Generalizations of Lašnev's theorem

by

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Abstract. Lašnev's theorem asserts that any closed mapping defined on a metric space can be decomposed into the union of a perfect mapping and a mapping with a σ-discrete range. We prove that the theorem holds if the domain is a regular σ-space or a Čech complete space satisfying certain additional conditions. We construct examples showing that the assumption that the domain is a Hausdorff σ-space or a p-space is not sufficient.

We investigate spaces X with the following property:

$(*)$ for any Y and any closed mapping $f\colon X \to Y$, $Y = Y_0 \cup \bigcup_{n \geq 1} Y_n$, where $f^{-1}(y)$ is compact for $y \in Y_0$ and Y_n is closed and discrete in Y for $n \geq 1$.

It has been proved by Lašnev [L] that metric spaces satisfy $(*)$. A list of generalizations of Lašnev's result with exact references may be found in [B, section 4] (see also [W1], [A] and [D]).

In the first section we prove that regular σ-spaces satisfy $(*)$ and construct a Hausdorff σ-space and a σ-compact space which do not satisfy $(*)$.

In the second section we prove that in the class of Čech complete spaces $(*)$ is a consequence of the following property:

$(**)$ for any closed subset X' of X and any closed mapping f of X' onto a compact space Y', $\operatorname{Fr} f^{-1}(y)$ is compact for $y \in Y'$.

From the results of the first section (see [W1]), it follows that $(*)$ does not imply $(**)$ in general.

The difficulties in replacing Čech completeness by the p-space property seem to be connected with the fact that p-spaces are not preserved by perfect mappings. We modify the example of a perfect mapping not preserving the p-space property from [W2] in order to construct a p-space which does not satisfy $(*)$.

In the third section we discuss a property $(*')$ which is a weakening of $(*)$ obtained by replacing the condition of compactness in $(*)$ by the Lindelöf property. We obtain results explaining why the mappings in the examples constructed in the first two sections have the Lindelöf fibers.

We use the terminology and notation from [E]. All mappings are assumed to be continuous and onto. All spaces are assumed to be regular unless it is stated otherwise.

A space X is said to be a *σ-space* if it has a σ-locally finite closed network [O].

A space X is said to be a *strong Σ-space* [N] if it has a sequence $\{\mathscr{E}_n\}_{n\geqslant 1}$ of locally finite closed collections and a cover $\mathscr{K}$ consisting of compact sets such that if $K\in\mathscr{K}$ and U is an open set containing K then $K\subset E\subset U$ for a certain $E\in\bigcup_{n\geqslant 1}\mathscr{E}_n$. Any sequence $\{\mathscr{E}_n\}_{n\geqslant 1}$ satisfying the above conditions will be called a *σ-locally finite network for* $\mathscr{K}$.

1. Lašnev's theorem for σ-spaces. The main result of this section is

THEOREM 1.1. *If X is a σ-space, then X satisfies* (∗).

Proof. Let $\{\mathscr{E}_n\}_{n\geqslant 1}$ be an increasing sequence of locally finite closed covers of X such that $\bigcup_{n\geqslant 1}\mathscr{E}_n$ is a network of X.

For $n\geqslant 1$ put $\mathscr{F}_n=\{f(E)\colon E\in\mathscr{E}_n\}$ and $F_n(y)=\bigcap\{F\in\mathscr{F}_n\colon y\in F\}$. Since f is a closed mapping, it follows that $\mathscr{F}_n$ and, consequently, $\{F_n(y)\colon y\in Y\}$ are closure-preserving. Therefore, $Y_n=\{y\in Y\colon F_n(y)=\{y\}\}$ is closed and discrete in Y for $n\geqslant 1$.

It remains to prove that $f^{-1}(y)$ is compact for $y\in Y_0$, where $Y_0=Y\setminus\bigcup_{n\geqslant 1}Y_n$.

Fix a point $y\in Y_0$ and observe that

1) $\{F_n(y)\}_{n\geqslant 1}$ is a decreasing network on y in Y. Thus $y\in Y_0$ and the fact that Y is a T_1-space imply that each $F_n(y)$ is an infinite set. Therefore, we can choose a sequence $\{y_n\}_{n\geqslant 1}$ of distinct points of Y such that $y_n\in F_n(y)$. From the definition of the sets $F_n(y)$ it follows that for $n\geqslant m\geqslant 1$.

2) $E\in\mathscr{E}_m$ and $E\cap f^{-1}(y)\neq\emptyset$ imply $E\cap f^{-1}(y_n)\neq\emptyset$.

We shall first show that $f^{-1}(y)$ has the Lindelöf property (in fact, a countable network). This will follow from the fact that $\{E\in\mathscr{E}_n\colon E\cap f^{-1}(y)\neq\emptyset\}$ is finite for $n\geqslant 1$.

Assume that for an $m\geqslant 1$ the above set is infinite and let $E_m, E_{m+1},\ldots$ be distinct elements of $\mathscr{E}_m$ intersecting $f^{-1}(y)$. From 2) it follows that for $n\geqslant m$, $E_n\cap f^{-1}(y_n)\neq\emptyset$. Therefore, since f, is a closed mapping, $\{y_n\}_{n\geqslant m}$ does not have any accumulation point in Y. This is a contradiction, for 1) shows that this sequence converges to y. The contradiction shows that $f^{-1}(y)$ has the Lindelöf property.

Assume that $f^{-1}(y)$ is not compact. Then there exists a countable family $\mathscr{U}$ of open subsets of X covering $f^{-1}(y)$ such that no finite union of the closures of elements of $\mathscr{U}$ covers $f^{-1}(y)$. Thus we can construct an increasing sequence $\{U_k\}_{k\geqslant 1}$ of open subsets of X covering $f^{-1}(y)$ such that $f^{-1}(y)\cap U_{k+1}\setminus\bar{U}_k\neq\emptyset$.

Choose $x_k\in f^{-1}(y)\cap U_{k+1}\setminus\bar{U}_k$ and $E_k\in\mathscr{E}_{n_k}$ for an $n_k>n_{k-1}$ such that $x_k\in E_k\subset X\setminus\bar{U}_k$. By virtue of 2), $E_k\cap f^{-1}(y_{n_k})\neq\emptyset$. Let z_k be an element of this intersection. The sequence $\{f(z_k)\}_{k\geqslant 1}$, as a subsequence of $\{y_n\}_{n\geqslant 1}$, converges to y. Thus $\{z_k\}_{k\geqslant 1}$ has an accumulation point $z\in f^{-1}(y)$. Then $z\in U_{k_0}$ for a certain $k_0\geqslant 1$ and $x_k\notin U_{k_0}$ for $k>k_0$. The contradiction shows that $f^{-1}(y)$ is compact.

COROLLARY 1.1 [W1]. *Moore spaces satisfy* (∗).

Remark 1.1. A space obtained by adding to a countable discrete space N a maximal family of almost disjoint infinite subsets of N [E,3.0.I] is a σ-space (in fact, a Moore space) which does not satisfy (∗∗) [K]. Thus (∗) does not imply (∗∗) in general. However, there is a connection between (∗) for σ-spaces and some properties of the type (∗∗), for the last part of our proof, showing that $f^{-1}(y)$ is compact provided that it has the Lindelöf property, can be deduced by considering the restriction of f to $X'=f^{-1}(y)\cup\bigcup_{n\geqslant 1}f^{-1}(y_n)$ and using either Lemma 3.7 from [WW] or, as observed by T. Przymusiński, the fact that any countable discrete collection of points of $f^{-1}(y)$ has an expansion open and discrete in X' [SA].

We shall illustrate Theorem 1.1 by constructing two spaces which can be mapped onto the interval $I=[0,1]$ by a closed mapping with no compact fibre. Thus these spaces will not satisfy (∗). The spaces will be obtained by modifying the topology of $I\times I$ and the mapping will be the projection onto the first factor. The modifications resemble the construction of the Alexandroff double circle [E,3.1.G].

The first example shows that Theorem 1.1 does not hold for Hausdorff σ-spaces.

EXAMPLE 1.1. A Hausdorff space X with a countable network and a closed mapping f of X onto the interval with no compact fibre.

The space X is obtained from $I\times I$ by adding to the collection of the closed subsets all the sets of the form $\{s\}\times(0,1]$. Clearly, X and the projection f of X onto the first factor have the above properties.

The next example shows that (∗) does not hold for σ-compact spaces. In particular, it follows that (∗) does not hold for strong Σ-spaces.

EXAMPLE 1.2. A σ-compact space X having a closed mapping f onto the interval with no compact fibre.

The space X is $I\times I$ with a topology such that the intervals of the form $\{s\}\times(0,1]$ are open subsets of X and have their usual topology while the base of neighbourhoods of $(s,0)\in X$ consists of the sets of the form $V\times I\setminus(\{s\}\times(0,1])$, where V is a neighbourhood of s in I.

It is easy to check that $I\times(\{0\}\cup[1/n,1])$ is a compact subset of X. Thus X

is a σ-compact space. Clearly, the projection f of X onto the first factor is a closed mapping.

Observe that the space X is not a perfect preimage of any σ-space.

PROBLEM 1.1. Do perfect preimages of σ-spaces satisfy $(*)$?

Do perfect strong Σ-spaces satisfy $(*)$? (see [Ch1,3.3]).

2. Lašnev's theorem for Čech complete spaces. We shall use property $(**)$ in order to generalize and unify the known results asserting that Čech complete spaces satisfying certain additional conditions satisfy $(*)$.

Assume that $f\colon X \to Y$ is a closed mapping and $\mathscr{U}$ is an open cover of X. Let $Y(\mathscr{U})$ denote the set of points $y \in Y$ such that no finite subcollection of $\mathscr{U}$ covers $f^{-1}(y)$.

LEMMA 2.1. *If* $\operatorname{Fr} f^{-1}(y)$ *is compact for* $y \in Y$, *then* $Y(\mathscr{U})$ *is closed and discrete in* Y.

Proof. If $\mathscr{U}'$ is a finite subcollection of $\mathscr{U}$ covering $\operatorname{Fr} f^{-1}(y)$, then $\{y\} \cup \{y' \in Y\colon f^{-1}(y') \subset \bigcup \mathscr{U}'\}$ is a neighbourhood of y containing at most one point of $Y(\mathscr{U})$.

THEOREM 2.1. *If a Čech complete space X satisfies $(**)$, then X satisfies $(*)$.*

Proof. Let $f\colon X \to Y$ be a closed mapping and let $\{\mathscr{U}_n\}_{n \geq 1}$ be a complete sequence of open covers of X [E,3.9.2].

Put $Y_n = Y(\mathscr{U}_n)$ and observe that Lemma 2.1 and $(**)$ imply that the intersection of Y_n with any compact subset of Y is finite. Since Y is a k-space, it follows that Y_n is closed and discrete in Y.

If $y \in Y_0 = Y \setminus \bigcup_{n \geq 1} Y_n$, then $f^{-1}(y)$ has a complete sequence consisting of finite open covers, which implies that $f^{-1}(y)$ is compact.

Spaces satisfying $(**)$ are discussed in [SA,1.10]. In particular, Theorem 2.1 yields

COROLLARY 2.1 [D,3.8]. *Dieudonné complete, Čech complete spaces satisfy* $(*)$.

COROLLARY 2.2. *If countably compact closed subsets of a normal Čech complete space X are compact, then X satisfies $(*)$.*

Using the method of proof of Lemma 3.7 from [WW], one can show that meta-Lindelöf spaces satisfy $(**)$. This gives (see [A])

COROLLARY 2.3. *Meta-Lindelöf Čech complete spaces satisfy* $(*)$.

For locally compact spaces we obtain

THEOREM 2.2. *If a locally compact space X* satisfies $(**)$ *and $f\colon X \to Y$ is a closed mapping, then the set of the points $y \in Y$ such that $f^{-1}(y)$ is not compact is closed and discrete in Y.*

The difficulties in extending the above results to the class of p-spaces are illustrated by the following

EXAMPLE 2.1. A p-space X and a closed mapping $f\colon X \to Y$ such that $\operatorname{Fr} f^{-1}(y)$ is compact for $y \in Y$ but the set of the points $x \in Y$ such that $f^{-1}(y)$ is not compact is not σ-discrete in Y.

Let Y be a space obtained by adding to an uncountable discrete space D a maximal almost disjoint family $\varDelta$ of countable subsets of D. The topology of Y is such that the points of D are isolated and each neighbourhood of $\delta \in \varDelta$ contains all but a finite number of points of $\delta \subset D$ [E,3.6.I].

Consider $X = Y \times M \setminus (D \times \{0\})$, where M is a sequence convergent to the point 0.

The fact that X is a p-space can be proved as in [W2] or [Ch2,3.2].

The projection f of X onto Y is a closed mapping and $\operatorname{Fr} f^{-1}(y)$ is compact for $y \in Y$. The set D of the elements of Y with a noncompact inverse image is not σ-discrete in Y because, by virtue of the maximality of $\varDelta$, no infinite subset of D is closed in Y.

Observe that the space X does not have property $(**)$ because the composition of f and a mapping identifying $\varDelta$ to a point maps X onto the Alexandroff compactification of D and the only non-isolated fibre of this composition has the noncompact boundary.

Applying the method of proof of Theorem 2.1, one can deduce that X is not Čech complete. We do not know any example showing that $(**)$ is essential in Theorem 2.1.

We conjecture that in the class of submetacompact ($=\theta$-refinable) p-spaces property $(**)$ implies $(*)$. This conjecture is supported by the fact that submetacompact p-spaces are preserved by perfect mappings [W2]. In fact, one can modify the proof of this fact in order to show that any closed mapping f defined on a submetacompact p-space has a decomposition as in $(*)$ if the boundaries of the fibres of f are compact.

3. A weak form of Lašnev's theorem. The mappings defined in the examples of the first section have Lindelöf fibres. This fact can be explained by the following two propositions, which can be deduced from the first part of the proof of Theorem 1.1.

PROPOSITION 3.1. *If X is a T_1 σ-space, then X satisfies* $(*')$ *(we do not assume that the Lindelöf fibres are regular).*

PROPOSITION 3.2. *If X is a strong Σ-space and $f\colon X \to Y$ is a closed mapping, then the set of $y \in Y$ such that $f^{-1}(y)$ does not have the Lindelöf property is contained in a countable union of closed subsets of Y having a closure-preserving cover by finite sets and, consequently, is weakly σ-discrete* [Y].

If X is a k-space, then one can use the reasoning from [F] in order to improve Proposition 3.2 (in fact, it is sufficient to assume that X is an av-space [7]).

PROPOSITION 3.3. *If a k-space X is a strong Σ-space, then X satisfies* $(*')$.

Proof. Let $\mathcal{K}$ be a cover of X consisting of compact sets and let $\{\mathcal{E}_n\}_{n\geqslant 1}$ be a σ-locally finite network for $\mathcal{K}$.

Denote by Y_n the set of the points $y \in Y$ such that $\{E \in \mathcal{E}_n\colon E \cap f^{-1}(y) \neq \emptyset\}$ is infinite.

If Y_n is not a discrete closed subset of Y, then from the fact that Y is a k-space it follows that Y_n contains a countable subset $\{y_m\colon m \geqslant 1\}$ which is not closed in Y. This is a contradiction, for one can find a sequence $\{E_m\}_{m\geqslant 1}$ of distinct elements of $\mathcal{E}_n$ such that $E_m \cap f^{-1}(y_m) \neq \emptyset$ and f is a closed mapping.

It is easy to see that $y \notin \bigcup_{n\geqslant 1} Y_n$ implies that $f^{-1}(y)$ has the Lindelöf property.

The following example shows that locally finite collections cannot be replaced by closure-preserving collections in Propositions 3.2 and 3.3.

Example 3.1. A paracompact space X with a closure-preserving cover by compact sets and a closed mapping f of X onto the interval with no Lindelöf fibre.

The space X is obtained from the space X constructed in Example 1.2 by isolating the points of $I \times (0, 1]$. For each $t \in (0, 1]$, the set $K_t = I \times \{0, t\}$ is a compact subset of X and $\{K_t\colon t \in (0, t]\}$ forms a closure-preserving cover of X.

It is easy to check that the projection f of X onto the first factor is a closed mapping and the fibres of f do not have the Lindelöf property.

The reasoning used in the proof of Theorem 2.1 gives

Proposition 3.4. *If a p-space X satisfies $(**)$, then X satisfies $(*')$.*

The same reasoning explains why the mapping f defined in Example 2.1 has Lindelöf fibres.

Observe that, by virtue of Proposition 3.3, subparacompact p-spaces satisfy $(*')$.

Problem 3.1. Do subparacompact p-spaces satisfy $(*)$?

Do perfect preimages of Moore spaces satisfy $(*)$?

Do perfectly subparacompact p-spaces satisfy $(*)$?

(see [Ch1,3.2]).

References

[A] A. V. Arhangel'skiĭ, *On closed mappings, bicompact spaces and a problem of P. Alexandroff*, Pacific J. Math. 18 (1968), pp. 201–208.

[B] D. K. Burke, *Closed mappings*, Surveys in General Topology, Academic Press 1980, pp. 1–32.

[Ch1] J. Chaber, *Closed mappings onto metric spaces*, preprint.

[Ch2] — *Perfect images of p-spaces*, preprint.

[D] N. Dykes, *Mappings and realcompact spaces*, Pacific J. Math. 31 (1969), pp. 347–358.

[E] R. Engelking, *General Topology*, Warszawa 1977.

[F] V. V. Filippov, *On the "degree of noncompactness" of a closed mapping of a paracompactum*, Moscow Univ. Math. Bull. 27 (1972), Nr. 3–4 (1973), pp. 78–79.

[K] J. Kofner, *Closed mappings and quasi-metrics*, Proc. Amer. Math. Soc. 80 (1980), pp. 333–336.

[L] N. Lašnev, *Continuous decompositions and closed mappings of metric spaces*, Soviet Math. Dokl. 6 (1965), pp. 1504–1506.

[N] K. Nagami, *Σ-spaces*, Fund. Math. 55 (1969), pp. 169–192.

[O] A. Okuyama, *Some generalizations of metric spaces, their metrization theorems and product spaces*, Sci. Rep. Tokyo, Kyoiku, Daigaku 9 (1967), pp. 236–254.

[SA] M. K. Singal and S. P. Arya, *On a theorem of Michael–Morita–Hanai*, General Topology and its relations to Modern Analysis and Algebra IV, Prague 1977, pp. 434–444.

[W1] J. M. Worrell, Jr., *On boundaries of elements of upper semicontinuous decompositions I*, Notices Amer. Math. Soc. 12 (1965), p. 219.

[W2] — *A perfect mapping not preserving the p-space property*, preprint.

[WW] — and H. H. Wicke, *Perfect mappings and certain interior images of M-spaces*, Trans. Amer. Math. Soc. 181 (1973), pp. 23–35.

[Y] Y. Yajima, *On spaces which have a closure-preserving cover by finite sets*, Pacific J. Math. 69 (1977), pp. 571–578.

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