(A_n) \leqslant k$ for almost all n.*

(5.7) Theorem. *If A_n $-\mathrm{mo}(\mathscr{P}_p)\to$ A_0 and A_0 is Z-dominated by a compactum B, then almost all A_n are Z-dominated by B.*

Proof Without loss of generality we can assume that $X \subset Q$ and $B \subset Q$. By (3.7) and [Č3, (2.3)], we can select neighborhoods U and V of A_0 in Q, $V \subset U$, and an index n_0 such that $\mathscr{P}(U, V; A_0)$, $\mathscr{P}_h(Q, U; A_n)$ and $\mathscr{P}(U, V; A_n)$ hold for all $n \geqslant n_0$.

Let $n \geqslant n_0$. We claim that A_n is Z-dominated by B. Indeed, let W be an arbitrary neighborhood of A_n in Q. Select a compact ANR neighborhood W_0 of A_n in $W \cap V$. Since A_0 is Z-dominated by B, there is a compact ANR neighborhood W' of B in Q such that for every smaller neighborhood W_0' of B in Q, there is a neighborhood V_0 of A_0 in Q and maps $g\colon V_0 \to W_0'$ and $f\colon W' \to V$ with $f \circ g$ homotopic in V to the inclusion $i_{V_0,V}$. The choice of V gives us homotopies $h_t\colon W_0 \to U$, $0 \leqslant t \leqslant 1$, and $k_t\colon W' \to U$, $0 \leqslant t \leqslant 1$, such that $h_0 = i_{W_0,V}$, $h_1(W_0) \subset V_0$, $k_0 = f$, and $k_1(W') \subset W_0$. Thus $i_{W_0,U}$ and $k_1 \circ g \circ h_1$ are $\mathscr{P}$-maps into W_0 which are homotopic in U. Hence, they are homotopic in W and our claim is proved.

The last theorem in the present section might look out of place to the reader but in view of (6.13) below it has a similar role as Theorems (5.1)–(5.7).

(5.8) Theorem. *If a compactum A Z-dominates a compactum B and $p_k(A; G) \leqslant m$ [Č3], then $p_k(B; G) \leqslant m$.*

Proof. We can assume that A and B lie in the Hilbert cube and that $\{U_1, U_2, \ldots\}$ and $\{V_1, V_2, \ldots\}$ are nested sequences of compact ANR neighborhoods of A and B in Q, respectively, with $\bigcap_{i=1}^{\infty} U_i = A$ and $\bigcap_{i=1}^{\infty} V_i = B$. Observe that $\check{H}^k(A; G) = \varinjlim H^k(U_i; G)$ and $\check{H}^k(B; G) = \varinjlim H^k(V_i; G)$.

If $m = \infty$, then there is nothing to prove. Hence, we assume that $m < \infty$. We shall show that every set $\{[x_1], [x_2], \ldots, [x_{m+1}]\}$ of $m+1$ elements of $\check{H}^k(B; G)$ is linearly dependend over the integers.

For every $i \in \{1, \ldots, m+1\}$ there is an index n_i and an element x_i of $H^k(V_{n_i}; G)$ representing $[x_i]$. If $n_0 = \max_i \{n_i\}$, then we can assume that $x_i \in H^k(V_{n_0}; G)$. Now, select an index s_0 such that for every $s \geqslant s_0$ there is an $n \geqslant n_0$ and maps $g\colon V_n \to U_s$ and $f\colon U_{s_0} \to V_{n_0}$ with $f \circ g \simeq i_{V_n,V_{n_0}}$ in V_{n_0}. Since $H^k(U_{s_0}; G)$ is a finite abelian group and the rank of $\check{H}^k(A; G)$ equals at most m, there is an $s \geqslant s_0$ such that every collection of $m+1$ elements of $H^k(U_{s_0}; G)$ is linearly dependent in $H^k(U_s; G)$. Then pick $n \geqslant n_0$ and maps f and g and put $y_i = (f \circ g)^*(x_i)$ $(i = 1, \ldots, m+1)$. It is easy to check that elements $\{y_i\}$ in $H^k(V_n; G)$ are linearly dependent and that they represent classes $\{[x_i]\}$.

6. The metrics d_1 and d_2. We can define metrics on the collections $\mathrm{mo}(\mathscr{C}_p, X)$ and $\mathrm{mo}(\mathscr{C}_p^*, X)$ of all weakly $\mathscr{C}_p$-movable and all $\mathscr{C}_p$-movable compacta in a metric space X, respectively, in a number of ways. Here we shall consider how to introduce metrics on these hyperspaces that will preserve $\mathrm{mo}(\mathscr{C}_p)$-regular and $\mathrm{mo}(\mathscr{C}_p^*)$-regular convergence. Using Begle's method in [Be] we shall define a metric d_1 on $\mathrm{mo}(\mathscr{C}_p, X)$ and a metric d_2 on $\mathrm{mo}(\mathscr{C}_p^*, X)$ in such a way that $\lim d_1(A_n, A_0) = 0$ iff $\lim d_Z(A_n, A_0) = 0$ and $A_n - \mathrm{mo}(\mathscr{C}_p) \to A_0$ and $\lim d_2(A_n, A_0) = 0$ iff $\lim d_Z(A_n, A_0) = 0$ and $A_n - \mathrm{mo}(\mathscr{C}_p^*) \to A_0$.

In fact, it is clear from the explanation on the page 444 in [Be] that such metrics can be introduced provided we can prove the analogues of Lemmas 1, 3, 4, and 5 in [Be] for the functions $\delta(\varepsilon, A)$ and $\delta^*(\varepsilon, A)$ (defined in (6.1)) corresponding to Begle's function $\delta_n(\varepsilon, P)$. The analogue of Lemma 4 was established in (3.2) while Lemmas (6.2), (6.3), and (6.4) below correspond to Lemmas 1, 3, and 5, respectively.

Throughout this section, without loss of generality, we assume that X lies in the Hilbert cube Q which has diameter 1 in a metric d defined on it.

(6.1) Definition. For a compact subset A of Q and an $\varepsilon > 0$, let $\delta(\varepsilon, A)$ be the least upper bound of all numbers δ, $\delta \leqslant \varepsilon$, such that $\mathscr{C}_p(N(\varepsilon, A), N(\delta, A); A)$ holds. Similarly, $\delta^*(\varepsilon, A)$ is the least upper bound of those $\delta \leqslant \varepsilon$ for which $\mathscr{C}_p^*(N(\varepsilon, A), N(\delta, A); A)$ holds.

It is clear that for each compactum A in Q, $\delta(\varepsilon, A)$ and $\delta^*(\varepsilon, A)$ always exist and are non-negative monotone non-decreasing, and hence measurable, functions on the half-open interval $I^* = (0, 1]$. Observe that A is (weakly) $\mathscr{C}_p$-movable iff $(\delta(\varepsilon, A) > 0)$ $\delta^*(\varepsilon, A) > 0$ everywhere in I^*.

The connection between Definitions (3.1) and (6.1) is provided by the following.

(6.2) Lemma. *If* $\lim d_H(A_n, A_0) = 0$, *then the sequence* $\{A_n\}$ *converges (weakly)* $\mathscr{C}_p$*-movably regularly to* A_0 *iff* $(\liminf \delta(\varepsilon, A) > 0)$ $\liminf \delta^*(\varepsilon, A) > 0$ *for each* ε *in* I^*.

Proof. The proof is similar to the proof of Lemma (4.2) in [Č3].

(6.3) Lemma. *Let* $\lim d_H(A_n, A_0) = 0$ *and let* $\lim Z_{A_0}(A_n) = 0$. *Then* $\limsup \delta(\varepsilon, A_n) \leqslant \delta(\varepsilon, A_0)$ *and* $\limsup \delta^*(\varepsilon, A_n) \leqslant \delta^*(\varepsilon, A_0)$ *for all but countably many points* ε *in* I^*.

Proof. We shall consider the function $\delta(-, -)$. The proof for the function $\delta^*(-, -)$ is similar.

We shall prove that $\limsup \delta(\varepsilon_0, A_n) > \delta(\varepsilon_0, A_0)$ at the point $\varepsilon_0 \in (0, 1)$ implies that the function $\delta(\varepsilon, A_0)$ is not continuous at the point ε_0. Since $\delta(\varepsilon, A_0)$ is a monotone function, there are at most countably many points in $(0, 1)$ at which this can happen.

Suppose $\limsup \delta(\varepsilon_0, A_n) > \delta(\varepsilon_0, A_0)$ and take an e, $0 < 2e < 1 - \varepsilon_0$, and a subsequence $\{B_i\}$ of $\{A_n\}$ such that $\delta(\varepsilon_0, B_i) \geqslant \delta(\varepsilon_0, A_0) + e$ for all $i > 0$. Let b, $0 < b < e/2$, be an arbitrary number.

We claim that $\mathscr{C}_p\big(N(\varepsilon_0 + b, A_0), N(\delta(\varepsilon_0, A_0) + (e/8), A_0); A_0\big)$ holds. This would imply that for every b, $0 < b < e/2$, $\delta(\varepsilon_0 + b, A_0) \geqslant \delta(\varepsilon_0, A_0) + (e/8)$, i.e., that the function $\delta(\varepsilon, A_0)$ has a jump at least $e/8$ at the point ε_0.

Let W be an arbitrary open neighborhood of A_0 in Q, let W^* be a compact ANR neighborhood of A_0 in W, and let $\eta = d(Q - W, W^*)$. Pick an ξ, $0 < \xi < b/4$, such that ξ-close maps into $N(\delta(\varepsilon_0, A_0) + (e/4), A_0)$ are $(\eta/2)$-homotopic in $N(\delta(\varepsilon_0, A_0) + (e/2), A_0)$. Select an integer j such that $Z_{A_0}(B_j) < \xi$, $d_H(B_j, A_0) < \xi$, and $B_j \subset \operatorname{int} W^*$. Let W_0', $W_0' \subset W^* \cap N(\delta(\varepsilon_0, B_j), B_j)$, be chosen with respect to W using $\mathscr{C}_p\big(N(\varepsilon_0, B_j), N(\delta(\varepsilon_0, B_j), B_j); B_j\big)$. Finally, pick a neighborhood W_0 of A_0 in Q, $W_0 \subset W \cap N\big(\delta(\varepsilon_0, A_0) + (e/8), A_0\big)$ for which there is a ξ-map $f\colon W_0 \to W_0'$.

Consider a $\mathscr{C}_p$-map $g\colon (K, K_0) \to \big(N(\delta(\varepsilon_0, A_0) + (e/8), A_0), W_0\big)$. Then $g' = g|K_0$ and $f \circ g'$ are ξ-close maps into $W^* \cap N(\delta(\varepsilon_0, A_0) + (e/8) + \xi, A_0) \subset W^* \cap N(\delta(\varepsilon_0, A_0) + (e/4), A_0)$. Hence, they are $(\eta/2)$-homotopic in $N(\delta(\varepsilon_0, A_0) + (e/2), A_0)$ and thus homotopic in W. Applying the HET in $N(\delta(\varepsilon_0, A_0) + (e/2), A_0)$, we see that there is a homotopy $g_t\colon (K, K_0) \to \big(N(\delta(\varepsilon_0, A_0) + (e/2), A_0), W\big)$, $0 \leqslant t \leqslant 1/3$, such that $g_0 = g$ and $g_{1/3}|K_0 = f \circ g'$.

Since $N(\delta(\varepsilon_0, A_0) + (e/2), A_0) \subset N(\delta(\varepsilon_0, B_j), B_j)$, $g_{1/3}$ is a $\mathscr{C}_p$-map into $\big(N(\delta(\varepsilon_0, B_j), B_j), W_0'\big)$. By assumption, there is a homotopy $f_t\colon K \to N(\varepsilon_0, B_j)$, $1/3 \leqslant t \leqslant 2/3$, with $f_{1/3} = g_{1/3}$, $f_{2/3}(K) \subset W$, and $f_{2/3}|K_0 = g_{1/3}|K_0$.

Applying the HET again, this time to the map $f_{2/3}$ and the partial homotopy $g_{1-t}|K_0$, $2/3 \leqslant t \leqslant 1$, we get a homotopy $h_t\colon K \to W$, $2/3 \leqslant t \leqslant 1$, such that $h_{2/3} = f_{2/3}$ and $h_1|K_0 = g'$. Since $N(\varepsilon_0, B_j) \subset N(\varepsilon_0 + b, A_0)$, the join of homotopies g_t, f_t, and h_t shows that $\mathscr{C}_p\big(N(\varepsilon_0 + b, A_0), N(\delta(\varepsilon_0, A_0) + (e/8), A_0); A_0\big)$ indeed holds.

(6.4) Lemma. *Let* $\lim d_H(A_n, A_0) = 0$ *and* $\lim Z_{A_n}(A_0) = 0$. *If* $A_n - \mathrm{mo}(\mathscr{C}_p) \to A_0$ $(A_n - \mathrm{mo}(\mathscr{C}_p^*) \to A_0)$, *then* $\delta(\varepsilon_0, A_0) \leqslant \liminf \delta(\varepsilon_0, A_n)$ $(\delta^*(\varepsilon_0, A_0) \leqslant \liminf \delta^*(\varepsilon_0, A_n))$ *at every point* $\varepsilon_0 \in I^*$ *in which the function* $\delta(\varepsilon, A_0)$ $(\delta^*(\varepsilon, A_0))$ *is continuous.*

Proof. Let us consider a point $\varepsilon_0 \in I^*$ at which the function $\delta(\varepsilon, A_0)$ is continuous. Suppose that $\delta(\varepsilon_0, A_0) > \liminf \delta(\varepsilon_0, A_n)$. Then there is an e, $0 < e < \varepsilon_0$, and a subsequence $\{B_i\}$ of $\{A_n\}$ such that $\delta(\varepsilon_0, B_i) + e < \delta(\varepsilon_0, A_0) - e$ for all $i > 0$. Since the function $\delta(\varepsilon, A_0)$ is continuous at ε_0, there is a number d, $0 < d < e$, such that $\delta(\varepsilon, A_0) \in (\delta(\varepsilon_0, A_0) - e, \delta(\varepsilon_0, A_0) + e)$ for all $\varepsilon \in (\varepsilon_0 - 2d, \varepsilon_0 + 2d) \cap I^*$. In particular, $\delta(\varepsilon_0 - d, A_0) > \delta(\varepsilon_0, B_i) + e$ for all $i > 0$.

We claim that there is an index k such that $\mathscr{C}_p\big(N(\varepsilon_0, B_k), N(\delta(\varepsilon_0, B_k) + e, B_k); B_k\big)$ holds. This would imply $\delta(\varepsilon_0, B_k) \geqslant \delta(\varepsilon_0, B_k) + e$, an obvious contradiction.

By using the fact that A_n $-\mathrm{mo}(\mathscr{C}_p)\to$ A_0, inside $N(\varepsilon_0-d, A_0)$ pick a compact ANR neighborhood V of A_0 and an index i_V so that $\mathscr{C}_p(N(\varepsilon_0-d, A_0), V; B_j)$ is true for all $j \geqslant i_V$. Then select a compact neighborhood V^* of A_0 in int V and let $\eta = d(V^*, Q-V)$. Let $\eta > \xi > 0$ be such that ξ-close maps into V are homotopic in V. Pick an integer k so large that $k \geqslant i_V$, $Z_{B_k}(A_0) < \xi$, $d_H(A_0, B_k) < d$, and $B_k \subset \operatorname{int} V$.

Consider now an arbitrary neighborhood W of B_k in Q. Select a neighborhood W_0' of A_0 in $V \cap N(\delta(\varepsilon_0-d, A_0), A_0)$ with respect to V using the fact that $\mathscr{C}_p\big(N(\varepsilon_0-d, A_0), N(\delta(\varepsilon_0-d, A_0), A_0); A_0\big)$ holds. Then pick a neighborhood W_0^* of B_k in Q, $W_0^* \subset V \cap W$, with respect to W applying $\mathscr{C}_p(N(\varepsilon_0-d, A_0), V; B_k)$. Finally, we take a neighborhood W_0 of B_k in $N(\delta(\varepsilon_0, B_k)+e, B_k) \cap W_0^*$ and an ξ-map $f\colon W_0 \to W_0'$.

Let $g\colon (K, K_0) \to \big(N(\delta(\varepsilon_0, B_k)+e, B_k), W_0\big)$ be a $\mathscr{C}_p$-map. Since $g' = g|K_0$ and $f \circ g'$ are ξ-close maps into V, they are homotopic in V via a homotopy $f_t\colon K_0 \to V$, $0 \leqslant t \leqslant 1/4$. Since $N(\delta(\varepsilon_0, B_k)+e, B_k) \subset N\big(\delta(\varepsilon_0-d, A_0), A_0\big)$, we can apply the HET in $N\big(\delta(\varepsilon_0-d, A_0), A_0\big)$ and get a homotopy $\tilde{f}_t\colon K \to N\big(\delta(\varepsilon_0, A_0), A_0\big)$ with $\tilde{f}_0 = g$ and $\tilde{f}_t|K_0 = f_t$, $0 \leqslant t \leqslant 1/4$.

By the choice of W_0', for a $\mathscr{C}_p$-map $\tilde{f}_{1/4}\colon (K, K_0) \to \big(N(\delta(\varepsilon_0-d, A_0), A_0), W_0'\big)$, there is a homotopy $g_t\colon K \to N(\varepsilon_0-d, A_0)$, $1/4 \leqslant t \leqslant 1/2$, such that $g_{1/4} = \tilde{f}_{1/4}$, $g_{1/2}(K) \subset V$, and $g_{1/2}|K_0 = f_{1/4}|K_0$.

Now, we consider a map $g_{1/2}$ of K into V and a partial homotopy $f_{(3/4)-t}$, $1/2 \leqslant t \leqslant 3/4$, on K_0 into V. By the HET, there is a homotopy $h_t\colon K \to V$ with $h_{1/2} = g_{1/2}$ and $h_t|K_0 = f_{(3/4)-t}$, $1/2 \leqslant t \leqslant 3/4$. This means that $h_{3/4}$ is a $\mathscr{C}_p$-map into (V, W_0). Hence, there is a homotopy $k_t\colon K \to N(\varepsilon_0-d, A_0)$, $3/4 \leqslant t \leqslant 1$, with $k_{3/4} = h_{3/4}$, $k_1(K) \subset W$, and $k_1|K_0 = h_{3/4}|K_0 = f_0 = g'$.

The join of homotopies f_t, g_t, h_t, and k_t shows that $\mathscr{C}_p\big(N(\varepsilon_0-d, A_0), N(\delta(\varepsilon_0, B_k)+e, B_k); B_k\big)$ holds. But $N(\varepsilon_0-d, A_0) \subset N(\varepsilon_0, B_k)$ so that $\mathscr{C}_p\big(N(\varepsilon_0, B_k), N(\delta(\varepsilon_0, B_k)+e, B_k); B_k\big)$ also holds as claimed.

The proof of the lemma for the function $\delta^*(-, -)$ is slightly different. In that case we define homotopies f_t, g_t, and h_t over the intervals $[0, 1/3]$, $[1/3, 2/3]$, and $[2/3, 1]$, respectively, while the homotopy g_t has the additional property that $g_t|K_0 = f \circ g'$, $1/3 \leqslant t \leqslant 2/3$. We shall now modify the join $\zeta_t\colon$ $K \to N(\varepsilon_0-d, A_0)$, $0 \leqslant t \leqslant 1$, of homotopies f_t, g_t, and h_t as follows.

Consider a map φ from the boundary of $I \times I$ into I defined by

$$\varphi(s, t) = \begin{cases} 3s, & t = 0,\ 0 \leqslant s \leqslant 1/3, \\ 1, & t = 0,\ 1/3 \leqslant s \leqslant 2/3, \\ 3-3s, & t = 0,\ 2/3 \leqslant s \leqslant 1, \\ 0, & t = 1, \text{ or } s = 0, \text{ or } s = 1. \end{cases}$$

Let $\tilde{\varphi}\colon I \times I \to I$ be a continuous extension of φ. Then define a partial homotopy H on a subset $K_0 \times I \times I \cup K \times I \times \{0\} \cup K \times \{0\} \times I \cup K \times \{1\} \times I$ of $K \times I \times I$ by

$$H(x, s, t) = \zeta_{\tilde{\varphi}(s,t)}(x), \quad (x, s, t) \in K_0 \times I \times I,$$
$$H|K \times I \times \{0\} = \zeta,$$
$$H|K \times \{0\} \times I = g,$$
$$H|K \times \{1\} \times I = \zeta_1.$$

Applying the HET in $N(\varepsilon_0-d, A_0)$, we see that g is homotopic rel K_0 (in $N(\varepsilon_0-d, A_0)$) to a $\mathscr{C}_p$-map $\zeta_1\colon (K, K_0) \to (V, W_0)$. This clearly suffices.

Combining the last two lemmas we have the following theorem.

(6.5) THEOREM. *If* $\lim d_Z(A_n, A_0) = 0$ *and* A_n $-\mathrm{mo}(\mathscr{C}_p)\to$ A_0 $(A_n$ $-\mathrm{mo}(\mathscr{C}_p^*)\to$ $A_0)$, *then* $\lim \delta(\varepsilon, A_n)$ $(\lim \delta^*(\varepsilon, A_n))$ *exists and equals* $\delta(\varepsilon, A_0)$ $(\delta^*(\varepsilon, A_0))$ *almost everywhere in* I^*.

We are now ready to introduce metrices d_1 and d_2 on hyperspaces $\mathrm{mo}(\mathscr{C}_p, X)$ and $\mathrm{mo}(\mathscr{C}_p^*, X)$ of all weakly $\mathscr{C}_p$-movable and all $\mathscr{C}_p$-movable compacta in a metric space X, respectively. Let E be the Banach space of all bounded measurable functions on the interval I^*, the norm of an element f in E being defined as:

$$\|f\| = \int_0^1 |f|\, d\varepsilon.$$

We define a correspondence between $\mathrm{mo}(\mathscr{C}_p, X)$ and a subset of $(2^X, d_Z) \times E$ by assigning to each element A of $\mathrm{mo}(\mathscr{C}_p, X)$ the element $(A, \delta(\varepsilon, A))$ of $2^X \times E$. This correspondence is one-to-one, so a metric d_1 is defined on $\mathrm{mo}(\mathscr{C}_p, X)$ by letting the distance between two points in $\mathrm{mo}(\mathscr{C}_p, X)$ be the distance between the corresponding points in $2^X \times E$. Specifically, if $A, B \in \mathrm{mo}(\mathscr{C}_p, X)$, then

$$d_1(A, B) = \Big[d_Z^2(A, B) + \Big(\int_0^1 |\delta(\varepsilon, A) - \delta(\varepsilon, B)|\, d\varepsilon\Big)^2\Big]^{1/2}.$$

Similarly, we can define a metric d_2 on $\mathrm{mo}(\mathscr{C}_p^*, X)$ by

$$d_2(A, B) = \Big[d_Z^2(A, B) + \Big(\int_0^1 |\delta^*(\varepsilon, A) - \delta^*(\varepsilon, B)|\, d\varepsilon\Big)^2\Big]^{1/2},$$

for $A, B \in \mathrm{mo}(\mathscr{C}_p^*, X)$.

With obvious modifications the argument on the page 444 in [Be] shows that the metric d_1 induces the same topology on $\mathrm{mo}(\mathscr{C}_p, X)$ as that naturally defined in terms of the metric d_Z and the weakly $\mathscr{C}_p$-movable convergence. Hence, we can prove the following two theorems.

(6.6) THEOREM. *There is a metric* d_1 *on the hyperspace* $\mathrm{mo}(\mathscr{C}_p, X)$ *of all weakly* $\mathscr{C}_p$*-movable compacta in a metric space* X *such that, for a sequence* $A_0, A_1, A_2, \ldots$ *in* $\mathrm{mo}(\mathscr{C}_p, X)$, $\lim d_1(A_n, A_0) = 0$ *iff* $\lim d_Z(A_n, A_0) = 0$ *and* A_n $-\mathrm{mo}(\mathscr{C}_p)\to$ A_0.

(6.7) THEOREM. *There is a metric d_2 on the hyperspace* $\mathrm{mo}(\mathscr{C}_p^*, X)$ *of all $\mathscr{C}_p$-movable compacta in a metric space X such that, for a sequence $A_0, A_1, A_2, \ldots$ in* $\mathrm{mo}(\mathscr{C}_p^*, X)$, $\lim d_2(A_n, A_0) = 0$ *iff* $\lim d_Z(A_n, A_0) = 0$ *and* $A_n - \mathrm{mo}(\mathscr{C}_p^*) \to A_0$.

(6.8) Remark. By (3.7) and results in [Č3] and [Č4], the metric d_1 on $\mathrm{mo}(\mathscr{P}_p, X)$ (i.e., on the collection of all FANR's in X) is equivalent to a metric d_0 defined as follows. For $A, B \in \mathrm{mo}(\mathscr{P}_p, X)$, put

$$d_0(A, B) = \Big[d_Z^2(A, B) + \Big(\int_0^1 |\gamma_{\mathscr{P}}(\varepsilon, A) - \gamma_{\mathscr{P}}(\varepsilon, B)|\, d\varepsilon\Big)^2\Big]^{1/2},$$

where $\gamma_{\mathscr{P}}(-, -)$ is a function defined in (4.1) in [Č4].

Both d_0 and d_1 are on $\mathrm{mo}(\mathscr{P}_p, X)$ equivalent to the metric d_{Z^*} as the corollary to our next theorem shows.

(6.9) THEOREM. *Suppose* $\lim d_Z(A_n, A_0) = 0$ *and* $A_n - \mathrm{ca}(\mathscr{P}) \to A_0$. *Then there is an index n_0 such that $A_n \in X[A_0]_Z$ for all $n \geq n_0$ and* $\lim d_{Z^*}(A_{n+n_0}, A_0) = 0$.

Proof. We shall prove that almost all A_n are Z-equivalent to A_0. The second part in the conclusion of the theorem will be clear from our proof.

By an improvement of (2.4) in [Č4] involving d_Z instead of d_F, A_0 is $\mathscr{P}$-calm. Assume that $X \subset Q$ and pick a compact ANR neighborhood V of A_0 in Q and an index n_0 such that $\mathscr{P}_h(Q, V; A_n)$ holds for all $n \geq n_0$ and for $n = 0$. Let $\varepsilon > 0$ has the property that (2ε)-close maps into V are homotopic in V. Finally, select $n_1 \geq n_0$ so that $d_Z(A_n, A_0) < \varepsilon$ for all $n \geq n_1$.

Let $n \geq n_1$ and let W be an arbitrary neighborhood of A_n in Q. Pick a neighborhood U of A_n in $V \cap W$ using $\mathscr{P}_h(Q, V; A_n)$. Then take a neighborhood W' of A_0 in Q and an ε-map $f\colon W' \to U$. If W_0' is a neighborhood of A_0 in W', pick a compact ANR neighborhood W_0 of A_n in U and an ε-map $g\colon W_0 \to W_0'$. The maps $f \circ g$ and $i_{W_0,W}$ are (2ε)-close maps of W_0 into V. Hence, they are homotopic in V and therefore also in W. This shows that A_0 Z-dominates A_n. In a similar way one proves that A_n Z-dominates A_0.

(6.10) COROLLARY. $\lim d_Z(A_n, A_0) = 0$ *and* $A_n - \mathrm{mo}(\mathscr{P}_p) \to A_0$ *iff there is an index n_0 such that $A_n \in X[A_0]_Z$ for all $n \geq n_0$ and* $\lim d_{Z^*}(A_{n+n_0}, A_0) = 0$ *and* $A_0 \in \mathrm{mo}(\mathscr{P}_p, X)$.

Proof. Apply (3.7) and (6.9) to prove necessity and (3.13) to prove sufficiency.

The results in §§ 3 and 5 imply the following corollaries.

(6.11) COROLLARY. *If a metric space (X, d) is homeomorphic to a metric space (Y, ϱ), then $(\mathrm{mo}(\mathscr{C}_p, X), d_1)$ is homeomorphic to $(\mathrm{mo}(\mathscr{C}_p, Y), \varrho_1)$ and $(\mathrm{mo}(\mathscr{C}_p^*, X), d_2)$ is homeomorphic to $(\mathrm{mo}(\mathscr{C}_p^*, Y), \varrho_2)$.*

Proof. The proof is similar to the proof of (4.7) in [Č3].

(6.12) COROLLARY. *The inclusions* $(\mathrm{mo}(\mathscr{C}_p, X), d_{Z^*}) \to (\mathrm{mo}(\mathscr{C}_p, X), d_1)$, $(\mathrm{mo}(\mathscr{C}_p, X), d_1) \to (2^X, d_Z)$, $(\mathrm{mo}(\mathscr{C}_p^*, X), d_2) \to (2^X, d_Z)$, *and* $(\mathrm{mo}(\mathscr{C}_p^*, X), d_2) \to (\mathrm{mo}(\mathscr{C}_p, X), d_1)$ *are continuous.*

Proof. For the first inclusion use (3.13) while the last three inclusions are obviously continuous.

(6.13) COROLLARY. *Let α be a property preserved by Z-domination or Z-equivalence. Then the collection of all elements of* $\mathrm{mo}(\mathscr{P}_p, X)$ $(\mathrm{mo}(\mathscr{P}_p^*, X))$ *which have property α constitute a closed and open subset of* $\mathrm{mo}(\mathscr{P}_p, X)$ $(\mathrm{mo}(\mathscr{P}_p^*, X))$.

Proof. Use (6.10) and (6.12).

The proofs in § 5 and in [ČŠ, § 2] show that every property considered in § 5 is preserved by Z-domination. Observe that results in § 5 and in § 5 of [Č3] allow a formulation of results similar to (6.13) for some other classes $\mathscr{C}_p$.

(6.14) COROLLARY. *Let a class $\mathscr{C}_p$ quasi-dominates a class $\mathscr{D}_p$. Then the inclusions* $(\mathrm{mo}(\mathscr{C}_p, X), d_1) \to (\mathrm{mo}(\mathscr{D}_p, X), d_1)$ *and* $(\mathrm{mo}(\mathscr{C}_p^*, X), d_2) \to (\mathrm{mo}(\mathscr{D}_p^*, X), d_2)$ *are continuous.*

Proof. See (3.4).

We leave many questions concerning the topological structure of metric spaces $(\mathrm{mo}(\mathscr{C}_p, X), d_1)$ and $(\mathrm{mo}(\mathscr{C}_p^*, X), d_2)$ open. The most natural problem would be to see what properties of X are carried over onto $(X[A]_Z, d_1)$ and $(X[B]_Z, d_2)$, where $A \in \mathrm{mo}(\mathscr{C}_p, X)$ and $B \in \mathrm{mo}(\mathscr{C}_p^*, X)$. In particular, are those spaces separable (topologically complete) if X is separable (topologically complete). The last two questions are in view of (6.12) and the method of proofs for Theorems 2 and 3 in [Be] equivalent to the following questions.

(6.15) If X is a separable (topologically complete) metric space and $A \in \mathrm{mo}(\mathscr{C}_p, X)$, is $(X[A]_Z, d_Z)$ also separable (topologically complete)?

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