Using the maximal inequality (1) and the reverse maximal inequality (2), we can prove the following dominated ergodic theorem.

THEOREM 3. *Let* $(X, \mathscr{F}, \mu)$ *and* $(T_g\colon g \in Z_+^d)$ *be as in Theorem* 1. *Then* $f \in R_w(\mu)$ *if and only if* $f^* \in R_{w+1}(\mu)$.

Proof. By Fubini's theorem we have

$$\int_{\{f^*>t\}} f^*(\log(f^*/t))^w d\mu = \int_{\{f^*>t\}} d\mu(x) \int_t^{f^*(x)} \left([\log(s/t)]^w + tw[\log(s/t)]^{w-1}\right) ds$$
$$= \int_t^{\infty} \left([\log(s/t)]^w + tw[\log(s/t)]^{w-1}\right) \mu\{f^* > s\}\, ds.$$

Thus we may apply (1) together with a well-known argument (see e.g. [5], p. 676) to infer that $f \in R_{w+1}(\mu)$ implies $f^* \in R_w(\mu)$. Similarly, (2) may be applied to infer that $f^* \in R_w(\mu)$ implies $f \in R_{w+1}(\mu)$. The details are omitted. (Cf. [10].)

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Applications of autoreproducing kernel moduli to the study on interpolability and minimality of a class of stationary Hilbertian varieties

by

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Abstract. The autoreproducing kernel modulus $\mathscr{H}(F)$ of an operator-valued spectral measure F is constructed. All of its elements are operator-valued measures. These measures are said (as suggested by the results obtained in [23]) to be Hellinger square integrable relative to F. Then the class of Hilbert space-valued stationary processes $(X_g)_{g\in G}$, having operator valued spectral densities is considered. For such processes, some characterizations of $\mathscr{H}(F)$ are given and compared to that obtained by Makagon in the recent paper [9] on Hellinger square integrable vector measures. From the results on $\mathscr{H}(F)$, the interpolable and minimal processes $(X_g)_{g\in G}$ are then analytically characterized.

Introduction. It is shown in [23] the impossibility for a minimal $\mathscr{S}(U, H)$-valued stationary processes to be of full rank. However this notion can be defined for Banach space-valued stationary processes. So a special class of these processes is considered here.

First it is constructed from a spectral bimeasure a unique autoreproducing kernel Loynes modulus $\mathscr{H}(F)$, all the elements of which are operator-valued measures (cf. Theorem 2). When F is a spectral measure, by analogy with the results obtained in [23], the measures in $\mathscr{H}(F)$ are said to be Hellinger square integrable with respect to (w.r.t.) F.

Then, Hilbert space-valued stationary processes $(X_g)_{g\in G}$, possessing operator-valued spectral densities are considered. The operator time-domains of these processes are proved to have the Radon–Nikodym property w.r.t. F (Theorem 4) and some characterizations of $\mathscr{H}(F)$ are obtained (Theorem 5). Afterwards, analytic conditions for interpolability and minimality studied by [19], [20], [23], [24], [27] are extended to the processes $(X_g)_{g\in G}$, and a criterion for such processes $(X_g)_{g\in G}$ to be minimal of full rank is also given.

We learned quite recently that Makagon in [9] has given a definition and a criterion for vector measures to be Hellinger square integrable w.r.t. a spectral measure. His definition is proved to be equivalent to our Definition 3 (cf. Theorem 3) whereas our criterion (Theorem 5) may be considered as an operator version of Makagon's criterion ([9], Theorem 1.5).

Notations. All the vector spaces considered in this study are complex vector spaces and the following notations will be used:

Let U, V be two Banach spaces.

$\mathcal{O}(V, U)$ is the space of all linear operators from V into U.

$\mathcal{L}(V, U)$ is the space of U-valued bounded linear operators defined on V.

U^* is the Banach adjoint space of U, i.e. $U^* = \{\bar{f}; f \in U'\}$ where $\bar{f}$ means the complex conjugate of f and U' is the topological dual of U.

Let $a \in \mathcal{L}(V, U)$; then the adjoint a^* of a is the bounded linear operator belonging to $\mathcal{L}(U^*, V^*)$ that is defined by $a^*(v^*) = v^* \cdot a$ $\forall v^* \in V^*$.

$a \in \mathcal{L}(U, U^*)$ is said to be a *hermitian operator* if for any $u_1, u_2 \in U$

$$\big(a(u_1)\big)(u_2) = \overline{\big(a(u_2)\big)(u_1)}.$$

Generally a hermitian operator is not self-adjoint, but if U is a reflexive Banach space (specially if U is a Hilbert space) then the notion of hermitian operators and self-adjoint operators are equivalent (when U is reflexive, it will be identified U^{**} with U).

$a \in \mathcal{L}(U, U^*)$ is said to be a *non-negative operator* if a is a hermitian operator such that for any $u \in U$ $\big(a(u)\big)(u) \geqslant 0$.

$\mathcal{L}^+(U, U^*)$ denotes the class of non-negative operators in $\mathcal{L}(U, U^*)$.

Let $\{W(\lambda)\}_{\lambda\in\Lambda}$ be a family in $\mathcal{L}(U) = \mathcal{L}(U, U)$; then $\mathrm{Sm}_{VU}(W)$ denotes the right $\mathcal{L}(V, U)$-modulus spanned by the family $\{W(\lambda)\}_{\lambda\in\Lambda}$ that is

$$\mathrm{Sm}_{VU}(W) \overset{\text{def}}{=} \Big\{\sum_{i=1}^{n} W(\lambda_i)a_i;\ \lambda_i \in \Lambda,\ a_i \in \mathcal{L}(V, U),\ i = 1, \ldots, n\Big\}.$$

Let H be a Hilbert space and $\{u_i\}_{i\in I}$ be a family in H; then $\mathrm{sp}(u)$ is the vector subspace in H, spanned by $\{u_i\}_{i\in I}$ and $\overline{\mathrm{sp}}(u)$ is its closure in H. The scalar product and norm in H will be denoted by $\langle .,. \rangle_H, |\cdot|_H$, resp.

1. Autoreproducing kernel Loynes-modulus for $\mathcal{L}(U, U^*)$-valued non-negative kernel. Let Λ be a set, U be a Banach space and U^* be its Banach adjoint space.

DEFINITION 1 (cf. [3], [13], [14]). A map K: $\Lambda\times\Lambda \to \mathcal{L}(U, U^*)$ is said to be an *$\mathcal{L}(U, U^*)$-valued non-negative kernel on Λ* if

$$\forall n \in \boldsymbol{N}^* = \boldsymbol{N}-\{0\},\ \forall \lambda_i \in \Lambda,\ \forall u_i \in U;\ i = 1, \ldots, n,$$

$$\sum_{i,j=1}^{n}\sum \big(K(\lambda_i, \lambda_j)u_i\big)(u_j) \geqslant 0.$$

Let K be an $\mathcal{L}(U, U^*)$-valued non-negative kernel on Λ; then there exists a Hilbert space H and a mapping β: $\Lambda \to \mathcal{L}(U, H)$ such that

(1) $$K(\lambda_1, \lambda_2) = \beta(\lambda_2)^*\beta(\lambda_1)\ \ \forall \lambda_1, \lambda_2 \in \Lambda$$

(cf. [3], [13], [14]). Let $\mathcal{L}(U, H)$ be provided with the weak topology; then it is associated with it a uniform structure $(\mathcal{U})$ on $\mathcal{L}(U, H) \times \mathcal{L}(U, H)$. This uniform structure corresponds to the weak topology on $\mathcal{L}(U, U^*)$, more precisely for any family $\{x_l\}_{l\in L}$, $\{y_{l'}\}_{l'\in L'}$ in $\mathcal{L}(U, H)$ the net $\{y_{l'}, x_l\}_{(l',l)\in L'\times L}$ converges to zero in $(\mathcal{U})$ if for any vectors v_1, v_2 in U

$$\lim_{(l,l')\in L\times L'} \langle x_l v_1, y_{l'} v_2 \rangle_H = 0.$$

Hence if Z denotes the space $\mathcal{L}(U, U^*)$, provided with its weak topology and if $\mathcal{L}(U, H)$ is provided with the Z-valued product or simply Z-product y^*x for $x, y \in \mathcal{L}(U, H)$ then $\mathcal{L}(U, H)$ is an LVH-*space* or *Loynes space* (cf. [6], [25]).

Let us note that the above procedure to define the uniform structure $(\mathcal{U})$ on $\mathcal{L}(U, H)$ is a direct and natural manner, and proceeds in the opposite sense of [6].

Now, as in [23], we are interested in construction of an autoreproducing kernel Loynes modulus for K that is more general than grammian moduli in [23].

Let V be any Banach space.

(a) We consider

$$\mathcal{H}_0 = \Big\{\sum_{i=1}^{m} \beta(\lambda_i)a_i;\ \lambda_i \in \Lambda,\ a_i \in \mathcal{L}(V, U);\ i = 1, \ldots, m;\ m \in \boldsymbol{N}^*\Big\}$$

and define on $\mathcal{H}_0 \times \mathcal{H}_0$ the Z-product $[f, g] = g^*f$. Then the vector space $\mathcal{H}_0$ which is provided with the uniform structure $(\mathcal{U})$ admits an unique completion $\overline{\mathcal{H}_0}$ up to an isomorphism (cf. [5], Theorem 2).

(b) Now we show that $\overline{\mathcal{H}_0} \subset \mathcal{L}(V, H)$. Let $M \in \overline{\mathcal{H}_0}$; then there exists a sequence $\{N_k\}_{k\in\boldsymbol{N}} \subset \mathcal{H}_0 \subset \mathcal{L}(V, H)$ such that

(2) $$\lim_{k\to\infty} [M-N_k, M-N_k] = 0 \quad \text{in} \quad Z = \mathcal{L}(V, V^*)$$

provided with the weak topology and

(3) $$\lim_{k,m\to\infty} [N_k-N_m, N_k-N_m] = 0 \quad \text{in} \quad Z,$$

hence $\forall v \in V$ $\lim |(N_k - N_m)v|_H = 0$; consequently there exists an element $N(v) \in H$ such that

(4) $$\lim_{k\to\infty} |N_k v - N(v)|_H = 0.$$

But

$$\{N_k\}_{k\in\mathbf{N}} \subset \mathscr{L}(V, H). \tag{5}$$

It follows, according to the Banach–Steinhaus theorem or the uniform boundedness principle that from (4), (5), we have

$$N \in \mathscr{L}(V, H) \quad \text{and} \quad \{N_k\}_{k\in\mathbf{N}} \text{ converges to } N \text{ in } \mathscr{L}(V, H). \tag{6}$$

However it is obvious from (2) and (4) that $N = M$.

(c) Let us choose $H = H(K)$ and $\beta(\lambda) = K(\lambda, \cdot)$ $\forall \lambda \in \Lambda$, where $H(K)$ is the autoreproducing kernel Hilbert space for K, that is the Hilbertian completion in $\mathrm{Appl}(\Lambda, U^*)$ of

$$H_0(K) = \Big\{\sum_{i=1}^{m} K(\lambda_i, \cdot)v_i;\ v_i \in U, \lambda_i \in \Lambda; i = 1, \ldots, n\Big\}$$

for the scalar product

$$\Big\langle \sum_{i=1}^{m} K(\lambda_i, \cdot)v_i, \sum_{j=1}^{n} K(\nu_j, \cdot)v'_j \Big\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n} \big(K(\lambda_i, \nu_j)v_i\big)(v'_j)$$

so that $\forall f \in H(K)$ $f(\lambda)u = \langle f, K_\lambda u\rangle_H \forall u \in U$ (cf. [14]), where for any two sets D, T; $\mathrm{Appl}(T, D)$ is the set of all mappings from T to D.

As for any $\lambda \in \Lambda$, $K_\lambda \overset{\mathrm{def}}{=} K(\lambda, \cdot)$: $\Lambda \mapsto \mathscr{L}(U, U^*)$, so any element $N \in \mathscr{H}_0$ can be considered either as an operator in $\mathscr{L}(V, H)$ or as a mapping N: $\Lambda \mapsto \mathscr{L}(V, U^*)$.

Now we show that we can choose $\overline{\mathscr{H}_0}$ in $\mathrm{Appl}\big(\Lambda, \mathscr{L}(V, U^*)\big)$. Indeed, let $\{N_k\}_{k\in\mathbf{N}}$ be a Cauchy sequence for $(\mathscr{U})$ in $\mathscr{H}_0$ and $M \in \overline{\mathscr{H}_0}$ be the limit of $\{N_k\}_{k\in\mathbf{N}}$ for $(\mathscr{U})$. Set $\tilde{M}(\lambda) = K_\lambda^* M$ $\forall \lambda \in \Lambda$; then according to (b), $\tilde{M} \in \mathrm{Appl}\big(\Lambda, \mathscr{L}(V, U^*)\big)$. Moreover, for any λ in Λ, any (v, u) in $V \times U$

$$\lim_{k\to\infty} \big(N_k(\lambda)v\big)(u) = \lim_{k\to\infty} \langle N_k v, K_\lambda u\rangle = \big(\tilde{M}(\lambda)v\big)(u).$$

Hence $\tilde{M}$ uniquely represents the class of Cauchy sequences one representative of which is $\{N_k\}_{k\in\mathbf{N}}$.

If $\mathscr{H}_{VU}(K)$ denotes the space of such $\tilde{M}$, then it is obvious that one can extend, as for $\mathscr{H}_0$ the uniform structure $(\mathscr{U})$ to $\mathscr{H}_{VU}(K)$. From now on, $\mathscr{H}_{VU}(K)$ is provided with the structure $(\mathscr{U})$. And it is easy to verify that

$$\langle \tilde{M}(\cdot)v, K_\lambda u, \rangle = \lim_{k\to\infty} \langle N_k v, K_\lambda u \rangle = \lim_{k\to\infty} \big(N_k(\lambda)v\big)u = \big(\tilde{M}(\lambda)v\big)(u). \tag{7}$$

that is for any v in V, $\tilde{M}(\cdot)v \in H(K)$.

Now we sum up all the previous results in the following theorem, which is an extension of Theorem 2 and Proposition 2 in [23].

THEOREM 1. *Let Λ be a set, V and U be two Banach spaces and K be an $\mathscr{L}(U, U^*)$-valued mapping defined on $\Lambda \times \Lambda$.*

Then K is a non-negative kernel if and only if (iff) there exists a unique Loynes right $\mathscr{L}(V)$-modulus $\mathscr{H}_{VU}(K)$ that is the completion of $\mathscr{H}_0$ in the $\mathscr{L}(V)$-modulus $\mathrm{Appl}\big(\Lambda, \mathscr{L}(V, U^*)\big)$ *for the uniform structure $(\mathscr{U})$ which satisfies the following autoreproducing property*:

$$\forall M \in \mathscr{H}_{VU}(K)\ \forall \lambda \in \Lambda\ \ M(\lambda) = K_\lambda^* M \tag{8}$$

where $\mathrm{Appl}\big(\Lambda, \mathscr{L}(V, U^*)\big)$ *is the $\mathscr{L}(V)$-modulus of all mappings from Λ into $\mathscr{L}(V, U^*)$.*

$\mathscr{H}_{VU}(K)$ will be called the *autoreproducing kernel Loynes* (a.k.L) modulus of K.

We now establish a deep relationship between $\mathscr{H}_{VU}(K)$ and $H(K)$.

PROPOSITION 1. (i) *Let v be any non null vector in V; then the Hilbert space $H(K)$ is exactly the linear space $H(v)$ spanned by the subset $\{Mv;\ M \in \mathscr{H}_{VU}(K)\}$ in $H(K)$.*

(ii) $\mathscr{H}_{VU}(K) = \mathscr{L}\big(V, H(K)\big)$.

Proof. (i): Let $v_0 \in V$, $v_0 \neq 0$; then there exists $v'_0 \in V'$ such that $v'_0(v_0) = 1$. For any $u \in U$, let us associate with it the operator a defined by $a(v) = v'_0(v)u$. It is obvious that $a \in \mathscr{L}(V, U)$. Hence, given any vector $g = \sum_{i=1}^{n} K(\lambda_i, \cdot)u_i \in H_0(K)$, it will be associated with it the operator $\hat{N} = \sum_{i=1}^{n} K(\lambda_i, \cdot)a_i$ in $\mathscr{H}_0$.

Now let f be any vector in $H(K)$; then there exists a sequence $\{g_n\}_{n\in\mathbf{N}} \subset H_0(K)$ such that $\lim_{n,m\to\infty} |g_n - g_m|_{H(K)} = 0$. Hence

$$\lim_{n,m\to\infty} |\hat{N}_n v - \hat{N}_m v|_{H(K)} = |v'_0(v)| \lim_{n,m\to\infty} |g_n - g_m|_{H(K)} = 0;$$

i.e. there exists an operator $\hat{M} \in \mathscr{H}_{VU}(K)$ such that $\hat{M}v_0 = f$. Consequently $H(v_0) = H(K)$.

(ii): If $\mathscr{H}_{VU}(K) \underset{\neq}{\subset} \mathscr{L}\big(V, H(K)\big)$ then there exists an operator M in $\mathscr{L}\big(V, H(K)\big)$ and a vector v_0 in V such that

$$\forall N \in \mathscr{H}_{VU}(K)\ Nv_0 \neq Mv_0. \tag{9}$$

But $Mv_0 \in H(K)$, hence from (i) there exists $N_0 \in \mathscr{H}_{VU}(K)$ such that $N_0 v_0 = Mv_0$. This contradicts assertion (9). So the proposition is proved. ∎

Remark. Let us notice from the previous proof that:

(a) $\mathrm{sp}\{Mv;\ M\in\mathscr{H}_{VU}(K),\ v\in V\} = H(K)$.

(b) For any $f\in H(K)$, $v_0\in V$, the operator $M = af$, with $a(v) = v_0'(v)f$, belongs to $\mathscr{H}_{VU}(K)$.

We now apply the previous results to deduce autoreproducing kernel Loynes moduli for spectral measures.

Let $(D, \mathscr{B})$ be a measurable space, U and V be two Banach spaces and H be a Hilbert space.

PROPOSITION 2 (cf. [22], ch. I). (i) *Let β be an $\mathscr{L}(U, H)$-valued weakly countably additive* (c.a.) *measure on $(D, \mathscr{B})$; then there exists a unique $\mathscr{L}(U, U^*)$-valued set function Θ defined on $\mathscr{B}\times\mathscr{B}$ by the relation*:

(10) $\quad \big(\Theta(A\times B)u\big)(v) = \langle\beta(A)u, \beta(B)v\rangle, \qquad \forall A, B\in\mathscr{B}, \forall u, v\in U.$

Furthermore, the previous set function Θ possesses the following properties:

(BM1) *Θ is an $\mathscr{L}(U, U^*)$-valued non-negative kernel on $\mathscr{B}$.*

(BM2) *Θ is additive.*

(BM3) $\forall u, v\in U$, $\lim\limits_{m,n\to\infty}\big(\Theta(A_n\times B_m)v\big)(u) = 0$, *for any decreasing sequences $\{A_n\}_{n\in\mathbf{N}}$ and $\{B_m\}_{m\in\mathbf{N}}$ of sets in $\mathscr{B}$ with void intersection.*

(ii) *Reciprocally, given any $\mathscr{L}(U, U^*)$-valued set function Θ defined on $\mathscr{B}\times\mathscr{B}$, and provided with properties* (BM1), (BM2), (BM3), *then there exists a Hilbert space H and an $\mathscr{L}(U, H)$-valued weakly* c.a. *measure β on $(D, \mathscr{B})$ which verifies relation* (10).

DEFINITION 2. Let β be any $\mathscr{L}(U, H)$-valued weakly c.a. measure on $(D, \mathscr{B})$; then the $\mathscr{L}(U, U^*)$-valued set function Θ associated with it in Proposition 2 will be called its *spectral bimeasure*.

THEOREM 2. *Let Θ be an $\mathscr{L}(U, U^*)$-valued spectral bimeasure on $\mathscr{B}$ and β be an $\mathscr{L}(U, H)$-valued weakly* c.a. *measure on $(D, \mathscr{B})$ associated with Θ. If $\mathfrak{M}\big(\mathscr{B}, \mathscr{L}(V, U^*)\big)$ denotes the space of all $\mathscr{L}(V, U)$-valued weakly* c.a. *measures on $\mathscr{B}$ and $\mathscr{M}_{VU}(\beta)$ the completion up to an isomorphism for $(\mathscr{U})$ of*

$$\mathrm{Sm}_{VU}(\beta) \overset{\mathrm{def}}{=} \Big\{\sum_{i=1}^{n}\beta(A_i)a_i;\ a_i\in\mathscr{L}(V, U),\ A_i\in\mathscr{B},\ i = 1, \ldots, n\Big\}$$

then

(i) *The $\mathscr{L}(V)$-modulus $\mathscr{H}_{VU}(\Theta)$ is contained in $\mathfrak{M}\big(\mathscr{B}, \mathscr{L}(V, U^*)\big)$.*

(ii) *The Loynes $\mathscr{L}(V)$-moduli $\mathscr{M}_{VU}(\beta)$ and $\mathscr{H}_{VU}(\Theta)$ are isomorphic and the following relationship holds*:

(11) *For each $M\in\mathscr{H}_{VU}(\Theta)$, there exists a unique X in $\mathscr{M}_{VU}(\beta)$ such that for any A in $\mathscr{B}$*

$$M(A) = [X, \beta(A)] = \beta^*(A)X.$$

Proof. Let us define

$$J\Big\{\sum_{i=1}^{n}\Theta_{A_i}u_i\Big\} = \sum_{i=1}^{n}\beta(A_i)u_i$$

for any $n\in\mathbf{N}^*$, $A_i\in\mathscr{B}$, $u_i\in U$ $(i = 1, \ldots, n)$ where Θ_A denotes $\Theta(A\times\cdot)$; then from relation (10) and the definition of the uniform structure $(\mathscr{U})$ we easily conclude that J is an isomorphism from $\mathscr{H}_{VU}(\Theta)$ onto $\mathscr{M}_{VU}(\beta)$. Hence for each $M\in\mathscr{H}_{VU}(\Theta)$ there exists a unique $X\in\mathscr{M}_{VU}(\beta)$ such that for any $A\in\mathscr{B}$ and any $(v, u)\in V\times U$,

$$([M, \Theta_A]v)(u) = \langle Xv, \beta(A)u\rangle.$$

But from the autoreproducing property we have $M(A) = [M, \Theta_A]$, so the theorem is established. ■

Remarks 1. (a) The previous theorem means that any element of the modulus $\mathscr{M}_{VU}(\beta)$ can be regarded as an $\mathscr{L}(V, U^*)$-valued weakly c.a. measure defined on $(D, \mathscr{B})$ by relationship (11).

So, in particular, if β is the measure associated with a V-bounded $\mathscr{L}(U, H)$-valued process $(X_g)_{g\in G}$ over a locally compact abelian group G, i.e.

$$\langle X_g u, h\rangle = \int_D\langle g, \lambda\rangle d\big(\langle\beta(\lambda)u, h\rangle\big) \qquad \forall u\in U, h\in H, g\in G$$

with D being the dual group of G, $\mathscr{B}$ the Borelian σ-algebra of D and $\langle\cdot, \lambda\rangle$ is a character on G, then the time-domain of $(X_g)_{g\in G}$, that is, $\mathscr{M}_{VU}(\beta)$, may be avantageously interpreted in some cases as the modulus of $\mathscr{L}(V, U^*)$-valued measures on $(D, \mathscr{B})$, verifying (11).

(b) Let $A\in\mathscr{B}$ and $\overline{\beta(A)U} = \overline{\mathrm{sp}}\{\beta(A)u;\ u\in U\}$ be the closure of $\beta(A)U$ in $H(= H(\Theta))$. Consider an element $M\in\mathscr{H}_{VU}(\Theta)$ and $v\in V$; then from Theorem 2,

$$\big(M(A)v\big)(u) = \langle Mv, \beta(A)u\rangle_H \qquad \forall u\in U.$$

But, for fixed $M\in\mathscr{H}_{VU}(\Theta)$, $v\in V$, the previous relationship defines a continuous functional on the Hilbert subspace $\overline{\beta(A)U}$; hence there exists a vector $\xi(v)\in\overline{\beta(A)U}$ such that

$$\big(M(A)v\big)(u) = \langle\xi(v), \beta(A)u\rangle_H = \big(\beta^*(A)\xi(v)\big)(u) \qquad \forall u\in U.$$

Consequently,

$$\forall M\in\mathscr{H}_{VU}(\Theta)\ \forall v\in V,\ \forall A\in\mathscr{B}\ \ M(A)v\in\beta^*(A)\{\overline{\beta(A)U}\}.$$

From now on, we consider the special case where Θ is defined by

(12) $\qquad \Theta(A\times B) = F(A\cap B) \qquad \forall A, B\in\mathscr{B}$

where F is an $\mathscr{L}^+(U, U^*)$-valued weakly c.a. measure on $(D, \mathscr{B})$. The measure F is then said to be an *operator-valued spectral measure* or shortly a *spectral measure* on $(D, \mathscr{B})$. (Notice that in [9], F is called a *semi-spectral measure*.)

In this case, any $\mathscr{L}(U, H)$-valued measure on $(D, \mathscr{B})$ associated with Θ by (10) is a quasi-isometric measure W (cf. [12]). Henceforth we shall denote $\mathscr{H}_{VU}(F)$ (resp. $\mathscr{M}_{VU}(W)$) instead of $\mathscr{H}_{VU}(\Theta)$ (resp. $\mathscr{M}_{VU}(\beta)$).

According to Theorem 2, an extension of the so-called *Kolmogorov isomorphism theorem* for Banach space (cf. [7]) may be obtained as an answer to the question whether the Loynes modulus $\mathscr{M}_{VU}(W)$ has the Radon–Nikodym property with respect to (w.r.t.) an $\mathscr{L}(U, H)$-valued quasi-isometric measure W.

In [23] it is shown for nuclear operator-valued spectral measure F that the autoreproducing kernel grammian modulus $\mathscr{H}_{VU}(F)$ is exactly the class of operator-valued measures that are Hellinger square integrable w.r.t. F.

So by analogy, we state the following definitions

DEFINITION 3. (i) An $\mathscr{L}(V, U^*)$-valued weakly c.a. measure M on $(D, \mathscr{B})$ is said to be *Hellinger square integrable w.r.t. F* if M belongs to $\mathscr{H}_{VU}(F)$.

(ii) Let M be a measure in $\mathscr{H}_{VU}(F)$; then the non-negative operator [M, M] is called the *Hellinger square integral of M w.r.t. F*.

Remark 2. Quite recently Makagon in [9], p. 197 has given a direct definition for U^*-valued measures to be Hellinger square integrable w.r.t. F. For the clarity, let us quote his definition hereunder:

For any $A \in \mathscr{B}$, let $\|F(A)u\|_A^2 = (F(A)u)(u)$ $\forall u \in U$ and $S(A)$ be the closure in U^* of $F(A)\,U$ in the $\|\cdot\|_A$-norm; then any U^*-valued measure m on $(D, \mathscr{B})$ is said to be *Hellinger square integrable w.r.t. F* if

(i) $\forall A \in \mathscr{B}$ $m(A) \in S(A)$,

(ii) $\|m\|_F^2 = \sup\limits_{\sigma\in\mathscr{F}} \sum\limits_{A\in\sigma} \|m(A)\|_A^2 < \infty$

where $\mathscr{F}$ is the family of all finite measurable partitions σ of D. We remark that if $M \in \mathscr{H}_{VU}(F)$ then $\forall v \in V$, $M(\cdot)v$ is Hellinger square integrable w.r.t. F in the previous sense of Makagon.

Indeed, according to Remark 1 (b) and the property that F is a spectral measure, it follows that $M(\cdot)v$ satisfies (i). On the other hand, from Theorem 1.3 in [9] it is obvious that relation (ii) holds for every $M(\cdot)v$.

In other words, a U^*-valued measure m is Hellinger square integrable w.r.t. F in the sense of Makagon iff $m \in H(F)$. Consequently, from Proposition 1, we deduce the next result.

THEOREM 3. *An $\mathscr{L}(V, U^*)$-valued weakly* c.a. *measure M on $(D, \mathscr{B})$ is Hellinger square integrable w.r.t. F iff $\forall v \in V$, $M(\cdot)v$ is Hellinger square integrable w.r.t. F in the sense of Makagon.*

In [9] Makagon is interested in the construction of the Radon–Nikodym derivative of $M(\cdot)v$ w.r.t. F, whereas our aim, in what follows, is to show that under certain conditions the modulus $\mathscr{M}_{VU}(W)$ has the Radon–Nikodym property and also to express under these conditions, for any $M \in \mathscr{H}_{VU}(F)$, the integral form of the operator $[M, M]$.

2. A special Kolmogorov isomorphism theorem for the modulus $\mathscr{M}_{VU}(W)$.

Let U, V be two separable Hilbert spaces and $(D, \mathscr{B})$ be a measurable space.

DEFINITION 4 (cf. [4], [10], [11]). (i) An $\mathscr{L}(V, U)$-valued function φ on D is said to be *$\mathscr{B}$-measurable* if there exists a sequence of $\mathscr{L}(V, U)$-valued $\mathscr{B}$-simple functions $\{\varphi_n\}_{n\in N}$ such that for any $\lambda \in D$ and any $v \in V$,

$$\lim_{n\to\infty} |\varphi_n(\lambda)v - \varphi(\lambda)v|_U = 0.$$

(ii) An $\mathscr{O}(V, U)$-valued function φ on D is said to be *$\mathscr{B}$-measurable* if there exists a sequence of $\mathscr{L}(V, U)$-valued $\mathscr{B}$-measurable functions $\{\varphi_n\}_{n\in N}$ such that for any $\lambda \in D$ and any v belonging to the domain $\mathrm{Dom}\,(\varphi(\lambda))$ of $\varphi(\lambda)$,

$$\lim_{n\to\infty} |\varphi_n(\lambda)v - \varphi(\lambda)v|_U = 0.$$

Let W be a quasi-isometric measure associated with the spectral measure F (cf. [12]); then the more general Kolmogorov isomorphism theorem was obtained by Makagon in [8] for the vector time-domain

$$\overline{\mathrm{sp}}\{W(A)u;\ A \in \mathscr{B}, u \in U\}.$$

Since, herein, we are interested in construction of isomorphisms for the operator time domain $\mathscr{M}_{VU}(W)$ of W, we shall consider more restrictive assumptions on F.

Additional assumptions. From now on, we assume that the spectral measure F has a Radon–Nikodym (R-N) derivative $f = \dfrac{dF}{d\mu}$ w.r.t. a positive σ-finite measure μ on $(D, \mathscr{B})$ and that $f \in \mathscr{L}^+(U)$ μ-almost everywhere (μ-a.e.).

Then it is known that there exists a separable Hilbert space $\mathscr{K}$ and an $\mathscr{L}(U, \mathscr{K})$-valued strongly measurable function Q on $(D, \mathscr{B})$ such that $f(\lambda) = Q^*(\lambda)\,Q(\lambda)$ μ-a.e. (cf. [14], Theorem 3.2 and [15], a more general result is obtained in [7], [21]).

Let us consider the space $\mathscr{L}^2_{VU}(F)$ of all $\mathcal{O}(V, U)$-valued $\mathscr{B}$-measurable functions φ such that

$$Q\varphi \in \mathscr{L}(V, \mathscr{K}) \quad \mu\text{-a.e.}$$

and for each $v \in V$,

$$\int_D |Q\varphi v|^2_{\mathscr{K}} d\mu < \infty.$$

DEFINITION 5. (i) A sequence $\{\varphi_n\}_{n\in N}$ in $\mathscr{L}^2_{VU}(F)$ is said to be a *Cauchy sequence in* $\mathscr{L}^2_{VU}(F)$ if for each $v \in V$

$$\lim_{n,m\to\infty} \int_D |Q\varphi_n(v) - Q\varphi_m(v)|^2_{\mathscr{K}} d\mu = 0.$$

(ii) A sequence $\{\varphi_n\}_{n\in N}$ in $\mathscr{L}^2_{VU}(F)$ is said to be $(\mathscr{U})$-*convergent to an element* $\varphi \in \mathscr{L}^2_{VU}(F)$ if for each $v \in V$

$$\lim_{n\to\infty} \int_D |Q\varphi_n(v) - Q\varphi(v)|^2_{\mathscr{K}} d\mu = 0.$$

(iii) A subset S of $\mathscr{L}^2_{VU}(F)$ is said to be *dense in* $\mathscr{L}^2_{VU}(F)$ if for every $\varphi \in \mathscr{L}^2_{VU}(F)$ there exists a sequence $\{\varphi_n\}_{n\in N}$ in S that is $(\mathscr{U})$-convergent to φ.

(iv) $\mathscr{L}^2_{VU}(F)$ is said to be $(\mathscr{U})$-*complete* if every Cauchy sequence in $\mathscr{L}^2_{VU}(F)$ is $(\mathscr{U})$-convergent in $\mathscr{L}^2_{VU}(F)$.

LEMMA 1. *The space of all $\mathscr{L}(V, U)$-valued $\mathscr{B}$-simple functions is dense in $\mathscr{L}^2_{VU}(F)$.*

Proof. Let $\varphi \in \mathscr{L}^2_{VU}(F)$; then from Definition 4 (i) and (ii) it is easy to see from the diagonal procedure that there exists a sequence of $\mathscr{L}(V, U)$-valued $\mathscr{B}$-simple functions $\{\varphi_n\}_{n\in N}$ over D such that for any $\lambda \in D$ and any vector $v \in \mathrm{Dom}\big(\varphi(\lambda)\big)$,

$$\lim_{n\to\infty} |\varphi_n(\lambda)v - \varphi(\lambda)v|_U = 0. \tag{13}$$

But

$$|Q(\lambda)\varphi_n(\lambda)v - Q(\lambda)\varphi(\lambda)v|_{\mathscr{K}} \leqslant \|Q\|\, |\varphi_n(\lambda)v - \varphi(\lambda)v|_U. \tag{14}$$

Hence from (13) and (14), we deduce that for any $\lambda \in \Lambda$, the two sequences $\{f_n(\lambda)\}_{n\in N}$, $\{h_n(\lambda)\}_{n\in N}$ converge to $f(\lambda)$ where for every $n \in N$,

$$f_n = \langle Q\varphi_n v, Q\varphi_n v\rangle, \qquad h_n = \langle Q\varphi_n v, Q\varphi v\rangle \qquad \text{and} \qquad f = \langle Q\varphi v, Q\varphi v\rangle.$$

Since f belongs to $L^1(\mu)$, according to the Lebesgue convergence theorem we deduce that the sequences $\{f_n\}_{n\in N}$ and $\{h_n\}_{n\in N}$ converge in the norm of $L^1(\mu)$, i.e.

$$\lim_{n\to\infty} \int_D |f_n - f|\, d\mu = \lim_{n\to\infty} \int_D |h_n - f|\, d\mu = 0. \tag{15}$$

However

$$\int_D |Q\varphi_n v - Q\varphi v|^2_{\mathscr{K}} d\mu = \int_D (f_n - h_n - \bar{h}_n + f)\, d\mu. \tag{16}$$

So from (15) and (16) the lemma is established. ∎

THEOREM 4 (Kolmogorov isomorphism theorem). (i) *The space of $\mathscr{L}(V, U)$-valued $\mathscr{B}$-simple functions is dense in $\mathscr{L}^2_{VU}(F)$.*

(ii) *$\mathscr{L}^2_{VU}(F)$ is $(\mathscr{U})$-complete.*

(iii) *The relationship on $\mathscr{L}^2_{VU}(F) \times \mathscr{L}^2_{VU}(F)$, the kernel of which is the subspace*

$$\Big\{\varphi \in \mathscr{L}^2_{VU}(F);\ \forall v \in V,\ \int_D |Q\varphi v|^2_{\mathscr{K}} d\mu = 0\Big\}$$

is an equivalence relation.

(iv) *The quotient space $L^2_{VU}(F)$ of $\mathscr{L}^2_{VU}(F)$ for the above equivalence relation is isomorphic to $\mathscr{M}_{VU}(W)$ so that for any $\varphi, \psi \in L^2_{VU}(F)$,*

$$[\varphi, \psi] \overset{\text{def}}{=} \int_D \psi^* f \varphi\, d\mu = [W(\varphi), W(\psi)] = W(\psi)^* W(\varphi). \tag{17}$$

Proof. (i): Cf. Lemma 1.

(ii): Let $\{\varphi_n\}_{n\in N}$ be a Cauchy sequence in $\mathscr{L}^2_{VU}(F)$ that is for each $v \in V$,

$$\lim_{n,m\to\infty} \int_D |(Q\varphi_n - Q\varphi_m)v|^2 d\mu = 0.$$

But $L^2(\mu, \mathscr{K})$ is a Hilbert space; then there exists a unique $g(v) \in L^2(\mu, \mathscr{K})$ such that

$$(Q\varphi_{n_k})v \underset{k\to\infty}{\to} g(v) \quad \mu\text{-a.e.} \tag{18}$$

and

$$\int_D |Q\varphi_n v - g(v)|^2_{\mathscr{K}} d\mu \underset{n\to\infty}{\to} 0. \tag{19}$$

From (18) it is easy to see that $g \in L(V, \mathscr{K})$ μ-a.e. Indeed, for any $\lambda \in D$ such that (18) holds, $\{Q\varphi_{n_k}(\lambda)\}_{k\in N} \subset \mathscr{L}(V, \mathscr{K})$; then according to the Banach–Steinhaus theorem it is concluded that $g \in \mathscr{L}(V, \mathscr{K})$ μ-a.e. Now since $Q \in \mathscr{L}(U, \mathscr{K})$ μ-a.e., hence according to [18], p. 540, Q admits a unique generalized inverse $Q^{\neq}$ which is an $\mathcal{O}(\mathscr{K}, U)$-valued $\mathscr{B}$-measurable function over D.

Let us set $\varphi = Q^{\neq} g$; then from the properties of the generalized inverses (cf. [10], [11], [18]) we have

$$Q\varphi = QQ^{\neq} g = P_{R(Q)} g \tag{20}$$

where $R(Q)$ is the closure of the range of Q and P_S is the orthogonal projection on the closed space S in $\mathscr{K}$.

Moreover, for every $n \in N$ $R(Q\varphi_n) \subseteq R(Q)$ (cf. [18]); then from (18) it is obvious that $R(g) \subseteq R(Q)$. Hence

$$P_{R(Q)}g = g. \tag{21}$$

Consequently from (20) and (21) it is deduced that for any $v \in V$

$$\int_D |Q\varphi v|^2_{\mathscr{K}} d\mu = \int_D |g(v)|^2_{\mathscr{K}} d\mu$$

and

$$\lim_{n\to\infty} \int_D |(Q\varphi_n - Q\varphi)v|^2_{\mathscr{K}} d\mu = \lim_{n\to\infty} \int_D |Q\varphi_n v - g(v)|^2_{\mathscr{K}} d\mu = 0.$$

Hence we conclude (ii).

(iii): This part is obvious.

(iv): The isomorphism between $L^2_{VU}(F)$ and $\mathscr{M}_{VU}(W)$ is deduced from (i) and (ii) in the usual manner as in the proof of Theorem 2. ∎

THEOREM 5. (i) *For each* $M \in \mathscr{H}_{VU}(F)$ *there exists a unique* $\varphi_M \in L^2_{VU}(F)$ *such that for any* $A \in \mathscr{B}$

$$M(A) = \int_A f\varphi_M d\mu \tag{22}$$

that is

$$dM/d\mu = f\varphi_M \in \mathscr{L}(V, U) \ \mu\text{-a.e.}$$

(ii) *For any* M *and* N *in* $\mathscr{H}_{VU}(F)$ *there exists a unique pair* (φ_M, φ_N) *in* $L^2_{VU}(F)$ *such that*

$$[M, N] = \int_D (dN/d\mu)^*(dF/d\mu)^{\neq}(dM/d\mu)\,d\mu = [\varphi_M, \varphi_N] \in \mathscr{L}(V) \tag{23}$$

where $(dF/d\mu)^{\neq}$ *is the generalized inverse for* f.

Proof. (i): It is obvious from both Theorems 2 and 4 that (22) holds. So $dM/d\mu = f\varphi_M$ μ-a.e. But $Q\varphi_M \in \mathscr{L}(V, \mathscr{K})$ (cf. the proof of Theorem 4). Hence $f\varphi_M = Q^*Q\varphi_M \in \mathscr{L}(V, U)$ μ-a.e.

(ii): From the property of the generalized inverse, we have $f^{\neq} = f^{\neq}ff^{\neq}$. Hence (23) is obtained. ∎

Let $\tilde{H}^2_F(\mu)$ be the subclass in $\mathfrak{M}(\mathscr{B}; \mathscr{L}(V, U))$ such that if $M \in \tilde{H}^2_F(\mu)$, then $dM/d\mu$ exists, $Q^{*\neq}\, dM/d\mu \in \mathscr{L}(V, \mathscr{K})$ μ-a.e. and $\langle [M, M]_\mu v, v\rangle < \infty\ \forall v \in V$, where

$$[M, M]_\mu \overset{\text{def}}{=} \int_D (dM/d\mu)^*(dF/d\mu)^{\neq}(dM/d\mu)\,d\mu$$

(the integral is meant in the weak sense).

Notice that from Theorem 5, $\mathscr{H}_{VU}(F) \subset \tilde{H}^2_F(\mu)$ hence $\tilde{H}^2_F(\mu)$ is not a void class.

We now prove that reversally $\tilde{H}^2_F(\mu)$ is contained in $\mathscr{H}_{VU}(F)$. So the integral form of the operators $[M, M]$ is established for any $M \in \mathscr{H}_{VU}(F)$.

RECIPROCAL OF THEOREM 5. *Let* $\mathscr{R}$ *be the relationship on* $\tilde{H}^2_F(\mu) \times \tilde{H}^2_F(\mu)$ *the kernel of which is* $\{M \in \tilde{H}^2_F(\mu);\ [M, M]_\mu = 0\}$ *and let* $H^2_F(\mu)$ *be the quotient modulus of* $\tilde{H}^2_F(\mu)$ *for the equivalence relationship* $\mathscr{R}$. *Then*

$$\mathscr{H}_{VU}(F) = H^2_F(\mu).$$

Proof. Let $M \in \tilde{H}^2_F(\mu)$; then it is obvious that

$$\varphi_M = (dF/d\mu)^{\neq}(dM/d\mu) \in \mathscr{L}^2_{VU}(F)$$

and $[M, M]_\mu = [\varphi_M, \varphi_M]_F$. Hence the spaces $\tilde{H}^2_F(\mu)$ and $\mathscr{L}^2_{VU}(F)$ are isomorphic. Since $\mathscr{H}_{VU}(F)$ is contained in $H^2_F(\mu)$ and is isomorphic onto $L^2_{VU}(F)$, the theorem is established. ∎

The results in Theorem 5 and in its reciprocal can easily be summed up as below:

COROLLARY 1. *If* U, V *are two separable Hilbert spaces, if* F *admits an* R-N *derivative* $dF/d\mu$ *w.r.t. a non-negative* σ*-finite measure* μ *on* $(D, \mathscr{B})$ *such that* $dF/d\mu \in \mathscr{L}^+(U)$ μ*-a.e. and if* M *is an* $\mathscr{L}(V, U)$*-valued measure on* $(D, \mathscr{B})$, *then the following assertions are equivalent:*

(i) M *is Hellinger square integrable w.r.t.* F *(i.e.* $M \in \mathscr{H}_{VU}(F)$*).*

(ii) $Q^{*\neq} dM/d\mu \in \mathscr{L}(V, \mathscr{K})$ μ*-a.e. and*

$$\int_D (dM/d\mu)^*(dF/d\mu)^{\neq}(dM/d\mu)\,d\mu \in \mathscr{L}^+(V)$$

(the integral is meant in the weak sense).

(iii) *There exists a function* $\varphi \in L^2_{VU}(F)$ *such that for any* $A \in \mathscr{B}$,

$$M(A) = \int_A (dF/d\mu)\varphi\,d\mu$$

(the integral being meant in the weak sense).

Remark. Let M be an operator-valued Hellinger square integrable w.r.t. F; then we immediately obtain

(α) $\forall v \in V, \forall A \in \mathscr{B}$ $M(A)v = \int_A Q^*(Q\varphi_M)v\,d\mu$ (in the weak sense) with $Q\varphi_M(\cdot)v$ being a $\mathscr{K}$-valued square μ-integrable function on $(D, \mathscr{B})$. (φ_M and M being related by relation (iii) in Corollary 1.) Under very more general assumptions, Makagon has shown in [9], Theorem 1.5 that a vector-valued measure m is Hellinger square integrable w.r.t. F iff it satisfies an analoguous condition to (α).

So the previous equivalence (i)$\Leftrightarrow$(iii) is somewhat (under more restrictive assumptions) an operator version of Theorem 1.5 in [9].

3. Interpolability and minimality of Hilbert space-valued weakly stationary processes. Let G be a locally compact abelian (LCA) group, $\hat{G}$ be its dual group; G, $\hat{G}$ are provided with their Borel σ-algebras $\mathscr{B}$, $\hat{\mathscr{B}}$ and their Haar measures dg, $d\lambda$, respectively.

Let U, V be two separable Hilbert spaces, H be any Hilbert space and X be a weakly continuous mapping from G into $\mathscr{L}(U, H)$ such that its correlation kernel $K(g, h) = X_h^* X_g$ is a function $K(g-h)$ dependent only on the difference $(g-h)$.

Such a mapping X is said to be *a continuous U-valued weakly stationary process over G* or to be a *U-valued stationary Hilbertian variety* according to Masani's terminology in [13], (according to [13], a Hilbert variety is an $\mathscr{L}(B, H)$-valued mapping defined on G, B being any Banach space).

For such processes $(X_g)_{g\in G}$ it is well known that there exists a unique $\mathscr{L}^+(U)$-valued weakly c.a. spectral measure F on $(\hat{G}, \hat{\mathscr{B}})$ such that for any v, $u \in U$ and $g \in G$,

$$\big(K(g)u\big)(v) = \int_{\hat{G}} \overline{\langle g, \lambda\rangle} d\big(F(\lambda)u\big)v \quad \text{(Bochner theorem).}$$

(For more details and further results the reader is referred to [1], [4] [25], [26].)

Here it is assumed that there exists an $\mathscr{L}^+(U)$-valued $\hat{\mathscr{B}}$-measurable function f over $\hat{G}$ such that for any $A \in \hat{\mathscr{B}}$ and any u, $v \in U$ we have

$$\big(F(A)u\big)(v) = \int_A \big(f(\lambda)u\big)(v)\, d\lambda.$$

Let us denote by

$\mathscr{M}_{VU}(X) = \overline{\mathrm{Sm}}_{VU}(X)$ the Loynes $\mathscr{L}(V)$-modulus generated by the process $(X_g)_{g\in G}$ (time-domain of $(X_g)_{g\in G}$),

$\mathscr{H}_{VU}(K)$ the autoreproducing Loynes modulus of K,

$\mathscr{H}_{VU}(F)$ the autoreproducing Loynes modulus of F.

It is obvious from Theorems 2 and 4 that these three spaces are isomorphic. Moreover, from the same arguments as in [23], Theorem 6, we can state the next result.

THEOREM 6. *For every $y \in \mathscr{H}_{VU}(K)$ there exists a unique $M_y \in \mathscr{H}_{VU}(F)$ such that for any $g \in G$*

$$y(g) = \int_{\hat{G}} \overline{\langle g, \lambda\rangle}\, dM_y(\lambda) \tag{24}$$

(the integral is meant in the weak sense).

The following lemma is fundamental for the study of interpolability and minimality of U-valued stationary processes.

LEMMA 2. *Let $L^1_{VU}(dg)$ be the space of $\mathscr{L}(V, U)$-valued weakly dg-integrable functions defined over G. For every $y \in L^1_{VU}(dg)$, its Fourier inverse transform is*

$$\hat{y}(\lambda) = \int_G \langle g, \lambda\rangle\, y(g)\, dg$$

that is for any $\lambda \in \hat{G}$ and $(v, u) \in V\times U$

$$\hat{y}(\lambda)(v, u) = \int_G \langle g, \lambda\rangle\, \big(y(g)v\big)(u)\, dg.$$

If $y \in L^1_{VU}(dg) \cap \mathscr{H}_{VU}(K)$ for a given Haar measure dg on G then

(i) *$\hat{y}$ is a $B(V, U)$-valued weakly $d\lambda$-integrable function and there exists a Haar measure $d\lambda$ on $\hat{G}$ such that*

$$y(g) = \int_{\hat{G}} \overline{\langle g, \lambda\rangle}\, \hat{y}(\lambda)\, d\lambda, \quad g \in G.$$

(ii) *The measure $N_{\hat{y}}$, defined by*

$$N_{\hat{y}}(A) = \int_A \hat{y}(\lambda)\, d\lambda$$

for any $A \in \hat{\mathscr{B}}$, belongs to $\mathscr{H}_{VU}(F)$.

All the integrals in the lemma are meant in the weak sense and $B(V, U)$ denotes the space of all sesquilinear functionals on $V \times U$.

Proof. It is analogous as for Lemma 4 and Proposition 4 in [23]. However, let us give some complements.

(i): Since $y \in L^1_{VU}(dg) \cap \mathscr{H}_{VU}(K)$, then for any $(v, u) \in V\times U$, $\hat{y}(\lambda)(v, u)$ is defined for any $\lambda \in \hat{G}$ and

$$\big(y(g)v\big)(u) = \int_{\hat{G}} \overline{\langle g, \lambda\rangle}\, d\big(M(\lambda)v\big)(u)$$

for a measure M in $\mathscr{H}_{VU}(F)$ (cf. Theorem 6). Hence by the inversion formula (cf. [28], p. 22) we obtain

$$\big(y(g)v\big)(u) = \int_{\hat{G}} \overline{\langle g, \lambda\rangle}\, \hat{y}(\lambda)(v,u)\, d\lambda.$$

(ii): For any $A \in \hat{\mathscr{B}}$ and $(v, u) \in V\times U$, let

$$N_{\hat{y}}(A)(v, u) = \int_A \hat{y}(\lambda)(v, u)\, d\lambda.$$

Then

$$\big(y(g)v\big)(u) = \int_{\hat G} \langle \overline{g, \lambda}\rangle dN_{\hat y}(\lambda)(v, u).$$

But $y \in \mathscr{H}_{VU}(K)$; then there exists a unique measure $M \in \mathscr{H}_{VU}(F)$ (cf. Theorem 6) such that for any $(v, u) \in V \times U$ and $g \in G$,

$$\big(y(g)v\big)(u) = \int_{\hat G} \langle \overline{g, \lambda}\rangle d\big(M(\lambda)v\big)(u),$$

hence

$$\int_{\hat G} \langle \overline{g, \lambda}\rangle d\big(M(\lambda)v\big)(u) = \int_{\hat G} \langle \overline{g, \lambda}\rangle dN_{\hat y}(\lambda)(v, u).$$

Now it follows from the uniqueness theorem in [28], p. 17, that $\big(M(A)v\big)(u) = N_{\hat y}(A)(v, u)$, i.e. $M = N_{\hat y}$. ■

Now let $\mathscr{D}$ be the class of all non-empty compact subsets of G. For any $\Gamma \in \mathscr{D}$, $\mathscr{M}_{VU}(X; G-\Gamma)$ denotes the space

$$\overline{\mathrm{Sm}}\{X_g a;\ g \in G,\ g \notin \Gamma,\ a \in \mathscr{L}(V, U)\}$$

that is the completion for the uniform structure $(\mathscr{U})$ of the space spanned by $\{X_g a;\ g \in G,\ g \notin \Gamma, a \in \mathscr{L}(V, U)\}$.

Let us note that for general Loynes spaces there is no theorem guaranteeing the existence of orthogonal projection hence of orthogonal complements, but for the special Loynes modules studied herein the previous difficulty does not arise (cf. [17], [25]).

So let $\mathscr{N}(\Gamma)$ be the orthogonal complement of $\mathscr{M}_{VU}(X; G-\Gamma)$ in $\mathscr{M}_{VU}(X)$.

DEFINITION 6 (cf. [19], [20], [24]).

(i) A compact subset $\Gamma \in \mathscr{D}$ is *interpolable w.r.t.* $(X_g)_{g\in G}$ if $\mathscr{N}(\Gamma) = \{0\}$.

(ii) $(X_g)_{g\in G}$ is *interpolable* if for any compact subset Γ belonging to $\mathscr{D}$, $\mathscr{N}(\Gamma) = \{0\}$.

(iii) $(X_g)_{g\in G}$ is *minimal* if for any $h \in G$, $\mathscr{N}(\{h\}) \neq \{0\}$.

LEMMA 3. *For any $\Gamma \in \mathscr{D}$, the space $\mathscr{N}(\Gamma)$ is isomorphic to*

$$\mathscr{C}\big(\Gamma; \mathscr{L}(V, U)\big) \cap \mathscr{H}_{VU}(K)$$

where for any subset S in G, $\mathscr{C}\big(S; \mathscr{L}(V, U)\big)$ denotes the space of all weakly continuous functions ξ from S into $\mathscr{L}(V, U)$, i.e.

$$\forall (v, u) \in V \times U \quad \xi\colon s \mapsto \big(\big(\xi(s)\big)v\big)(u)$$

is a continuous function on S.

Proof. Let $\mathscr{N}'(\Gamma)$ be the subspace in $\mathscr{H}_{VU}(K)$ that is isomorphic onto $\mathscr{N}(\Gamma)$; then for each $y \in \mathscr{N}'(\Gamma)$, there exists a unique $Y \in \mathscr{N}(\Gamma)$ such that for any $g \in G$, $y(g) = X_g^* Y$, hence from the weak continuity of the process $(X_g)_{g\in G}$,

$$\forall (v, u) \in V \times U \quad g \in G \mapsto \big(y(g)v\big)(u)$$

is a continuous function with a compact support contained in Γ.

The reciprocal is obvious. ■

Following the same arguments as in [23], Theorem 10 and Corollary 5, the next theorem and corollary are easy to deduce.

THEOREM 7. (i) *A subset $\Gamma \in \mathscr{D}$ is interpolable w.r.t. $(X_g)_{g\in G}$ iff for any $y \in \mathscr{C}\big(\Gamma; \mathscr{L}(V, U)\big)$ either $y = 0$ or for every $v \in V$,*

$$\int_{\hat G} \langle [\hat y(\lambda)]^* f^{\neq}(\lambda)\ \hat y(\lambda)v, v\rangle d\lambda = \infty.$$

(ii) *$(X_g)_{g\in G}$ is interpolable iff for every non-null function y in $\mathscr{K}\big(G; \mathscr{L}(V, U)\big)$,*

$$\int_{\hat G} \langle \hat y(\lambda)^* f^{\neq}(\lambda)\hat y(\lambda)v, v\rangle d\lambda = \infty, \quad v \in V$$

where $\mathscr{K}\big(G; \mathscr{L}(V, U)\big)$ is the space of $\mathscr{L}(V, U)$-valued weakly continuous functions with compact support in G.

COROLLARY 2. *Let G be a discrete Abelian group. $(X_g)_{g\in G}$ is interpolable iff for any trigonometric polynomial $\hat y$ with $\mathscr{L}(V, U)$-coefficients*

$$\forall v \in V \int_{\hat G} \langle [\hat y(\lambda)]^* f^{\neq}(\lambda)\hat y(\lambda)v, v\rangle\, d\lambda = \infty \ \text{ or } \ \hat y = 0.$$

Let us note that as in [23], [24], minimal continuous U-valued weakly stationary processes only can exist over discrete groups. So we now assume that G is a discrete abelian group. Since $(X_g)_{g\in G}$ is stationary, $\mathscr{N}(\{0\}) \neq \{0\}$ iff for any $h \in G$ $\mathscr{N}(\{h\}) \neq \{0\}$. Hence, following [19], [20], let us consider the projection Y_0 of X_0 on $\mathscr{N}(\{0\})$, then $[Y_0, Y_0] = Y_0^* Y_0 \in \mathscr{L}^+(U)$ and $[Y_0, Y_0]^{\neq}$ exists in $\mathscr{O}(U)$. Hence

$$y_0 = R(Y_0) = [Y_0, Y_0]\delta_0 \quad \text{and} \quad \hat y_0(\lambda) = \int_G y_0(g)\,dg = [Y_0, Y_0]$$

where R denotes the isomorphism between $\mathscr{M}_{VU}(X)$ and $\mathscr{H}_{VU}(K)$. So

$$N_{\hat y_0}(A) = \int_A [Y_0, Y_0] d\lambda = [Y_0, Y_0] d\lambda(A),$$

$A \in \hat{\mathscr{B}}$ and $N_{\hat{y}_0} \in \mathscr{H}_{VU}(F)$ that is

$$[N_{\hat{y}_0}, N_{\hat{y}_0}] = \int_{\hat{G}} [Y_0, Y_0](dF/d\lambda)^{\neq}[Y_0, Y_0]d\lambda \in \mathscr{L}^+(U).$$

(All these integrals are meant in the weak sense.)

Let us also consider the orthogonal projection J on the closure of the range of $[Y_0, Y_0]$, i.e. $J = [Y_0, Y_0]^{\neq}[Y_0, Y_0]$.

Remark. Let us note that the minimality theorem of Makagon–Weron ([27], Theorem 4.6) for q-variate stationary processes can easily be extended to U-valued stationary processes. So only the minimality for stationary processes of full rank is studied below.

DEFINITION 7. A U-valued weakly stationary process $(X_g)_{g\in G}$ over a discrete group G is said to be *of full rank* if the inverse of $[Y_0, Y_0]$ exists in $\mathscr{L}(U)$.

The following theorem is an operator extension of the result in [19] p. 309, [24], p. 180. (Another extension is studied by Miamee–Salehi in [16].)

THEOREM 8. (i) *$(X_g)_{g\in G}$ is minimal of full rank iff for almost all λ in $\hat{G}$, $f(\lambda)$ has an inverse in $\mathscr{L}(U)$ and $\int_{\hat{G}} f^{-1}(\lambda)d\lambda$ exists in the weak sense.*

(ii) *If $(X_g)_{g\in G}$ is minimal of full rank then $[Y_0, Y_0]$ has an inverse in $\mathscr{L}(U)$ and*

$$[\varepsilon_0, \varepsilon_0] = \int_{\hat{G}} f^{-1}(\lambda)d\lambda \quad \text{with} \quad \varepsilon_0 = Y_0[Y_0, Y_0]^{-1}.$$

Necessity. If $[Y_0, Y_0]^{-1}$ exists in $\mathscr{L}(U)$ then $Y_0 \neq 0$, so $(X_g)_{g\in G}$ is minimal. Moreover

$$Y_0 \in \mathscr{N}(\{0\}) \Leftrightarrow \varepsilon_0 = Y_0[Y_0, Y_0]^{-1} \in \mathscr{N}(\{0\})$$

(that is not the case only if $[Y_0, Y_0]^{\neq}$ exists).

Let e be the corresponding element of ε_0 in $\mathscr{H}_{UU}(K)$; then $e = I\delta_0$ and its Fourier inverse transform is $\hat{e} = I$ where I is the unit operator in $\mathscr{L}(U)$ and

$$\delta_0(g) = \begin{cases} 1 & \text{if} \quad g = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$N_{\hat{e}}(A) = \int_A I d\lambda \in \mathscr{H}_{UU}(F), \qquad A \in \hat{\mathscr{B}}. \tag{25}$$

But $N_{\hat{e}} \in \mathscr{H}_{UU}(F)$ iff there exists a unique $\varphi_e \in L^2_{UU}(F)$ such that

$$N_{\hat{e}}(A) = \int_A f\varphi_e d\lambda, \qquad A \in \hat{\mathscr{B}}. \tag{26}$$

From (25) and (26) it is deduced that

$$f\varphi_e = I \quad d\lambda\text{-a.e.} \tag{27}$$

Moreover,

$$N_{\hat{e}}(A) = [N_{\hat{e}}, \Theta_A] = \int_A (dF/d\lambda)(dF/d\lambda)^{\neq}(dN_{\hat{e}}/d\lambda)\, d\lambda$$

(cf. Theorem 2); then from the uniqueness of φ_e in $L^2_{UU}(F)$ we get $\varphi_e = f^{\neq}dN_{\hat{e}}/d\lambda$, hence $\varphi_e^* = \varphi_e$, so (27) becomes $f\varphi_e = \varphi_e f = I$ $d\lambda$-a.e. This means that f^{-1} exists $d\lambda$-a.e. and is equal to $\varphi_e d\lambda$-a.e.

Sufficiency. If f^{-1} exists in $\mathscr{L}(U)$ such that $\int_{\hat{G}} f^{-1}d\lambda$ exists (in the weak sense), then $N_I \in \mathscr{H}_{UU}(F)$ where

$$N_I(A) = \int_A I d\lambda \quad \text{for} \quad [N_I, N_I] = \int_{\hat{G}} I(f^{-1})I\, d\lambda$$

exists. Hence the Fourier transform e of I is equal to $e = I\delta_0$ and belongs to $\mathscr{H}_{UU}(K)$. So it is immediate that the isomorphic element ε_0 of e in $\mathscr{M}_{UU}(X)$ belongs to $\mathscr{N}(\{0\})$ for $[\varepsilon_0, X_g] = e(g)$, $g \in G$. Therefore $\mathscr{N}(\{0\}) \neq \{0\}$ that is $(X_g)_{g\in G}$ is minimal and $Y_0 \neq 0$ (otherwise X_0 should be orthogonal to $\mathscr{N}(\{0\})$, but this fact contradicts $[\varepsilon_0, X_0] = I \neq 0$).

Let a be an operator in $\mathscr{L}(U)$ such that for any $g \in G$

$$[\varepsilon_0 a - Y_0, X_g] = 0, \quad \text{i.e.} \quad I\delta_0(g)a = [Y_0, X_g]. \tag{28}$$

It is obvious from this condition that $a = [Y_0, X_0] = [Y_0, Y_0]$ and $a \neq 0$ because $Y_0 \neq 0$.

Moreover, condition (28) leads to the equality $Y_0 = \varepsilon_0 a$. Indeed, relationship (28) holds iff for any $v, u \in U$ and $g \in G$,

$$\langle(\varepsilon_0 a - Y_0)u, X_g v\rangle = ([\varepsilon_0 a - Y_0, X_g]u)(v) = 0.$$

But $(\varepsilon_0 a - Y_0)(u) \in \overline{\mathrm{sp}}\{X_g v;\ v \in U,\ g \in G\}$ which is the closed subspace in H, spanned by $\{X_g v;\ v \in U, g \in G\}$; then $(\varepsilon_0 a - Y_0)u = 0$ $\forall u \in U$, i.e. $Y_0 = \varepsilon_0 a$.

Now $I = [\varepsilon_0, X_0] = [\varepsilon_0, X_0 - Y_0 + Y_0] = [\varepsilon_0, Y_0] = [\varepsilon_0, \varepsilon_0][Y_0, Y_0]$. Consequently, $[Y_0, Y_0]$ is invertible in $\mathscr{L}(U)$ and

$$[Y_0, Y_0]^{-1} = [\varepsilon_0, \varepsilon_0] = [N_I, N_I] = \int_{\hat{G}} f^{-1}(\lambda)d\lambda. \blacksquare$$

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