STUDIA MATHEMATICA, T. LXXV, (1983)

Spline characterizations of H^1

by

S U N - Y U N G A. C H A N G* (Los Angeles, Calif.)
and
Z. C I E S I E L S K I (Sopot)

Abstract. There are given characterizations of the Hardy H^1 space on the circle in terms of the orthogonal system of polynomial splines of fixed order treated earlier by one of the authors. One of the results says: a function from L^1 belongs to H^1 if and only if its corresponding spline Fourier series converges unconditionally in L^1. Moreover, two more characterizations of H^1 are given in terms of the corresponding square and "area" functions.

1. Introduction. Before describing the main result we introduce the basic definitions. The one-dimensional torus in this paper is identified with $T = [-1/2, 1/2)$. In the Hardy space $H^1(\Delta)$, $\Delta = \{z\colon |z| < 1\}$, we have the norm

$$\|f\|_{H^1(\Delta)} = \sup_{0<r<1} \|f(re^{2\pi i x})\|_{L^1(T)}.$$

To each $f \in H^1(\Delta)$ there corresponds a function on T, $f(e^{2\pi i x}) = u(x) + iv(x)$. Notice that $\operatorname{Im} f(0) = 0$ iff $\int_T v(x)\,dx = 0$ and if it is the case, then

$$v(x) = \tilde{u}(x) = \lim_{\varepsilon \to 0+} \int_{\varepsilon}^{1/2} \frac{u(x+y) - u(x-y)}{\tan(\pi y)}\,dy.$$

Thus

$$H_0^1(\Delta) = \{f \in H^1(\Delta)\colon \operatorname{Im} f(0) = 0\}$$

is linearly isomorphic to

$$H^1(T) = \{u \in L^1(T)\colon \tilde{u} \in L^1(T)\}$$

* Research supported in part by the National Science Foundation.

with the norm

$$\|u\|_{H^1(T)} = \|u\|_{L^1(T)} + \|\tilde{u}\|_{L^1(T)}.$$

We know that the dual space $(H^1(T))^*$ can be identified with BMO(T) by the result of C. Fefferman [10]. The norm in BMO(T) is defined by

$$\|f\|_{\mathrm{BMO}(T)} = \Big|\int_T f\Big| + \sup (1/|I|) \int_I |f - f_I|,$$

where $f_I = (1/|I|) \int_I f$ and the sup is taken over all arcs $I \subset T$.

In this note we will give characterizations of $H^1(T)$ using periodic spline bases of order r, $r \geqslant 2$. Our paper is motivated by the work [11] of B. Maurey, where existence of unconditional bases for $H^1(T)$ is proved, and also by the works of L. Carleson [3] and P. Wojtaszczyk [14], where explicit unconditional bases for $H^1(T)$ are given. It should also be pointed out that our result is a continuous version of theorems of Burkholder–Gundy [2] and Davis [9] corresponding to the martingale case, i.e. to $r = 1$. Related work in H^p spaces for $0 < p \leqslant 1$ was recently done by Sjölin and Strömberg in [13].

This note is organized as follows. In Section 2, we indicate some definitions and basic facts about spline systems, where the reader is referred to [6], [7] for more details. In Section 3, we state and prove our main result. And in the last Section 4, we make some comments on our result and indicate a generalization of it to the polydisc.

2. Preliminaries. We will use the following notation. To each natural integer $n \geqslant 1$, we assign dyadic interval (n) as follows:

$$(n) = \begin{cases} (-1/2, 1/2) & \text{for} \quad n = 1, \\ ((\nu-1)/2^m - 1/2, \nu/2^m - 1/2) & \text{for} \quad n = 2^m + \nu, 1 \leqslant \nu \leqslant 2^m, m \geqslant 0. \end{cases}$$

The orthonormal in $L^2(T)$ periodic spline system of order r is denoted by $(F_n^{(r)}, n \geqslant 1)$ and it is defined as follows:

$$F_n^{(r)}(x) = \sqrt{2\pi} F_n^{(m)}(\pi x),$$

where $r = m+2$ and $F_n^{(m)}$ on $[-\pi, \pi)$ is defined as in [6]. Thus, $F_1^{(r)} = 1$ and $(1, F_n^{(r)}) = 0$ for $n > 1$. Using the technique as developed in [7] we can prove for some constants c_r, $0 < q_r < 1$, depending on r only, that

$$(2.1) \qquad |F_n^{(r)}(x)| \leqslant c_r n^{1/2} q_r^{nd(x,(n))},$$

and

$$(2.2) \qquad |F_n^{(r)}(x) - F_n^{(r)}(y)| \leqslant c_r d(x, y) n^{3/2} q_r^{n(d(x,(n)) \wedge d(y,(n)))},$$

where d is the usual distance on the torus T and $a \wedge b = \min(a, b)$.

As a simple consequence of (2.1) we find that for each $r \geqslant 1$ there is a constant $a_r > 0$ such that

$$(2.3) \qquad \int_{d(x,(n)) \leqslant a_r/n} (F_n^{(r)}(x))^2 dx \geqslant 1/2 \quad \text{for} \quad n \geqslant 1.$$

The following lemma can be proved by simple compactness argument.

LEMMA 1. *Let $r \geqslant 1$ and $0 < \delta < 1$ be given. Then there is a constant $c_{r,\delta}$ such that for any $w \in P_r$ (a polynomial of degree $\leqslant r-1$), any interval I, any measurable set $E \subset I$, $|E| \geqslant \delta |I|$, we have*

$$\int_I |w| \leqslant c_{r,\delta} \int_E |w|.$$

3. Main result. Before we state our main result let us introduce some more notation. For $f \in L^1(T)$, the square function is defined to be

$$Q^{(r)} f(x) = \Big(\sum_{n=1}^{\infty} \big((f, F_n^{(r)}) F_n^{(r)}(x)\big)^2\Big)^{1/2}$$

and the "area" function

$$A^{(r)} f(x) = \Big(\sum_{x \in (n)} (f, F_n^{(r)})^2 / |(n)|\Big)^{1/2}.$$

Notice that when $r = 1$, i.e. when the $F_n^{(r)}$'s are the Haar functions, $A^{(1)} f = Q^{(1)} f$. Moreover, for given $\varepsilon = (\varepsilon_n)$, $\varepsilon_n = \pm 1$, we define

$$f_\varepsilon = \sum_{n=1}^{\infty} \varepsilon_n (f, F_n^{(r)}) F_n^{(r)}.$$

The result can now be formulated.

THEOREM. *Let $f \in L^1(T)$ and let $r \geqslant 2$; then the following properties are equivalent:*

$$(3.1) \qquad f \in H^1(T);$$

$$(3.2) \qquad Q^{(r)} f \in L^1(T);$$

$$(3.3) \qquad \sup_\varepsilon \|f_\varepsilon\|_{L^1(T)} < \infty;$$

(3.4) *The series* $\sum_{n=1}^{\infty} (f, F_n^{(r)}) F_n^{(r)}$ *converges unconditionally in* $L^1(T)$;

$$(3.5) \qquad A^{(r)} f \in L^1(T).$$

Furthermore, the norms corresponding to (3.2), (3.3) *and* (3.5) *are equivalent norms in* $H^1(T)$.

Before we pass to the proof of the theorem, we will first establish some auxiliary results.

LEMMA 2. *If $r \geqslant 2$, then there is a finite number c_r such that*

$$\|F_n^{(r)}\|_{H^1(T)} \leqslant c_r n^{-1/2}, \quad n \geqslant 1.$$

Proof. Since the dual space to $H^1(T)$ is the BMO(T), it suffices to verify that

$$\Big|\int_T F_n^{(r)} \varphi\, dx\Big| \leqslant c_r n^{-1/2} \|\varphi\|_{\mathrm{BMO}(T)} \quad \text{for} \quad \varphi \in \mathrm{BMO}(T).$$

To see this, let us for given $\varphi \in \mathrm{BMO}(T)$ and n divide T into dyadic intervals of equal length $|(n)|$. Call this collection of intervals in its natural ordering $\{(n)_j\}$ with $(n)_0 = (n)$. Then by property (2.1) we have for $n>1$

$$\begin{aligned}\Big|\int_T F_n^{(r)} \varphi\, dx\Big| &= \Big|\int_T F_n^{(r)} (\varphi - \varphi_{(n)_0})\, dx\Big| \\ &= \Big|\sum_j \int_{(n)_j} F_n^{(r)} (\varphi - \varphi_{(n)_j})\, dx + \int_{(n)_j} F_n^{(r)} (\varphi_{(n)_j} - \varphi_{(n)_0})\, dx\Big| \\ &\leqslant \sum_j c_r n^{1/2} q_r^{|j|} n^{-1} \|\varphi\|_{\mathrm{BMO}(T)} + \sum_j c_r n^{1/2} n^{-1} q_r^{|j|} |j| \|\varphi\|_{\mathrm{BMO}(T)} \\ &\leqslant c_r n^{-1/2} \|\varphi\|_{\mathrm{BMO}(T)}.\end{aligned}$$

This is the desired estimate.

Remark. For a different proof of this lemma we refer to [14].

In what follows the H^1 and BMO spaces corresponding to the dyadic (martingale case) are denoted by H_d^1 and BMO$_d$, respectively.

LEMMA 3. *Let $f \in L^1(T)$ and let $f = \sum_{n=1}^{\infty} a_n F_n^{(r)}$. Then the condition*

$$\sum_{(m) \subset (n)} |a_m|^2 = O(|(n)|)$$

is equivalent to

(a) $f \in \mathrm{BMO}(T)$ *for* $r \geqslant 2$;

(b) $f \in \mathrm{BMO}_d$ *for* $r = 1$.

Part (a) is an extension of Carleson's [3] and Wojtaszczyk's [14] result to higher order splines, it can be proved similarly as in [3] and [14] using properties (2.1) and (2.2) of $\{F_n^{(r)}\}$ once we have Lemma 2. Part (b) is well-known.

Remark. In both cases we have (in case $r = 1$ we replace BMO by BMO$_d$)

$$\|f\|_{\mathrm{BMO}} \sim \sup_{(n)} \Big((1/|(n)|) \sum_{(m) \subset (n)} |a_m|^2 \Big)^{1/2}. \tag{3.6}$$

Using the fact that BMO is the dual of H^1 we obtain immediately

COROLLARY. *The system $(F_n^{(r)}, n \geqslant 1)$ is unconditional basis in $H^1(T)$ if $r \geqslant 2$ and in H_d^1 if $r = 1$.*

Of course the case $r = 1$ (Haar or martingale case) is well known.

About the following known abstract lemma we have learned from T. Figiel. With his permission we present it here with his proof.

LEMMA 4. *Let $\langle X, \|\cdot\|\rangle$ be a Banach space, and let (x_n) be a sequence in X such that $\sum_n x_n$ converges unconditionally. Then*

$$\sup_\varepsilon \Big\|\sum_n \varepsilon_n x_n\Big\| < \infty,$$

where the sup is taken over all $\varepsilon = (\varepsilon_n)$ with $\varepsilon_n = \pm 1$.

Proof. Define a linear operator $A\colon X^* \to l^1$ by the formula $Ax^* = (x^*(x_n))$. Since the series

$$\sum_n x^*(x_n)$$

converges unconditionally, it follows that it is absolutely convergent. The operator A has closed graph. According to the closed graph theorem, $\|A\| < \infty$. The Hahn–Banach theorem implies that for each ε there is x^* in X^* such that $\|x^*\| = 1$ and

$$\begin{aligned}\Big\|\sum_n \varepsilon_n x_n\Big\| = x^*\Big(\sum_n \varepsilon_n x_n\Big) = \sum_n \varepsilon_n x^*(x_n) \leqslant \sum_n |x^*(x_n)| \\ = \|Ax^*\|_{l^1} \leqslant \|A\| < \infty.\end{aligned}$$

Proof of the Theorem. The implications will be proved in the following order: (3.1) ⇒ (3.4) ⇒ (3.3) ⇒ (3.2) ⇒ (3.1) and (3.1) ⇔ (3.5).

(3.1) ⇒ (3.4): Apply the Corollary to Lemma 3.

(3.4) ⇒ (3.3): Apply Lemma 4.

(3.3) ⇒ (3.2): Denote by $r_n(t)$ the nth Rademacher function. Then by Khintchin's inequality $(I = (0, 1))$,

$$\begin{aligned}\sup \|f_\varepsilon\|_{L^1} &\geqslant \int_I \Big\|\sum_n r_n(t) a_n F_n^{(r)}\Big\|_{L^1(T)} dt \\ &= \int_T \Big[\int_I \Big|\sum_n r_n(t) a_n F_n^{(r)}(x)\Big| dt\Big] dx \\ &\geqslant C \|Q^{(r)} f\|_{L^1(T)}, \quad \text{where} \quad a_n = (f, F_n^{(r)}).\end{aligned}$$

(3.2) ⇒ (3.1): This proof is based on some probabilistic ideas (cf. [1] and also [4]). It is sufficient to prove with some constant $C_r < \infty$ that

$$|(f, g)| \leqslant C_r \|Q^{(r)} f\|_{L^1(T)} \|g\|_{\mathrm{BMO}(T)}, \tag{3.7}$$

for $g \in \mathrm{BMO}(T)$ and given $f \in L^1(T)$. Since $(F_n^{(r)})$ is a basis in $H^1(T)$, we have

$$f = \sum_n a_n F_n^{(r)}.$$

For a given dyadic interval (n) we denote by s_n the middle point of (n). Moreover, we will fix $\alpha > 0$, $\beta > 0$ (to be chosen later) and let

$$(n)_\alpha = \{x\colon |x - s_n| \leqslant \alpha|(n)|/2\}, \quad |(n)_\alpha| = \alpha|(n)|, \quad \Omega_k = \{Q^{(r)}f > 2^k\},$$
$$\tau_k = \{(n)\colon |(n)_\alpha \cap \Omega_{k-1}| > \beta\alpha|(n)|,\ |(n)_\alpha \cap \Omega_k| \leqslant \beta\alpha|(n)|\},$$
$$E_{k-1} = \{M(\chi_{\Omega_{k-1}}) > \beta\},$$

where M is the Hardy maximal operator. Since it is of weak type (1, 1), we have for some $C > 0$

$$|E_{k-1}| \leqslant (C/\beta)|\Omega_{k-1}|.$$

First we are going to prove that the following inequalities imply (3.7):

$$\sum_{(n)\in\tau_k} a_n^2 \leqslant C \int_{E_{k-1}\setminus\Omega_k} (Q^{(r)}f(x))^2\,dx, \tag{3.8}$$

where C is a constant depending only on our choice of α, β; and

$$\sum_{(n)\in\tau_k} b_n^2 \leqslant C|E_{k-1}|\,\|g\|^2_{\mathrm{BMO}(T)}, \tag{3.9}$$

where

$$f = \sum_{n=1}^{\infty} a_n F_n^{(r)}, \qquad g = \sum_{n=1}^{\infty} b_n F_n^{(r)}.$$

The proof of (3.7) goes as follows. Using (3.8) and (3.9) we obtain

$$\begin{aligned}\int_T fg &= \sum_{n=1}^{\infty} a_n b_n = \sum_k \sum_{(n)\in\tau_k} a_n b_n \\ &\leqslant C \sum_k \Big(\int_{E_{k-1}\setminus\Omega_k} (Q^{(r)}f)^2\Big)^{1/2} |E_{k-1}|^{1/2} \|g\|_{\mathrm{BMO}(T)} \\ &\leqslant C\|g\|_{\mathrm{BMO}(T)} \sum_k 2^k |E_{k-1}| \leqslant C\|g\|_{\mathrm{BMO}(T)} \sum_k 2^k |\Omega_{k-1}| \\ &\leqslant C\|Q^{(r)}f\|_{L^1(T)} \|g\|_{\mathrm{BMO}(T)}.\end{aligned}$$

We will now verify (3.8). First we notice

$$\int_{E_{k-1}\setminus\Omega_k} (Q^{(r)}f)^2(x)\,dx \geqslant \sum_{(n)\in\tau_k} a_n^2 \int_{E_{k-1}\setminus\Omega_k} (F_n^{(r)}(x))^2\,dx.$$

Now for $(n) \in \tau_k$ we have $(n)_\alpha \subset E_{k-1}$ by the way we define E_k's. Thus it suffices to prove

$$\int_{(n)_\alpha\setminus\Omega_k} (F_n^{(r)}(x))^2\,dx \geqslant C_r \quad \text{for some constant } C_r. \tag{3.10}$$

To see this, let $e = (n)_\alpha \setminus \Omega_k$. Suppose we choose $\alpha = 2p+1$, p is a natural number chosen later, then

$$(n)_\alpha = \bigcup_{|j|\leqslant p} \{(n)+j\}.$$

If we then choose $\beta < 1/4\alpha$, then

$$|(n)_\alpha - e| = |(n)_\alpha \cap \Omega_k| \leqslant (1/4)|(n)|,$$

hence $|e \cap \{(n)+j\}| \geqslant (3/4)|(n)|$ for each $|j| \leqslant p$. Thus the portion of e in each dyadic interval in which $F_n^{(r)}$ is a polynomial is at least $1/2$. Applying now Lemma 1, we obtain for each $|j| \leqslant p$

$$\int_{e\cap\{(n)+j\}} (F_n^{(r)}(x))^2\,dx \geqslant C_r \int_{\{(n)+j\}} (F_n^{(r)}(x))^2\,dx,$$

and therefore

$$\int_{(n)_\alpha\setminus\Omega_k} (F_n^{(r)}(x))^2\,dx = \int_e (F_n^{(r)}(x))^2\,dx \geqslant C_r \int_{(n)_\alpha} (F_n^{(r)}(x))^2\,dx. \tag{3.11}$$

It is now apparent that if we choose $\alpha = 2p+1$ so that

$$(n)_\alpha \supset \{d(x, (n)) \leqslant \alpha_r/n\},$$

where α_r is the constant appearing in inequality (2.3), then an application of (2.3) to the right hand side of (3.11) gives (3.10). From which the desired inequality (3.8) follows.

The proof of (3.9) is easier. Notice that $(n) \in \tau_k$ implies that $(n)_\alpha \subset E_{k-1}$ and this in turn implies $(n) \subset E_{k-1}$. Thus by Lemma 3

$$\sum_{(n)\in\tau_k} b_n^2 \leqslant C_r \|g\|^2_{\mathrm{BMO}(T)} |E_{k-1}|,$$

which completes the proof of (3.2) $\Rightarrow$ (3.1).

(3.1) $\Leftrightarrow$ (3.5): First we notice that if $a_n = (f, F_n^{(r)})$, then

$$(A^{(r)}f(x))^2 = \sum_{n=1}^{\infty} a_n^2 (1/|(n)|)\chi_{(n)}(x) = \sum_{n=1}^{\infty} a_n^2 (F_n^{(1)}(x))^2,$$

where the $F_n^{(1)}$'s are the Haar functions. If we name $g(x) = \sum_{n=1}^{\infty} a_n F_n^{(1)}(x)$;

then $\|A^{(r)}f\|_{L^1(T)} = \|Q^{(1)}g\|_{L^1(T)}$ while $\|Q^{(1)}g\|_{L^1(T)} \sim \|g\|_{H^1_d}$ by result of Davis [9]. However, by Lemma 3, we have $\|g\|_{H^1_d} \sim \|f\|_{H^1(T)}$, thus $\|A^{(r)}f\|_{L^1(T)} \sim \|f\|_{H^1(T)}$ as desired.

Remark. The implications (3.1) ⇒ (3.2) and (3.1) ⇒ (3.3) in the non-periodic case are treated in [12].

4. Some comments.

1. There is some difference between the real Banach space $H^1(T)$ used in this note and the space H^1 (which we will denote by $H^1(I)$) used in [12] and [14]. $H^1(I)$ is better defined if we use the atomic characterization of H^1 (cf. [8]) with atoms supported in subintervals of I. To describe the relation between $H^1(I)$ and $H^1(T)$, for each $u \in H^1(I)$ let

$$Pu(x) = \begin{cases} u(2x) & \text{for } 0 \leqslant x < 1/2, \\ u(-2x) & \text{for } -1/2 \leqslant x \leqslant 0. \end{cases}$$

Then

$$Pu(x+1) = Pu(x) \quad \text{and} \quad (Pu, Pv)_T = (u, v)_I,$$

where the last inner product is equal to $\int_I uv$.

PROPOSITION 1. *Let $H^1_+(T)$ be the subspace of even functions in $H^1(T)$. Then, $P\colon H^1(I) \to H^1_+(T)$ is a linear isomorphism and*

$$\|Pu\|_{H^1(T)} \leqslant \|u\|_{H^1(I)} \leqslant 2\|Pu\|_{H^1(T)}.$$

The proof is straightforward if we use the atomic definitions.

Suppose now that we have a basis (unconditional basis) $(\varphi_n, n \geqslant 1)$ in $H^1(I)$. Then the recipe for obtaining basis (unconditional basis) in $H^1(T)$ is the following: take the sequence $(P_{\varphi_n}, \tilde{P}_{\varphi_n}, n \geqslant 1)$. Thus, Wojtaszczyk's result [14] simply says that if $(\varphi_n, n \geqslant 1)$ is the non-periodic Franklin system, then $(P_{\varphi_n}, \tilde{P}_{\varphi_n}, n \geqslant 1)$ is an unconditional basis in $H^1(T)$.

2. Suppose now that we have basis (unconditional basis) $(\psi_n, n \geqslant 1)$ in $H^1(T)$; then $(\psi_n + i\tilde{\psi}_n, n \geqslant 1)$ is a basis (unconditional basis) in the real Banach space $H^1_0(\Delta)$. However, $H^1(\Delta) = \{i\} + H^1_0(\Delta)$ and therefore $(i, \psi_n + i\tilde{\psi}_n, n \geqslant 1)$ is a basis (unconditional basis) in the real Banach space $H^1(\Delta)$.

3. Let us now define for given $r \geqslant 2$

$$G_0(x) = i,$$

$$G_1(x) = 1,$$

$$G_n(x) = \big(F_n^{(r)}(x) + i\tilde{F}_n^{(r)}(x)\big)/\sqrt{2}.$$

After the remark 2 made above we get from our Theorem:

PROPOSITION 2. *The system $(G_n, n \geqslant 0)$ is an unconditional basis in the real Banach space $H^1(\Delta)$ and for $f \in H^1(\Delta)$ we have*

$$f = \sum_{n=0}^{\infty} \mathrm{Re}(f, G_n)_\Delta G_n$$

with $(f, g)_\Delta = \int_T f\bar{g}$.

4. Now, for each $E \subset \{0, 1, \ldots\}$ define a projection

$$P_E f = \sum_{n\in E} \mathrm{Re}(f, G_n)_\Delta G_n.$$

$(P_E, E \subset \{0, 1, \ldots\})$ is a uniformly bounded commuting Boolean algebra of projections acting in the subspace $H^1(\Delta)$ of $L^1(T; C)$. Now, the result of McCarthy [5] on product of bounded commuting Boolean algebras of projections in L^p spaces (it can be extended to subspaces of L^p spaces) implies

THEOREM 1. *The system $(G_{n_1} \otimes \ldots \otimes G_{n_d}, n_1 \geqslant 0, \ldots, n_d \geqslant 0)$ is an unconditional basis in the real $H^1(\Delta^d)$ for any $d \geqslant 1$.*

5. To construct an unconditional orthonormal basis in the complex space $H^1(\Delta^d)$ it is sufficient to produce a spline orthonormal basis in $H^1_+(\mathrm{T})$, e.g. as in [15].

References

[1] A. Bernard, *Espaces H^1 de martingales à deux indices. Dualité avec des martingales de type "BMO"*, Bull. Soc. Math. France 1979.

[2] D. L. Burkholder and R. F. Gundy, *Extrapolation and interpolation of quasi-linear operators on martingales*, Acta Math. 124 (1970), 249–304.

[3] L. Carleson, *An explicit unconditional basis in H^1*, Inst. Mittag-Leffler, Report No. 2, 1980.

[4] S.-Y. A. Chang and R. Fefferman, *A continuous version of the duality of H^1 with BMO on the polydisc*, Ann. of Math. 112 (1980), 179–201.

[5] C. A. McCarthy, *Commuting Boolean algebras of projections II. Boundedness in L_p*, Proc. Amer. Math. Soc. 15 (1964), 781–787.

[6] Z. Ciesielski, *Bases and approximation by splines*, in: *Proc. of the International Congress of Math.*, Vancouver 1974, 47–51.

[7] — *Constructive function theory and spline systems*, Studia Math. 53 (1975), 277–302.

[8] R. Coifman, *A real variable characterization of H^p*, ibid. 51 (1974), 269–274.

[9] B. Davis, *On the integrability of martingale square function*, Israel J. Math. 82 (1970), 187–190.

[10] C. Fefferman, *Characterizations of bounded mean oscillations*, Bull. Amer. Math. Soc. 77 (1971), 587–588.

[11] B. Maurey, *Isomorphismes entre espaces H_1*, Acta Math. 145 (1980), 79–120.

[12] F. Schipp and P. Simon, *Investigation of Haar and Franklin series in Hardy spaces*, Analysis Math. 8 (1982), 47–56.

[13] P. Sjölin and J. O. Strömberg, *Basis properties of Hardy spaces*, Preprint No. 19. Stockholm University, 1981.
[14] P. Wojtaszczyk, *The Franklin system is an unconditional basis in H_1*, Ark. Mat. 20 (1982), 293–300.
[15] S. Wronicz, *Construction of an orthonormal basis in the space of functions analytic in a disc and of class C^n on its closure.*

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
LOS ANGELES, CALIFORNIA 90024
and
MATHEMATICAL INSTITUTE
POLISH ACADEMY OF SCIENCES
SOPOT, POLAND

Received October 12, 1981 (1715)

STUDIA MATHEMATICA, T. LXXV. (1983)

H^∞ is a Grothendieck space

by

J. BOURGAIN (Brussels)

Abstract. It is shown that a non-weakly compact operator on H^∞ fixes an l^∞-copy. In particular, H^∞ has the Grothendieck property and l^∞ embeds in any infinite-dimensional complemented subspace of H^∞.

I. Introduction. This work is a continuation of [3] (cf. also [4]). Let us recall some definitions. Π denotes the circle and m its Haar measure. H_0^1 is the space of integrable functions f on Π such that $\hat{f}(n) = 0$ for $n \leqslant 0$. We use the notations $q\colon L^1 \to L^1/H_0^1$ and $\sigma\colon L^1/H_0^1 \to L^1$ for the quotient map and the minimum norm lifting, respectively. The duality

$$\langle f, \varphi\rangle = \int f\varphi\, dm$$

identifies the dual $(L^1/H_0^1)^*$ with the space H^∞ of bounded analytic functions on the unit disc D.

It was shown in [3] that H^∞ has the Dunford–Pettis property (DPP) and $(H^\infty)^*$ is weakly sequentially complete (WSC). We establish here the Grothendieck property (GP) of H^∞. Recall that a Banach space X has GP provided weak*-null sequences in X^* are weakly-null, or, equivalently, each operator $T\colon X \to c_0$ is weakly compact. In fact, a stronger result is obtained. If $T\colon H^\infty \to Y$ is an operator, then T is either weakly compact or there exists a subspace Z of H^∞, Z isomorphic to l^∞, on which T induces an isomorphism.

As corollary it follows that l^∞ embeds in any infinite dimensional complemented subspace of H^∞, solving one of the questions raised in [18].

Latter results where previously announced in [5].

II. Operators on H^∞ and the Grothendieck property. Classical examples of G-spaces are the $L^\infty(\mu)$-spaces. Next result, implying the G-property, emphasizes the same behaviour of H^∞ and L^∞ in several aspects.

THEOREM 1. *Assume $T\colon H^\infty \to Y$ is an operator. If T is not weakly compact, then T is an isomorphism when restricted to a subspace Z of H^∞, Z isomorphic to l^∞.*