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# Superspaces of $(s)$ with basis

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**Abstract.** The sequence space $(s)$ is characterized as the unique nuclear Fréchet space $F$ with basis and a continuous norm such that when $F$ is isomorphic to a subspace of a nuclear Fréchet space $E$ with a basis $(x_n)$, then a subsequence of $(x_n)$ generates a subspace isomorphic to $F$. Variations of this result are considered in the context of stable $D_1$ and $D_2$ spaces.

**1. Introduction.** A superspace of $(s)$, the space of rapidly decreasing sequences, is a nuclear Fréchet space which contains a subspace isomorphic to $(s)$. The simplest way to construct such a space is to take an arbitrary nuclear Fréchet space $E$ and form the cartesian product $E\times(s)$. We will show that if the superspace is assumed to have a basis, it actually always has this simple form. In fact, the basis of the superspace can be divided into two disjoint subsequences one of which generates a (necessarily complemented) subspace isomorphic to $(s)$. Among nuclear Fréchet spaces with basis and a continuous norm this property of superspaces characterizes $(s)$. We make a generalization to arbitrary stable $D_1$ spaces and consider as a natural dualization a stable $D_2$ space as a quotient space.

**2. Preliminaries.** If not otherwise stated, the subscripts and superscripts appearing are assumed to run through $\boldsymbol{N} = \{1, 2, \ldots\}$. The symbol $\boldsymbol{K}$ stands for the scalar field (real or complex numbers).

We refer to [12], [3] and [8] for the undefined concepts and the basic results used.

Let $E$ be a nuclear Fréchet space and let the topology of $E$ be defined by an increasing sequence $(p_k)$ of seminorms. Suppose the sequence $(x_n)$ of $E$ is a basis, that is, for every $x\in E$ there is a unique sequence $(\xi_n)$ of scalars such that

$$x = \sum_n \xi_n x_n. \tag{1}$$

By the absolute basis theorem ([12], 10.2.1) $(x_n)$ is absolute, i.e. in (1),

$$\sum_n |\xi_n| p_k(x_n) < \infty$$

for all $k$. Denote $a_n^k = p_k(x_n)$,

$$K(a) = \{(\xi_n) \in \mathbf{K}^{\mathbf{N}} \mid |(\xi_n)|_k = \sum_n |\xi_n| a_n^k < \infty \ \forall k\}. \tag{2}$$

If the sequence space $K(a)$ is equipped with the topology defined by the seminorms $|\cdot|_k$, then the assignment $x \mapsto (\xi_n)$ defines an isomorphism $E \to K(a)$. The space $K(a)$ is called the *Köthe space* associated with $E$, $(x_n)$. The infinite matrix $a = (a_n^k)$ is called a *Köthe matrix representing* $(x_n)$ and it has the properties $0 \leqslant a_n^k \leqslant a_n^{k+1}$, $\sup_k a_n^k > 0$, for each $k$ there is $l$ with $(a_n^k/a_n^l) \in l_1$ (we agree $0/0 = 0$). Conversely, every such matrix defines through (2) a nuclear Fréchet space with $(e_n)$, the sequence of coordinate vectors, as a basis. In particular, $(s) = K(n^k)$.

Consider another Köthe space

$$K(b) = \{(\eta_n) \mid \sum_n |\eta_n| b_n^k < \infty \ \forall k\}$$

and let $T\colon K(b) \to K(a)$ be a continuous linear mapping. If $T$ is represented by the matrix $(t_{in})$, that is,

$$Te_n = \sum_i t_{in} e_n,$$

then the adjoint $T'\colon K(a)' \to K(b)'$ is represented by the transpose of the matrix $(t_{in})$,

$$T'e_i' = \sum_n t_{in} e_n'.$$

Here $(e_i')$ (resp. $(e_n')$) is the sequence of coordinate functionals of the coordinate basis $(e_i)$ of $K(a)$ (resp. $(e_n)$ of $K(b)$). One easily checks that $(e_i')$ and $(e_n')$ are weak bases. We will use the identifications

$$K(a)' = \{(\xi_n) \mid \exists k, C\colon |\xi_n| \leqslant C a_n^k\}, \qquad K(b)' = \{(\eta_n) \mid \exists k, C\colon |\eta_n| \leqslant C b_n^k\}.$$

Here $C$ denotes a positive constant. Now

$$T'(\xi_i) = \Big(\sum_i t_{in} \xi_i\Big)_n.$$

In comparing two Köthe matrices $(a_n^k)$ and $(b_n^k)$ we use the following notations: $(a_n^k) \lesssim (b_n^k)$ means for all $k$ there is $l$ with $(a_n^k/b_n^l) \in l_\infty$, $(a_n^k) \sim (b_n^k)$ means $(a_n^k) \lesssim (b_n^k)$ and $(b_n^k) \lesssim (a_n^k)$. If $(a_n^k) \sim (b_n^k)$, then $K(a) = K(b)$.

A Köthe matrix $(a_n^k)$ is *regular* if the sequence $(a_n^k/a_n^{k+1})$ is non-increasing for every $k$. We say that $(p_n^k)$ is a $D_1$ *matrix* if it is regular, $p_n^1 = 1$ and $((p_n^k)^2) \lesssim (p_n^k)$. Correspondingly, $(q_n^k)$ is a $D_2$ *matrix* if it is regular, $q_n^k > 0$, $\lim_k q_n^k = 1$ and $(q_n^k) \lesssim ((q_n^k)^2)$. The definitions of the types $D_1$ and $D_2$ are equivalent to those in [1].



Let $E$ be a locally convex space and denote by $\mathscr{U}_E$ its neighborhood basis at 0 consisting of barrels. Denote by $d_n(V, U)$ $(n = 0, 1, 2, \ldots)$ the $n^{\text{th}}$ Kolmogorov diameter of $V \in \mathscr{U}_E$ with respect to $U \in \mathscr{U}_E$. Suppose $(p_n^k)$ is a $D_1$ matrix. We say that $E$ is $K(p, \mathbf{N})$-*nuclear* if for every $U$ and $k$ there is $V$ such that $(p_n^k d_{n-1}(V, U))_n \in l_\infty$. If $(q_n^k)$ is a $D_2$ matrix, $E$ is said to be $K(q)$-*nuclear* if for every $U$ there is $V$ and $k$ such that $((1/q_n^k) d_{n-1}(V, U))_n \in l_\infty$. The $K(p, \mathbf{N})$-nuclearity was introduced in [13] with the type $G_\infty$ in place of $D_1$. These two types are however equivalent and so are, respectively, $G_1$ and $D_2$ ([14], [16]). The above definition of $K(q)$-nuclearity is consistent with the general concept of $\lambda$-nuclearity ([4], [15]).

A locally convex space $E$ is *stable* if $E \times E \simeq E$. If $(p_n^k)$ is a $D_1$ matrix, the stability of $K(p)$ is equivalent to $(p_{2n}^k) \lesssim (p_n^k)$ and if $(q_n^k)$ is a $D_2$ matrix, the stability of $K(q)$ is equivalent to $(q_n^k) \lesssim (q_{2n}^k)$ (see [17]); we also call the Köthe matrix in question stable.

The following is proved in [5]:

PROPOSITION 2.1. (i) *If $(p_n^k)$ is a stable $D_1$ matrix, $K(a)$ is $K(p, \mathbf{N})$-nuclear and $a_n^1 = 1$, then there is a strictly increasing sequence $(i_n)$ of indices such that $(p_n^k) \lesssim (a_{i_n}^k)$ and $\sup_n i_n/n < \infty$.*

(ii) *If $(q_n^k)$ is a stable $D_2$ matrix, $K(a)$ is $K(q)$-nuclear and $0 < a_n^k < 1$, then there is a strictly increasing sequence $(i_n)$ of indices such that $(a_{i_n}^k) \lesssim (q_n^k)$ and $\sup_n i_n/n < \infty$.*

We will also need an important result from combinatorics called Hall's theorem.

THEOREM 2.2. (Hall) *Let $A$ be a set and suppose $(A_i)_{i \in I}$ is a family of finite subsets of $A$. There is a system of distinct representatives $a_i \in A_i$, $i \in I$, if and only if the following condition is satisfied: for all distinct $i_1, \ldots, i_k \in I$ the set $A_{i_1} \cup \ldots \cup A_{i_k}$ has at least $k$ elements.*

For a proof and a thorough discussion of theorems of this type we refer to [9].

**3. Superspaces of $(s)$.** We begin with a general result on imbedding one nuclear Fréchet space with basis into another.

PROPOSITION 3.1. *Let $E$ and $F$ be nuclear Fréchet spaces with bases $(x_n)$ and $(y_n)$, respectively and suppose $F$ has a continuous norm. If $F$ is isomorphic to a subspace of $E$, then there are representations $(a_n^k)$ and $(b_n^k)$ of $(x_n)$ and $(y_n)$, respectively, such that for each $k$ there is an injection $n \mapsto j_n^k$ and scalars $\mu_n^k > 0$ with*

$$b_n^k \leqslant \mu_n^k a_{j_n^k}^{k+1}, \quad n \in \mathbf{N}, \tag{i}$$

$$\mu_n^k a_{j_n^k}^l \leqslant b_n^{l+1}, \quad l, n \in \mathbf{N}. \tag{ii}$$

Proof. Let $(c_n^k)$ and $(d_n^k)$, $d_n^k > 0$, be representations of $(x_n)$ and $(y_n)$, respectively, and let $T\colon K(d)\to K(c)$ be an imbedding (i.e. isomorphism into) represented by a matrix $(t_{in})$. Set

$$U_k^* = \{(\xi_n) \in K(c) \mid \sup_n |\xi_n| c_n^k \leqslant 1\}, \quad V_k^* = \Big\{(\eta_n) \in K(d) \mid \sum_n |\eta_n| d_n^k \leqslant 1\Big\}.$$

It is a consequence of nuclearity that the scalar multiples of the neighborhoods $U_k^*$ form a neighborhood basis of $0 \in K(c)$. Since $T$ is an imbedding, there are strictly increasing sequences $(j_k)$ and $(l_k)$ of indices and decreasing sequences $(C_k)$ and $(D_k)$ of positive constants such that

$$T(D_{k+1}V_{j_{k+1}}^*) \subset T\big(K(d)\big) \cap (C_k U_{l_k}^*) \subset T(D_k V_{j_k}^*) \tag{3}$$

for all $k$. Set $a_n^k = C_k^{-1} c_n^{l_k}$, $b_n^k = D_k^{-1} d_n^{j_k}$. We may assume that $a_n^{k+1}/a_n^k \geqslant 2$. Let $U_k = C_k U_{l_k}^*$, $V_k = D_k V_{j_k}^*$. Then

$$U_k^\circ = \Big\{(\xi_n) \in K(a)' \mid \sum_n (|\xi_n|/a_n^k) \leqslant 1\Big\}, \qquad V_k^\circ = \{(\eta_n) \in K(b)' \mid |\eta_n| \leqslant b_n^k\}.$$

By (3),

$$T(V_{k+1}) \subset T\big(K(b)\big) \cap U_k \subset T(V_k). \tag{4}$$

Polarizing the right side of (4) we get

$$\big(T\big(K(b)\big) \cap U_k\big)^\circ \supset (T')^{-1}(V_k^\circ)$$

which by the surjectivity of $T'$ gives

$$V_k^\circ \subset T'\Big(\big(T\big(K(b)\big) \cap U_k\big)^\circ\Big).$$

It then follows from the Hahn–Banach theorem that

$$V_k^\circ \subset T'(U_k^\circ). \tag{5}$$

From (4) we also obtain

$$\sum_n |\eta_n| b_n^k \leqslant \sup_i \Big|\sum_n t_{in}\eta_n\Big| a_i^k \leqslant \sum_n |\eta_n| b_n^{k+1}$$

for all $(\eta_n) \in K(b)$. Setting $(\eta_n) = e_n$ we get

$$b_n^k \leqslant \sup_i |t_{in}| a_i^k \leqslant b_n^{k+1}. \tag{6}$$

Fix now $k$ and define

$$I_n = \{i \mid b_n^k \leqslant 2|t_{in}| a_i^k\}.$$

By (6) the set $I_n$ is non-empty and since $\lim_i |t_{in}|\, a_i^k = 0$ and $b_n^k > 0$, $I_n$ is finite. We will show that the family $(I_n)_{n\in N}$ satisfies the condition of Theorem 2.2.



Consider a union $I = I_{n_1} \cup \ldots \cup I_{n_{m+1}}$, where $n_1, \ldots, n_{m+1}$ are distinct and $m \geqslant 1$. We assume that $I = \{i_1, \ldots, i_q\}$ with $q \leqslant m$ and show that this leads to a contradiction. Denote $D = \{z = (z_1, \ldots, z_{m+1}) \in \boldsymbol{K}^{m+1} \mid |z_s| \leqslant b_{n_s}^k\}$. Note that since $b_{n_s}^k > 0$, the set $D$ is a compact neighborhood of $0 \in \boldsymbol{K}^{m+1}$. Pick $z \in D$ and define $(\eta_n) \in V_k^\circ$ by

$$\eta_n = \begin{cases} z_s, & n = n_s, \\ 0, & n \notin \{n_1, \ldots, n_{m+1}\}. \end{cases}$$

By (5) there is $(\xi_i) \in U_k^\circ$ with $(\eta_n) = T'(\xi_i)$. Hence,

$$z_s = \eta_{n_s} = \sum_i t_{in_s}\xi_i = \sum_{i\in I} t_{in_s}\xi_i + \sum_{i\notin I} t_{in_s}\xi_i, \qquad s = 1, \ldots, m+1.$$

Denote $u_p = (t_{i_p n_1}, \ldots, t_{i_p n_{m+1}}) \in \boldsymbol{K}^{m+1}$, $p = 1, \ldots, q$, $w_s = \sum_{i\notin I} t_{in_s}\xi_i$, $w = (w_1, \ldots, w_{m+1}) \in \boldsymbol{K}^{m+1}$. Then

$$z = (z_1, \ldots, z_{m+1}) = \sum_{p=1}^{q} \xi_{i_p} u_p + w.$$

Here

$$|w_s| \leqslant \sum_{i\notin I} |t_{in_s}|\,|\xi_i| \leqslant \sum_{i\notin I_{n_s}} |t_{in_s}|\,|\xi_i| = \sum_{i\notin I_{n_s}} |t_{in_s}|\, a_i^k (|\xi_i|/a_i^k)$$

$$\leqslant \Big(\sum_i |\xi_i|/a_i^k\Big) \sup_{i\notin I_{n_s}} |t_{in_s}| a_i^k \leqslant \tfrac{1}{2} b_{n_s}^k,$$

where the last estimate follows from $(\xi_i) \in U_k^\circ$ and the definition of $I_{n_s}$. Thus, $w \in \frac{1}{2}D$. Since $z$ was arbitrary, $D \subset L + \frac{1}{2}D$, where $L = \operatorname{sp}\{u_1, \ldots, u_q\}$ is a subspace of $\boldsymbol{K}^{m+1}$ of dimension at most $q \leqslant m$. That this is impossible can be seen for example by polarizing, $L^\circ \cap D^\circ \subset \frac{1}{2}D^\circ$, and choosing a linear form $z' \in (\boldsymbol{K}^{m+1})'$ with $L \subset \operatorname{Ker}(z')$, $\max\{|\langle z, z'\rangle| \mid z \in D\} = 1$.

Applying now Theorem 2.2 we choose distinct representatives $j_n^k \in I_n$, $n \in \boldsymbol{N}$. By the definition of $I_n$ and the assumption $a_n^{k+1}/a_n^k \geqslant 2$ we then obtain

$$b_n^k \leqslant 2|t_{j_n^k n}| a_{j_n^k}^k \leqslant |t_{j_n^k n}| a_{j_n^k}^{k+1}, \qquad n \in \boldsymbol{N}.$$

Setting $\mu_n^k = |t_{j_n^k n}|$ we get (i). For (ii) we use the right side of (6),

$$\mu_n^k a_{j_n^k}^l = |t_{j_n^k n}| a_{j_n^k}^l \leqslant \sup_i |t_{in}| a_i^l \leqslant b_n^{l+1}, \qquad l, n \in \boldsymbol{N}. \ \blacksquare$$

COROLLARY 3.2. *Suppose a nuclear Fréchet space $F$ with a basis $(y_n)$ and a continuous norm is isomorphic to a subspace of a nuclear Fréchet space $E$ with a basis $(x_n)$. Then there are representations $(a_n^k)$ and $(b_n^k)$ of $(x_n)$ and $(y_n)$, respectively and injections $n \mapsto j_n^k$, $k \in \boldsymbol{N}$, such that*

$$\frac{a_{j_n^l}^{k-1}}{a_{j_n^l}^{l+1}} \leqslant \frac{b_n^k}{b_n^l} \leqslant \frac{a_{j_n^k}^{k+1}}{a_{j_n^k}^{l-1}}, \quad n \in \boldsymbol{N},\ k, l \geqslant 2. \tag{7}$$

Proof. Using the notations of Proposition 3.1 we obtain

$$\frac{a_{j_n^l}^{k-1}}{a_{j_n^l}^{l+1}} = \frac{\mu_n^l a_{j_n^l}^{k-1}}{\mu_n^l a_{j_n^l}^{l+1}} \leqslant \frac{b_n^k}{b_n^l} \leqslant \frac{\mu_n^k a_{j_n^k}^{k+1}}{\mu_n^k a_{j_n^k}^{l-1}} = \frac{a_{j_n^k}^{k+1}}{a_{j_n^k}^{l-1}} \quad \text{for} \quad n \in \boldsymbol{N},\ k, l \geqslant 2. \ \blacksquare$$

Formula (7) could be regarded as a refinement of Dubinsky's fundamental inequality for subspaces ([3], III (1.3)) in that now the sequences $(j_n^k)_n$ are known to have no repetitions. For a similar result in a different context see [7], Lemma 5.*

THEOREM 3.3. *Let $E$ be be a nuclear Fréchet space with a basis $(x_n)$. If $(s)$ is isomorphic to a subspace of $E$, then a subsequence of $(x_n)$ generates a subspace isomorphic to $(s)$.*

Proof. Let $(a_n^k)$ and $(b_n^k)$ be representations of $(x_n)$ and the coordinate basis of $(s)$, respectively, as in Proposition 3.1. Since $(b_n^k) \sim (n^k)$, there is $k_0$ such that $b_n^{k_0} \geqslant 1$ for sufficiently large $n$. Set $j_n = j_n^{k_0}$, $\mu_n = \mu_n^{k_0}$. By (i) of Proposition 3.1, $\mu_n a_{j_n}^{k_0+1} \geqslant 1$ for large $n$. Since $n \mapsto j_n$ is injective, $K(\mu_n a_{j_n}^k)$ is nuclear. Now $K\big((n^k), \boldsymbol{N}\big)$-nuclearity is just ordinary nuclearity so that we can apply (i) of Proposition 2.1 to find a strictly increasing sequence $(i_n)$ with $(n^k) \lesssim (\mu_{i_n} a_{j_{i_n}}^k)$, $\sup_n i_n/n < \infty$. Thus,

$$(n^k) \lesssim (\mu_{i_n} a_{j_{i_n}}^k) \lesssim (b_{i_n}^k) \sim \big((i_n)^k\big) \lesssim (n^k),$$

where the second estimate follows from (ii) of Proposition 3.1. Hence, $(s) = K(\mu_{i_n} a_{j_{i_n}}^k) \simeq \overline{\mathrm{sp}}(x_{j_{i_n}})$. $\blacksquare$

Remark 3.4. The previous theorem shows in particular that if $T\colon (s) \to E$ is an imbedding, then $T\big((s)\big)$ is isomorphic to a complemented subspace of $E$. It may however happen that $T\big((s)\big)$ itself is not complemented. For the details of the following example we refer to ([10], Chapter 7).

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*Added in proof: Kondakov has recently reported a result essentially equivalent to Proposition 3.1.



Denote by $C_0^\infty[-1, 1]$ the space of infinitely differentiable functions on the closed interval $[-1, 1]$ which vanish together with all their derivatives at the endpoints. When equipped with the topology of uniform convergence in all derivatives on $[-1, 1]$ this space is isomorphic to $(s)$. By E. Borel's theorem the mapping

$$S\colon C_0^\infty[-1, 1] \to \boldsymbol{K}^{\boldsymbol{N}}, \qquad S(f) = \big(f^{(n)}(0)\big)_n$$

is surjective; of course it is continuous and linear. There is a natural isomorphism

$$S^{-1}(0) \simeq C_0^\infty[-1, 0] \times C_0^\infty[0, 1].$$

Also, $C_0^\infty[-1, 0] \simeq C_0^\infty[0, 1] \simeq (s)$ so that $S^{-1}(0) \simeq (s)$. If $S^{-1}(0)$ were complemented, $C_0^\infty[-1, 1] = S^{-1}(0) \oplus G$, then $S|G$ would be an isomorphism $G \to \boldsymbol{K}^{\boldsymbol{N}}$ (by the open mapping theorem) which is absurd since $G$ has a continuous norm but $\boldsymbol{K}^{\boldsymbol{N}}$ has not.

Next we show that the property of superspaces of $(s)$ demonstrated in Theorem 3.3 in fact characterizes $(s)$ among nuclear Fréchet spaces with basis and a continuous norm.

PROPOSITION 3.5. *Suppose the nuclear Fréchet space $F$ with basis and a continuous norm has the following property: if $F$ is isomorphic to a subspace of a nuclear Fréchet space $E$ with a basis $(x_n)$, then a subsequence of $(x_n)$ generates a subspace isomorphic to $F$. Then $F$ is isomorphic to $(s)$.*

Proof. By [2] or [6] $F$ is isomorphic to a subspace of $(s)^{\boldsymbol{N}}$. A basis of $(s)^{\boldsymbol{N}}$ is given by the family $(e_{mn})$, $e_{mn} = (0, \ldots, 0, e_n, 0, \ldots)$, where $e_n$ appears in the $m^{\text{th}}$ place. Let

$$V_k = \Big\{(\eta_n) \in (s) \,\Big|\, \sum_n |\eta_n| n^k \leqslant 1\Big\}$$

and set $W_k = (V_k)^k \times (s)^{\boldsymbol{N}}$. Then $(W_k)$ is a neighborhood basis of $0 \in (s)^{\boldsymbol{N}}$ and it gives rise to a representation $(a_{mn}^k)$ of $(e_{mn})$,

$$a_{mn}^k = \begin{cases} n^k, & 1 \leqslant m \leqslant k,\ n \in \boldsymbol{N}, \\ 0, & m > k. \end{cases} \tag{8}$$

By hypothesis, there is an injection $i \mapsto \big(m(i), n(i)\big)$ such that $K(a_{m(i),n(i)}^k) \simeq F$. Since $F$ has a continuous norm, there is $k_0$ with $a_{m(i),n(i)}^{k_0} > 0$ for all $i$. This implies by (8) that $m(i) \leqslant k_0$. Hence $F \simeq K\big(n(i)^k\big)$, where each value $n(i)$ is repeated at most $k_0$ times. It then follows that $F \simeq \Lambda_\infty(\alpha)$, where $\alpha = (\alpha_n)$ is a nuclear exponent sequence of infinite type, i.e. $\sup_n \log n/\alpha_n < \infty$. It remains to be shown that $\sup_n \alpha_n/\log n < \infty$.

We assume that $\sup_n \alpha_n/\log n = \infty$ and show that this leads to a contradiction. Choose a strictly increasing sequence $(n_i)$ of indices such that $n_1 = 1$ and

(9) $$\alpha_{n_{i-1}+1} < (1/i)\,\alpha_{n_i}, \quad i \geqslant 2,$$

(10) $$i^2 \log n_i \leqslant \alpha_{n_i}, \quad i \in \boldsymbol{N}.$$

Then define $m_i = \min\{m \mid (1/i)\,\alpha_{n_i} \leqslant \alpha_m\}$. Note that $m_1 = 1$ and by (9) and the definition of $m_i$,

(11) $$n_{i-1}+1 < m_i \leqslant n_i, \quad i \geqslant 2,$$

(12) $$\alpha_{m_i-1} < (1/i)\,\alpha_{n_i} \leqslant \alpha_{m_i}, \quad i \in \boldsymbol{N}.$$

Set $N_1 = \bigcup_{i=1}^{\infty} \{n \in \boldsymbol{N} \mid m_i \leqslant n \leqslant n_i\}$ and $N_2 = \boldsymbol{N} \setminus N_1$. Note that by (11), $N_2$ is infinite. Denote by $(\alpha_n^1)$ and $(\alpha_n^2)$ the subsequence of $\alpha$ corresponding to $N_1$ and $N_2$, respectively, that is,

$$\alpha^1\colon\ \alpha_1, \alpha_{m_2}, \alpha_{m_2+1}, \ldots, \alpha_{n_2}, \alpha_{m_3}, \alpha_{m_3+1}, \ldots, \alpha_{n_3}, \alpha_{m_4}, \ldots,$$

$$\alpha^2\colon\ \alpha_2, \alpha_3, \ldots, \alpha_{m_2-1}, \alpha_{n_2+1}, \alpha_{n_2+2}, \ldots, \alpha_{m_3-1}, \alpha_{n_3+1}, \ldots$$

Trivially, $\Lambda_\infty(\alpha) \simeq \Lambda_\infty(\alpha^1) \times \Lambda_\infty(\alpha^2)$. Now $\alpha^1$ is of finite type. In fact, for any $n$, $\alpha_n^1 = \alpha_{m_i+j}$ for some $i$ and $j \geqslant 0$, $n \leqslant m_i + j \leqslant n_i$, so that by (12) and (10),

$$\frac{\alpha_n^1}{\log n} \geqslant \frac{\alpha_{m_i}}{\log n_i} \geqslant \frac{\alpha_{n_i}}{i} \cdot \frac{1}{\log n_i} \geqslant \frac{i^2 \log n_i}{i \log n_i} = i.$$

Consequently, $\lim_n \alpha_n^1/\log n = \infty$. By ([3], III (2.4.5)) there is a stable finite type exponent sequence $(\beta_n)$ such that $\sup_n \beta_n/\alpha_n^1 < \infty$ so that by ([3], III (2.4.4)) there is an imbedding $\Lambda_\infty(\alpha^1) \to \Lambda_1(\beta)$. It follows that $\Lambda_\infty(\alpha) \simeq F$ is isomorphic to a subspace of $\Lambda_1(\beta) \times \Lambda_\infty(\alpha^2)$. By the hypothesis, a subsequence of the basis of $\Lambda_1(\beta) \times \Lambda_\infty(\alpha^2)$ (the union of the coordinate bases of $\Lambda_1(\beta)$ and $\Lambda_\infty(\alpha^2)$) generates a subspace isomorphic to $\Lambda_\infty(\alpha)$. This subsequence contains only finitely many basis vectors of $\Lambda_1(\beta)$ since otherwise $\Lambda_\infty(\alpha)$ would contain a subspace isomorphic to a finite type power series space and this is impossible ([3], III (2.4.3)). Thus $\Lambda_\infty(\alpha)$ is isomorphic to a subspace of $L \times \Lambda_\infty(\alpha^2)$, where $L$ is a finite dimensional subspace of $\Lambda_1(\beta)$. If $\dim L = n_0$, we obtain by computing the Kolmogorov diameters both in $\Lambda_\infty(\alpha)$ and $L \times \Lambda_\infty(\alpha^2)$ and using ([3], I (6.2.2)) that $\alpha_n^2 \leqslant C\alpha_{n+n_0}$, $n \in \boldsymbol{N}$, where $C$ is a constant. But suppose $i \geqslant n_0 + 2$ and that $\alpha_n^2 = \alpha_{n_i+1}$. Since $n \leqslant m_i - i + 1$ we get by (12),

$$\frac{\alpha_n^2}{\alpha_{n+n_0}} = \frac{\alpha_{n_i+1}}{\alpha_{n+n_0}} \geqslant \frac{\alpha_{n_i+1}}{\alpha_{m_i-1}} \geqslant \frac{\alpha_{n_i}}{\alpha_{m_i-1}} > i.$$



Thus we arrive at the contradiction $\sup_n \alpha_n^2/\alpha_{n+n_0} = \infty$. ∎

Remark 3.6. Proposition 3.5 holds if we only assume that $F$ is isomorphic to a complemented subspace of any of its superspaces. In fact, if $F$ is isomorphic to a complemented subspace of $(s)^{\boldsymbol{N}}$, we can use ([1], Theorem 2.2) to obtain a mapping $i \mapsto (m(i), n(i))$ (not necessarily injective) such that $F \simeq K(a^k_{m(i),n(i)})$ and, as before, $m(i) \leqslant k_0$. By the nuclearity of $F$ each value $n(i)$ occurs only for finitely many different $i$ so that $F \simeq \Lambda_\infty(\alpha)$. We conclude that $\Lambda_\infty(\alpha)$ is isomorphic to a complemented subspace of $\Lambda_1(\beta) \times \Lambda_\infty(\alpha^2)$. By ([11], Proposition II. 1.6) there is $n_0$ and a continuous linear surjection $\Lambda_\infty(\alpha^2) \to \Lambda_\infty((\alpha_n)_{n \geqslant n_0})$. The contradiction follows from ([3], I (6.2.3)).

**4. A generalization and a dualization.** We get immediately the following generalization of Theorem 3.3.

THEOREM 4.1. *Let $(p_n^k)$ be a stable $D_1$ matrix and suppose $E$ is a $K(p, \boldsymbol{N})$-nuclear Fréchet space with a basis $(x_n)$. If $K(p)$ is isomorphic to a subspace of $E$, then a subsequence of $(x_n)$ generates a subspace isomorphic to $K(p)$.*

Proof. Let $(a_n^k)$ and $(b_n^k) \sim (p_n^k)$ be representations of $(x_n)$ and the coordinate basis of $K(p)$, respectively as in Proposition 3.1. As in Theorem 3.3, we find $k_0$, an injection $n \mapsto j_n$ and scalars $\mu_n > 0$ such that $\mu_n a_{j_n}^{k_0+1} \geqslant 1$ for large $n$. Since $K(\mu_n a_{j_n}^k)$ is $K(p, \boldsymbol{N})$-nuclear, there is a strictly increasing sequence $(i_n)$ with $(p_n^k) \lesssim (\mu_{i_n} a_{j_{i_n}}^k)$, $\sup_n i_n/n < \infty$. Thus,

$$(p_n^k) \lesssim (\mu_{i_n} a_{j_{i_n}}^k) \lesssim (b_{i_n}^k) \sim (p_{i_n}^k) \lesssim (p_n^k),$$

where in the second estimate (ii) of Proposition 3.1 was used and the last one follows from the stability of $(p_n^k)$ and the fact that $(p_n^k)_n$ is non-decreasing. ∎

To obtain a dualization of the previous theorem we first prove the quotient space analogue of Proposition 3.1.

PROPOSITION 4.2. *Let $E$ and $F$ be nuclear Fréchet spaces with bases $(x_n)$ and $(y_n)$, respectively, and suppose $F$ has a continuous norm. If $F$ is isomorphic to a quotient space of $E$, then there are representations $(a_n^k)$ and $(b_n^k)$ of $(x_n)$ and $(y_n)$, respectively, such that for each $k$ there is an injection $n \mapsto m_n^k$ and scalars $\nu_n^k > 0$ with*

(i) $$\nu_n^k a^k_{m_n^k} \leqslant b_n^{k+1}, \quad n \in \boldsymbol{N},$$

(ii) $$b_n^l \leqslant \nu_n^k a^{l+1}_{m_n^k}, \quad n, l \in \boldsymbol{N}.$$

Proof. Let $(c_n^k)$ and $(d_n^k)$, $d_n^k > 0$, be representations of $(x_n)$ and $(y_n)$, respectively, and let $T\colon K(c) \to K(d)$ be a continuous linear surjection with a representing matrix $(t_{in})$. Set

$$U_k^* = \{(\xi_n) \in K(c) | \sum_n |\xi_n| c_n^k \leqslant 1\}, \quad V_k^* = \{(\eta_n) \in K(d) | \sup_n |\eta_n| d_n^k \leqslant 1\}.$$

The nuclearity of $K(d)$ implies that the scalar multiples of the neighborhoods $V_k^*$ form a neighborhood basis of $0 \in K(d)$. Since $T$ is open and continuous, there are strictly increasing sequences $(j_k)$ and $(l_k)$ of indices and decreasing sequences $(C_k)$ and $(D_k)$ of positive constants such that

$$T(C_{k+1} U_{l_{k+1}}^*) \subset D_k V_{j_k}^* \subset T(C_k U_{l_k}^*) \tag{13}$$

for all $k$. Set $a_n^k = C_k^{-1} c_n^{l_k}$, $b_n^k = D_k^{-1} d_n^{j_k}$. We assume that $b_n^{k+1}/b_n^k \geqslant 2$. Let $U_k = C_k U_{l_k}^*$, $V_k = D_k V_{j_k}^*$. By (13),

$$T(U_{k+1}) \subset V_k \subset T(U_k). \tag{14}$$

For every $i$, $(1/b_i^k) e_i \in V_k$, so that by the right side of (14) there is $(\xi_n) \in U_k$ with $T(\xi_n) = (1/b_i^k) e_i$. Thus,

$$1/b_i^k = \sum_n t_{in} \xi_n. \tag{15}$$

Choose $k_0$ with $\lim_n a_n^{k_0}/a_n^2 = \infty$. Fix $k \geqslant k_0$ and define

$$N_i = \{n | \ a_n^k < 2|t_{in}| b_i^k\}.$$

We show that $N_i$ is non-empty and finite. By the left side of (14),

$$\sup_i \Big| \sum_n t_{in} \xi_n \Big| b_i^k \leqslant \sum_n |\xi_n| a_n^{k+1} \tag{16}$$

which, by setting $(\xi_n) = e_n$, $k = 1$, gives

$$|t_{in}| b_i^1 \leqslant \sup_i |t_{in}| b_i^1 \leqslant a_n^2 \tag{17}$$

for all $i$ and $n$. Since $b_i^1 > 0$, $t_{in} \neq 0$ implies $a_n^k \geqslant a_n^2 > 0$. On the other hand, since $T$ is surjective, there are indices $n$ for which $t_{in} \neq 0$. By summing over these $n$ we get from (15),

$$1/b_i^k \leqslant \sum |t_{in}| \ |\xi_n| = \sum (|t_{in}|/a_n^k) |\xi_n| a_n^k$$
$$\leqslant \Big(\sum |\xi_n| a_n^k\Big) \sup (|t_{in}|/a_n^k) \leqslant \sup (|t_{in}|/a_n^k)$$

since $(\xi_n) \in U_k$. Thus, for some $n$, $|t_{in}|/a_n^k > 1/2b_i^k$. If $t_{in} \neq 0$ for infinitely many $n$, then (17) implies

$$\lim_{\substack{n\to\infty \\ t_{in}\neq 0}} (a_n^k/|t_{in}|) \geqslant \lim_{\substack{n\to\infty \\ t_{in}\neq 0}} b_i^1 (a_n^{k_0}/a_n^2) = \infty$$

so that $N_i$ is finite.



By deleting the $k_0 - 1$ first rows from the matrices $(a_n^k)$ and $(b_n^k)$ we can assume that $k_0 = 1$.

We show that the family $(N_i)_{i \in \mathbf{N}}$ satisfies the condition of Theorem 2.2. Consider a union $N = N_{i_1} \cup \ldots \cup N_{i_{m+1}}$, where $i_1, \ldots, i_{m+1}$ are distinct and $m \geqslant 1$. We assume that $N = \{n_1, \ldots, n_q\}$ with $q \leqslant m$ and show that this leads to a contradiction. Denote

$$D = \{z = (z_1, \ldots, z_{m+1}) \in \mathbf{K}^{m+1} | \ |z_s| b_{i_s}^k \leqslant 1\}.$$

Since $b_{i_s}^k > 0$, $D$ is a compact neighborhood of $0 \in \mathbf{K}^{m+1}$. Pick $z \in D$ and define $(\eta_i) \in V_k$ by

$$\eta_i = \begin{cases} z_s, & i = i_s, \\ 0, & i \notin \{i_1, \ldots, i_{m+1}\}. \end{cases}$$

Thus, by (14) there is $(\xi_n) \in U_k$ with $(\eta_i) = T(\xi_n)$ so that

$$z_s = \eta_{i_s} = \sum_n t_{i_s n} \xi_n = \sum_{n \in N} t_{i_s n} \xi_n + \sum_{n \notin N} t_{i_s n} \xi_n, \quad s = 1, \ldots, m+1.$$

Denote $u_p = (t_{i_1 n_p}, \ldots, t_{i_{m+1} n_p}) \in \mathbf{K}^{m+1}$, $p = 1, \ldots, q$, $w_s = \sum_{n \notin N} t_{i_s n} \xi_n$, $w = (w_1, \ldots, w_{m+1}) \in \mathbf{K}^{m+1}$. Then

$$z = (z_1, \ldots, z_{m+1}) = \sum_{p=1}^{q} \xi_{n_p} u_p + w.$$

Here

$$|w_s| b_{i_s}^k \leqslant \sum_{n \notin N} |t_{i_s n}| \ |\xi_n| b_{i_s}^k \leqslant \sum_{n \notin N_{i_s}} |t_{i_s n}| \ |\xi_n| b_{i_s}^k \leqslant \tfrac{1}{2} \sum_{n \notin N_{i_s}} |\xi_n| a_n^k \leqslant \tfrac{1}{2}$$

since $(\xi_n) \in U_k$. Hence, $w \in \frac{1}{2} D$ and because $z$ was arbitrary, $D \subset \operatorname{sp}\{u_1, \ldots, u_q\} + \frac{1}{2} D$. As in the proof of Proposition 3.1, this is impossible since $q \leqslant m$.

By Theorem 2.2 we can find distinct representatives $m_i^k \in N_i$, $i \in \mathbf{N}$. By the definition of $N_i$,

$$\nu_i^k a_{m_i^k}^k < 2 b_i^k \leqslant b_i^{k+1}, \quad i \in \mathbf{N},$$

where we defined $\nu_i^k = 1/|t_{i m_i^k}|$ and used the condition $b_n^{k+1}/b_n^k \geqslant 2$. This proves (i). For (ii) we use (16) with $(\xi_n) = e_{m_i^k}$,

$$(1/\nu_i^k) b_i^l = |t_{i m_i^k}| b_i^l \leqslant \sup_j |t_{j m_i^k}| b_j^l \leqslant a_{m_i^k}^{l+1}, \quad i, l \in \mathbf{N}. \ \blacksquare$$

COROLLARY 4.3. *Suppose a nuclear Fréchet space $F$ with a basis $(y_n)$ and a continuous norm is isomorphic to a quotient space of a nuclear Fréchet* *space $E$ with a basis $(x_n)$. Then there are representations $(a_n^k)$ and $(b_n^k)$ of $(x_n)$ and $(y_n)$, respectively, and injections $n \mapsto m_n^k$, $k \in \mathbf{N}$, such that*

$$\frac{a^{k-1}_{m_n^k}}{a^{l+1}_{m_n^k}} \leqslant \frac{b_n^k}{b_n^l} \leqslant \frac{a^{k+1}_{m_n^l}}{a^{l-1}_{m_n^l}}, \quad n \in \mathbf{N},\ k,\ l \geqslant 2. \tag{18}$$

Proof. We use the same notation as in Proposition 4.2 except that the injection corresponding to $k$ is denoted by $m_n^{k+1}$. Then

$$\frac{a^{k-1}_{m_n^k}}{a^{l+1}_{m_n^k}} = \frac{\nu_n^{k-1} a^{k-1}_{m_n^k}}{\nu_n^{k-1} a^{l+1}_{m_n^k}} \leqslant \frac{b_n^k}{b_n^l} \leqslant \frac{\nu_n^{l-1} a^{k+1}_{m_n^l}}{\nu_n^{l-1} a^{l-1}_{m_n^l}} = \frac{a^{k+1}_{m_n^l}}{a^{l-1}_{m_n^l}} \quad \text{for} \quad n \in \mathbf{N},\ k,\ l \geqslant 2. \ \blacksquare$$

Formula (18) should be compared to Dubinsky's fundamental inequality for quotient spaces ([3], IV (1.4)).

THEOREM 4.4. *Let $(q_n^k)$ be a stable $D_2$ matrix and suppose $E$ is a $K(q)$-nuclear Fréchet space with a basis $(x_n)$. If $K(q)$ is isomorphic to a quotient space of $E$, then a subsequence of $(x_n)$ generates a subspace isomorphic to $K(q)$.*

Proof. Let $(a_n^k)$ and $(b_n^k)$ be representations of $(x_n)$ and the coordinate basis of $K(q)$, respectively, as in Proposition 4.2. Since $\lim_n q_n^k = 0$, the equivalence $(b_n^k) \sim (q_n^k)$ implies $\lim_n b_n^k = 0$ for all $k$. By injectivity, $\lim_n m_n^k = \infty$. Thus, by (i) of Proposition 4.2 we can select indices $1 = n_1 < n_2 < \ldots$ such that

(19) if $\gamma_n = \nu_n^k a^k_{m_n^k}$ for $n_k \leqslant n < n_{k+1}$, $k \in \mathbf{N}$, then $\lim_n \gamma_n = 0$,

(20) if $M_k = \{m_n^k \mid n_k \leqslant n < n_{k+1}\}$, then $M_k \cap M_l = \emptyset$ for $l \notin \{k-1, k, k+1\}$.

Let $m_n = m_n^k$, $\nu_n = \nu_n^k$ for $n_k \leqslant n < n_{k+1}$, $k \in \mathbf{N}$. Note that by (20), each value in the sequence $(m_n)$ occurs for at most two different values of $n$. Also, by (16),

$$(1/\nu_n) b_n^l = |t_{n m_n^k}| b_n^l \leqslant \sup_i |t_{i m_n^k}| b_i^l \leqslant a^{l+1}_{m_n^k} = a^{l+1}_{m_n}$$

for all $n_k \leqslant n < n_{k+1}$, $k \in \mathbf{N}$. Thus,

$$(q_n^l) \sim (b_n^l) \lesssim (\nu_n a^l_{m_n}). \tag{21}$$

Further, for a fixed $l$ and $n_k \leqslant n < n_{k+1}$, $k \geqslant l$,

$$\nu_n a^l_{m_n} \leqslant \nu_n a^k_{m_n} = \gamma_n$$



so that $\lim_n \nu_n a^l_{m_n} = 0$. Using the stability of $(q_n^l)$ it is easy to see that the space $K(\nu_n a^l_{m_n})$ is $K(q)$-nuclear. Hence we can apply (ii) of Proposition 2.1 to find a strictly increasing sequence $(i_n)$ with $(\nu_{i_n} a^l_{m_{i_n}}) \lesssim (q_n^l)$, $\sup_n i_n/n < \infty$. Using then (21) we obtain

$$(q_n^l) \lesssim (q_{i_n}^l) \lesssim (\nu_{i_n} a^l_{m_{i_n}}) \lesssim (q_n^l),$$

where the first estimate follows from the stability of $(q_n^l)$ and the fact that $(q_n^l)_n$ is non-increasing. The proof is completed by deleting possible repetitions (at most two of each) from $(m_{i_n})$ by passing to a subsequence $(m_{i_{s_n}})$ with $s_n < 2n$,

$$(q_n^l) \lesssim (q_{s_n}^l) \lesssim (\nu_{i_{s_n}} a^l_{m_{i_{s_n}}}) \lesssim (q_{s_n}^l) \lesssim (q_n^l).$$

Thus, $K(q_n^l) = K(\nu_{i_{s_n}} a^l_{m_{i_{s_n}}}) \simeq \overline{\mathrm{sp}}(x_{m_{i_{s_n}}})$. $\blacksquare$

**References**


[1] C. Bessaga, *Some remarks on Dragilev's theorem*, Studia Math. 31 (1968), 307–318.
[2] C. Bessaga and A. Pełczyński, *On embedding of nuclear spaces in the space of all infinitely differentiable functions on the line*, Dokl. Akad. Nauk. SSSR 134 (1960), 745–748.
[3] E. Dubinsky, *The structure of nuclear Fréchet spaces*, Lecture Notes in Math. 720, Springer-Verlag, Berlin–Heidelberg–New York 1979.
[4] E. Dubinsky and M. S. Ramanujan, *On λ-nuclearity*, Mem. Amer. Math. Soc. 128 (1972).
[5] L. Holmström, *On stable $D_1$ and $D_2$ spaces*, Arch. Math. 36 (1981), 546–553.
[6] T. Kōmura and Y. Kōmura, *Über die Einbettung der nuklearen Räume in $(s)^A$*, Math. Ann. 162 (1966), 284–288.
[7] V. P. Kondakov, *On a certain generalization of power series spaces*, in: *Actual problems of mathematical analysis*, Rostov 1978, 92–99 (Russian).
[8] G. Köthe, *Topological vector spaces I*, Springer-Verlag, New York 1969.
[9] L. Mirsky, *Transversal theory*, Academic Press, New York and London 1971.
[10] B. S. Mitjagin, *Approximative dimension and bases in nuclear spaces*, Uspehi Mat. Nauk. 16 (4) (1961), 73–132 (Russian).
[11] K. Nyberg, *On subspaces of products of nuclear Fréchet spaces*, Ann. Acad. Sci. Fenn. Ser A I Math. Diss. 31 (1980), 1–56.
[12] A. Pietsch, *Nuclear locally convex spaces*, Springer-Verlag, Berlin–Heidelberg–New York 1972.
[13] M. S. Ramanujan and B. Rosenberger, *On λ(φ, F)-nuclearity*, Compositio Math. 37, 2 (1979), 113–125.
[14] W. Robinson, *Some equivalent classes of Köthe spaces*, Comment. Math. Prace Mat. 20 (1978), 449–451.

[15] — *Relationships between λ-nuclearity and pseudo-μ-nuclearity,* Trans. Amer. Math. Soc. 201 (1975), 291–303.
[16] T. Terzioğlu, *Die diametrale Dimension von lokal konvexen Räumen,* Collect. Math. 20 (1969), 49–99.
[17] — *Stability of smooth sequence spaces,* J. Reine Angew. Math. 276 (1975), 184–189.

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# A reverse maximal ergodic theorem

by

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**Abstract.** A reverse maximal ergodic theorem is proved for a $d$-parameter discrete semigroup $(T_g\colon g\in Z_+^d)$ of measure preserving transformations on a $\sigma$-finite measure space $(X,\mathscr{F},\mu)$ which is ergodic in the sense that if $E\in\mathscr{F}$ with $E\neq X$ is $T_g$-invariant for all $g\in Z_+^d$ then $\mu E=0$ or $\infty$. A continuous version follows from standard approximation arguments.

**1. Introduction.** Let $(X,\mathscr{F},\mu)$ be a $\sigma$-finite measure space and $(T_g\colon g\in Z_+^d)$ a $d$-parameter discrete semigroup of measure preserving transformations on $(X,\mathscr{F},\mu)$. For $0\leqslant f\in L_1(\mu)+L_\infty(\mu)$, the *maximal function* $f^*$ is defined by

$$f^*(x)=\sup_{n\geqslant 1} n^{-d}\sum_{g\in V_n} f(T_g x)\quad \text{where}\quad V_n=\{0,\ldots,n-1\}^d.$$

It is then known (cf. [11], [4], [1]) that the maximal inequality holds:

$$\mu\{f^*>a\}\leqslant \frac{1}{B_d a}\int_{\{f>B_d a\}} f\,d\mu \quad \text{for any } a>0 \tag{1}$$

where $B_d$ is a constant dependent only on the dimension $d$.

The purpose of this paper is to show that a reverse maximal inequality holds provided that the semigroup $(T_g\colon g\in Z_+^d)$ is *ergodic* in the sense that if $E\in\mathscr{F}$ with $E\neq X$ is $T_g$-invariant for all $g\in Z_+^d$ then $\mu E=0$ or $\infty$. Here it should be noted that N. Dang-Ngoc [2] has shown a similar inequality for an ergodic $d$-parameter *group* $(T_g\colon g\in Z^d)$ of measure preserving transformations on a probability measure space. However, the maximal function $f^\sim$ he considered is defined by

$$f^\sim(x)=\sup_{n\geqslant 1}(2n-1)^{-d}\sum_{g\in W_n} f(T_g x)\quad \text{where}\quad W_n=\{-n+1,\ldots,n-1\}^d,$$

and he remarked that his argument is not modified if $f^\sim$ is replaced by $f^*$. Nevertheless, we shall modify his argument to prove our result. For the particular case $(T^n\colon n\in Z_+^1)$ where $T$ is conservative and ergodic in the usual sense, the inequality was already obtained by Derriennic [3] in a