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H^p estimates for weakly strongly singular integral operators on spaces of homogeneous type

by

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Abstract. Let X be a normalized homogeneous space. We define "weakly strongly" singular kernel on $X\times X$, and we study the action of the "convolution" operator induced by this kernel on the atomic Hardy spaces $H^p(X)$, with $0<p\leqslant 1$. A boundedness result is obtained. These operators are analogues of the weakly strongly operators on $\boldsymbol{R}^n$ studied by C. L. Fefferman and E. M. Stein in [6].

1. Introduction. In this paper we study a generalization of convolution operators induced by weakly strongly singular integral kernels. Examples of these kernels, in the case of R^n are given by

$$k(x) = |x|^{-\beta}\psi(x)\exp i|x|^{a},$$

where $0<a<1$, $\beta>0$ and ψ is a C^∞ function on $\boldsymbol{R}^n$, which vanishes near zero and equals 1 outside a bounded set (see [5], page 21). The L^p theory, $1<p<+\infty$, for operators obtained by convolution with kernels $k(x)$, has been studied by I. I. Hirschmann [7], S. Wainger [12], C. L. Fefferman [5], C. L. Fefferman and E. M. Stein [6], J. E. Björk [1] and P. Sjölin [11].

Also in [6], C. L. Fefferman and E. M. Stein obtain boundedness results for $H^p(\boldsymbol{R}^n)$, $1\geqslant p>p_0(a,\beta,n)>1/2$. Estimates including the limiting case $p=p_0(a,\beta,n)$ were obtained by R. R. Coifman in [2] when $n=1$.

Here we consider a generalization of these kernels and the action of the induced operators on H^p spaces, $p\leqslant 1$, defined in terms of atoms on spaces of homogeneous type. First we define what we mean by a weakly strongly singular kernel on spaces of homogeneous type. In Theorem 3 we prove that the operator K induced by this kernel maps atoms into elements of H^p, $p\leqslant 1$. In the proof of this theorem we extend some techniques used by R. A. Macías and C. Segovia in [9]. The extension of the operator to the whole space H^p requires the introduction of an auxiliary operator, namely $K^{\#}$, acting on the space $\mathrm{Lip}(1/p-1)$ of classes of Lipschitz functions. This operator is an adaptation of the operator $K^{\#}$ considered in [9].

In Theorem 5 we show that $K^{\#}$ is a bounded operator from $\mathrm{Lip}(1/p-1)$ into $\mathrm{Lip}(1/p-1)$. This result is used in Theorem 6 in order to prove the H^p boundedness of the weakly strongly singular integral operator on X.

2. Preliminary definitions and notations. Let X be a set and $d(x,y)$ a function defined on $X\times X$ such that:

(i) $d(x,y)\geqslant 0$ and $d(x,y)=0$ if and only if $x=y$;

(ii) $d(x,y)=d(y,x)$; and

(iii) there exist a finite constant A such that $d(x,y)\leqslant A\big(d(x,z)+ +d(z,y)\big)$.

We shall suppose that there is a measure μ such that the balls $B(x,r) =\{y:\ d(x,y)<r\}$ are measurable. Moreover we shall assume that there exist two positive and finite constants b_1 and b_2 such that

$$b_1 r\leqslant \mu\big(B(x,r)\big)\leqslant b_2 r. \tag{2.1}$$

The function $d(x,y)$ satisfying (i), (ii) and (iii) shall be called a *quasi-distance* and the triple (X,d,μ) satisfying the above requirements shall be called a *normalized homogeneous space* (see [3] and [10]) and shall be denoted by X.

Let $\varphi(x)$ a be real or complex valued function on X; square integrable on bounded subsets of X. Let $m_B(\varphi)$ be the mean value of $\varphi(x)$ on a ball B, that is to say,

$$m_B(\varphi)=\mu(B)^{-1}\int_B \varphi(x)\,d\mu(x).$$

We shall say that φ *belongs to* $\mathrm{Lip}(\alpha)$, $0\leqslant\alpha\leqslant 1$, if there exists a positive and finite constant c such that, for every ball B on X,

$$\Big(\mu(B)^{-1}\int_B |\varphi(x)-m_B(\varphi)|^2\,d\mu(x)\Big)^{1/2}\leqslant c\cdot\mu(B)^{\alpha} \tag{2.2}$$

holds. The least constant c such that (2.2) holds shall be denoted by $\|\varphi\|_\alpha^*$. We shall denote by $\bar\varphi$ the class of functions which differ from φ in a constant. The space of the class $\bar\varphi$ shall be denoted by $\mathrm{Lip}(\alpha)$. The norm of $\bar\varphi$ shall be denoted by $|\|\bar\varphi\||_\alpha^*$.

Let $0<p\leqslant 1<q\leqslant\infty$. A *$(p,q)$-atom* on X is a function $a(x)$ with support contained in a ball B satisfying:

$$\Big(\mu(B)^{-1}\int_B |a(x)|^q\,d\mu(x)\Big)^{1/q}\leqslant\mu(B)^{-1/p}\quad\text{if}\quad q<+\infty. \tag{2.3}$$

$$\|a\|_\infty\leqslant\mu(B)^{-1/p}\quad\text{if}\quad q=\infty \tag{2.4}$$

and

$$\int_X a(x)\,d\mu(x)=0 \tag{2.5}$$

(see [2] and [8]). A (p,q)-atom can be identified with a linear functional on $\mathrm{Lip}(1/p-1)$, by

$$\langle F_a,\bar\varphi\rangle=\int_X a(x)\varphi(x)\,d\mu(x)$$

and we have $\|F_a\|\leqslant 1$.

We define H^p as the subspace of all linear functionals on $\mathrm{Lip}(1/p-1)$ that can be written as $\sum_{i=1}^{\infty}\alpha_i a_i$, where $\{a_i\}$ is a sequence of $(p,2)$-atoms and $\{\alpha_i\}$ is a sequence of real numbers such that $\sum_{i=1}^{\infty}|\alpha_i|^p<+\infty$. The "norm" of $f\in H^p$ shall be defined as

$$\|f\|_{H^p}=\inf\Big\{\Big(\sum_{i=1}^{\infty}|\alpha_i|^p\Big)^{1/p}:\ f=\sum_{i=1}^{\infty}\alpha_i a_i\Big\}.$$

This is not a norm in the ordinary sense, unless $p=1$. However, H^p with the metric defined by $\|\cdot\|_{H^p}^p$ becomes a complete metrizable topological vector space.

3. Results.

DEFINITION 1. Let $k(x,y)$ be a measurable function defined on $X\times X$. We shall say that $k(x,y)$ is a *weakly strongly singular kernel* if there exist constants $\theta,\gamma,\varepsilon$ satisfying $0<\theta<1$, $\theta/2<\gamma<1/2$, $1-\theta\leqslant\varepsilon\leqslant 1$ and a bounded function $\varphi_\delta(x,y)$, $\delta>0$, which vanishes when $d(x,y)<\delta/2$ and is equal to 1 when $d(x,y)>\delta$, such that if we define q as $1/q=1/2+\gamma$, then the following conditions hold:

(3.1) For any ball B and $\nu>0$ if

$$D=\{(x,y):\ d(x,y)>\nu\}\cap(B\times B),$$

then

$$\iint_D |k(x,y)|^{q'}\,d\mu(x)\,d\mu(y)<+\infty,$$

where $1/q+1/q'=1$.

(3.2) Let $k_\delta(x,y)=\varphi_\delta(x,y)\,k(x,y)$. For any function f in $L^q(X)$ with bounded support, the operator

$$K_\delta f(x)=\int_X k_\delta(x,y)f(y)\,d\mu(y)$$

satisfies

$$\|K_\delta f\|_2 \leqslant C_1 \|f\|_q,$$

where C_1 is a finite constant independent of $f(y)$ and δ.

(3.3) For any function $f(y)$ in $L^2(X)$ with bounded support we have

$$\|K_\delta f\|_2 \leqslant C_2 \|f\|_2,$$

where C_2 is a finite constant independent of $f(y)$ and δ.

(3.4) For any function $f(y)$ in $L^j(X)$, $j = q$ or $j = 2$, with bounded support, $\lim\limits_{\delta\to 0} K_\delta f = Kf$ exist in $L^2(X)$.

(3.5) $k(x, y)$ vanishes if $d(x, y) > 1$. If $d(y, x_0) \leqslant 1$ and $d(x, x_0) > 2d(y, x_0)^{1-\theta}$, then there exists a finite constant C_3 such that

$$|k(x, y) - k(x, x_0)| \leqslant C_3 d(y, x_0)^\varepsilon d(x, x_0)^{-1-\varepsilon/(1-\theta)}$$

holds.

(3.6) Let χ_R be the characteristic function of the ball $B(x_0, R)$, $R > 0$, where x_0 is an arbitrary point of X, and let K^* be the adjoint of the operator K in $L^2(X)$. Then the limit of $K^*(\chi_R)$ for R tending to infinity exist weakly in $L^{q'}$ on bounded sets and it is equal to a finite constant.

LEMMA 1. *Let ε, θ, q and γ be the constants from Definition* 1. *If we define*

$$\varrho = (\varepsilon+1-1/q)/\big(\varepsilon/(1-\theta)+1/2\big),$$

then the following inequalities hold:

$$\varrho < 1-\theta; \tag{3.7}$$

$$(2/q-1)/(1-\varrho) < \big(1+2\varepsilon/(1-\theta)\big); \tag{3.8}$$

and there exists p such that

$$0 < p \leqslant 1 \quad \textit{and} \quad 1/p < (1/q-\varrho/2)/(1-\varrho). \tag{3.9}$$

Moreover, if p satisfies (3.9), *we have*

$$1/p < 1+\varepsilon/(1-\theta) \tag{3.10}$$

and the interval

$$\big(2/p-1,\ (2/q-1)/(1-\varrho)\big) \tag{3.11}$$

is not empty.

Proof. To prove (3.7) we observe that $\gamma > \theta/2$ and therefore we have $1/q > 1/2+\theta/2$. Thus $1/q+\varepsilon+1/2 > \varepsilon+1+\theta/2$, which implies $\varrho < 1-\theta$.

The proofs of (3.8), (3.10) and (3.11) are simple and will be omitted. To prove (3.9) it is sufficient to show that $(1/q-\varrho/2)/(1-\varrho) > 1$, i.e., $\gamma/(1-\varrho) > 1/2$. From the expressions of ϱ and q we have

$$1-\varrho = \big(2\varepsilon\theta/(1-\theta)+\gamma\big)/\big(\varepsilon/(1-\theta)+1/2\big)$$

and since $\theta < 2\gamma$, (3.9) is proved.

THEOREM 1. *Let p satisfy* (3.9) *and let $k(x, y)$ be a weakly strongly singular integral kernel. Let $a(x)$ be a $(p, 2)$-atom and let $B = B(x_0, \sigma)$ be the ball containing the support of $a(x)$ such that*

$$\Big((\mu(B)^{-1} \int_B |a(x)|^2 d\mu(x)\Big)^{1/2} \leqslant \mu(B)^{-1/p},$$

given in the definition of $(p, 2)$-atom. Then we have:

(i) *If $\sigma \leqslant 1$ and s satisfies $2/p-1 < s < (2/q-1)/(1-\varrho)$, the function $M(x) = Ka(x)$ satisfies:*

$$\int_X |M(x)|^2 d\mu(x) \leqslant c\cdot\sigma^{(2/q-2/p)}, \tag{3.12}$$

$$\int_X |M(x)|^2 d(x, x_0)^s d\mu(x) \leqslant c\cdot\sigma^{(\varrho s+2/q-2/p)} \tag{3.13}$$

and

$$\int_X M(x)\, d\mu(x) = 0 \tag{3.14}$$

where c in (3.12) *and* (3.13) *is a finite constant independent of σ.*

(ii) *If $\sigma > 1$, the function $M(x) = Ka(x)$ satisfies* (3.14). *Moreover we have:*

$$\int_X |M(x)|^2 d\mu(x) \leqslant c\cdot\sigma^{(1-2/p)} \tag{3.15}$$

and

$$\int_X |M(x)|^2 d(x, x_0)^{2/p-1+\nu} d\mu(x) \leqslant c\cdot\sigma^\nu; \tag{3.16}$$

that is $M(x)$ is a (p, ν) molecule (see [4] and [9]).

Proof. Let us prove (i). First, we observe that since $1 < q < 2$, $a(x)$ is also a (p, q)-atom. By conditions (3.2) and (3.4) we get

$$\int_X |M(x)|^2 d\mu(x) \leqslant c\cdot\Big[\int_B |a(x)|^q d\mu(x)\Big]^{2/q} \leqslant c\cdot\sigma^{2/q-2/p}.$$

This ends the proof of (3.12). Let us show (3.13). Let $B_n = B(x_0, 2A2^n\sigma^\varrho)$.

We write

$$\int_X |M(x)|^2 d(x, x_0)^s d\mu(x) = \int_{B_0} |M(x)|^2 d(x, x_0)^s d\mu(x) + \int_{X\sim B_0} |M(x)|^2 d(x, x_0)^s d\mu(x) = I_1 + I_2.$$

Since $s > 2/p-1$ and $0 < p \leqslant 1$, we get $s > 0$ and hence from (3.12) it follows that $I_1 \leqslant c\cdot\sigma^{\varrho s-2/q-2/p}$.

Let us estimate I_2. We have

$$I_2 = \sum_{n=0}^{\infty} \int_{B_{n+1}\sim B_n} |M(x)|^2 d(x, x_0)^s d\mu(x) \leqslant \sum_{n=0}^{\infty} (2A2^{n+1}\sigma^{\varrho})^s \int_{B_{n+1}\sim B_n} |M(x)|^2 d\mu(x).$$

By the definition of $M(x)$ and (2.5) we get that

$$M(x) = \int_X \big(k(x, y) - k(x, x_0)\big) a(y)\, d\mu(y)$$

and therefore from (3.5) it results that

$$\int_{B_{n+1}\sim B_n} |M(x)|^2 d\mu(x) \leqslant c\cdot\sigma^{2s+2-2/p} (2^n\sigma^{\varrho})^{-1-2s/(1-\varrho)}.$$

Then, we have $I_2 \leqslant c\cdot\sigma^{\varrho s+2/q-2/p}$. Since this estimate for I_2 is the same that we got for I_1, (3.13) is proved.

Let us show (3.14). By conditions (3.12) and (3.13) the function $M(x)$ is absolutely integrable on X. Thus, if χ_R is the characteristic function of the ball $B(x_0, R)$, we get

$$\int_X Ka(x)\, d\mu(x) = \lim_{R\to\infty} \int_{B(x_0,R)} Ka(x)\, d\mu(x) = \lim_{R\to\infty} \int_X a(x) K^*(\chi_R)(x)\, d\mu(x).$$

Therefore, since $a(x)$ is supported on a bounded set, from (3.6) we get

$$\int_X M(x)\, d\mu(x) = c\cdot \int_X a(x)\, d\mu(x) = 0.$$

Let us prove (ii). It follows from (3.3), (3.4) and definition of $a(x)$,

that (3.15) holds. Let us show (3.16). We have

$$\int_X |M(x)|^2 d(x, x_0)^{2/p-1+\nu}\, d\mu(x) = \int_{B(x_0, 2A\sigma)} |M(x)|^2 d(x, x_0)^{2/p-1+\nu} d\mu(x) + \int_{X\sim B(x_0,2A\sigma)} |M(x)|^2 d(x, x_0)^{2/p-1+\nu} d\mu(x).$$

The last integral on the right hand side is zero since $k(x, y)$ vanishes when $d(x, x_0) > 2A\sigma$ and the first integral is bounded by $c\cdot\sigma^{\nu}$. This prove (3.16). (3.14) is proved in the same way of part (i).

Next, we shall prove a decomposition theorem for $Ka(x)$. We shall need the following lemma (see [9]):

LEMMA 2. *Let $\varphi \in \mathrm{Lip}(1/p-1)$ with $0 < p \leqslant 1$ and $\sigma_j = b^j\sigma$ be such that $\sigma > 0$, $b > 1$ and j a non-negative integer. If we denote by m_j the mean value $m_j = m_{B(x_0,\sigma_j)}(\varphi)$, then the following estimates hold:*

$$|m_j| \leqslant c\cdot\|\varphi\|^*_{1/p-1}(\sigma_j)^{1/p-1} + |m_0| \quad \text{if} \quad p < 1; \tag{3.17}$$

and

$$|m_j| \leqslant c\cdot\|\varphi\|^*_0 j + |m_0| \quad \text{if} \quad p = 1. \tag{3.18}$$

THEOREM 2. *Let p satisfy (3.9). Let $M(x)$ be a measurable function satisfying either the set of conditions (3.12), (3.13) and (3.14) with $\sigma \leqslant 1$ or the set of conditions (3.14), (3.15) and (3.16) with $\sigma > 1$. Then, for every $\varphi \in \mathrm{Lip}(1/p-1)$, the function $M(x)\varphi(x)$ is absolutely integrable on X. Moreover, the induced linear functional*

$$F_M(\bar\varphi) = \int_X M(x)\varphi(x)\, d\mu(x)$$

is well defined and bounded on $\mathrm{Lip}(1/p-1)$.

Proof. Let $B_n = B(x_0, b^n\sigma)$, where $b > 1$, $\sigma \leqslant 1$ and n a non-negative integer. We put $B_{-1} = \emptyset$. Let $\varphi \in \mathrm{Lip}(1/p-1)$ be such that $m_{B(x_0,\sigma)}(\varphi) = 0$. By Minkowski and Schwarz inequality and taking into account the definition of the Lipschitz norm of φ we get

$$\int_X |M(x)|\ |\varphi(x)|\, d\mu(x) \leqslant \sum_{n=0}^{\infty} \Big(\int_{B_n\sim B_{n-1}} |M(x)|^2 d\mu(x)\Big)^{1/2} \big(c\|\varphi\|^*_{1/p-1}\cdot\mu(B_n)^{1/p-1} + |m_n|\big)\mu(B_n)^{1/2}.$$

By an application of (3.13) when $n \geqslant 1$ and (3.12) when $n = 0$ we get for all $n \geqslant 0$

$$\Big(\int_{B_n\sim B_{n-1}} |M(x)|^2 d\mu(x)\Big)^{1/2} \leqslant c\cdot b^{-ns/2}\sigma^{1/q-1/p-(s/2)(1-\varrho)}. \tag{3.19}$$

Then replacing this estimate in the above inequality, by Lemma 2 and Lemma 1 we obtain

$$\int_X |M(x)|\ |\varphi(x)|\,d\mu(x) \leqslant c\|\varphi\|^*_{1/p-1}.$$

To obtain the last inequality for any function ψ in $\mathrm{Lip}(1/p-1)$ we observe that $m_{B(x_0,\sigma)}(\psi - m_{B(x_0,\sigma)}(\psi)) = 0$. Therefore since $M(x)$ is absolutely integrable on X, from $M(x)\psi(x) = M(x)(\psi(x)-m_{B(x_0,\sigma)}(\psi)) + M(x)m_{B(x_0,\sigma)}(\psi)$ it follows that $M(x)\psi(x)$ is an absolutely integrable function. Moreover, from condition (3.4) we have

$$\int_X M(x)\psi(x)\,d\mu(x) = \int_X M(x)\big(\psi(x)-m_{B(x_0,\sigma)}(\psi)\big)\,d\mu(x).$$

Hence

$$\Big|\int_X M(x)\psi(x)\,d\mu(x)\Big| \leqslant \int |M(x)|\,|\psi(x)-m_{B(x_0,\sigma)}(\psi)|\,d\mu(x)$$

$$\leqslant c\cdot\|\psi - m_{B(x_0,\sigma)}(\psi)\|^*_{1/p-1} = c\|\psi\|^*_{1/p-1}.$$

To finish the proof of the theorem we observe that the result for a measurable function $M(x)$ satisfying (3.14), (3.15) and (3.16) with $\sigma > 0$ was obtained in [9]. Its proof uses techniques similar to these used for the case $\sigma \leqslant 1$, and will not be repeated here. Next, we will show that F_M belongs to H^p and that there exists a finite constant C independent of $M(x)$ such that $\|F_M\| \leqslant C$.

THEOREM 3. *Let $M(x)$ satisfy the conditions of Theorem 2. Then we have:*

(i) *There exists a sequence $\{a_n\}$ of $(p, 2)$-atoms and a sequence $\{\lambda_n\}$ of real numbers satisfying $\sum_{n=1}^{\infty} |\lambda_n|^p \leqslant C$, such that $M(x) = \sum_{n=1}^{\infty} \lambda_n a_n(x)$, where C is a positive and finite constant independent of $M(x)$.*

(ii) *For every $\bar\varphi \in \mathrm{Lip}(1/p-1)$ the linear functional F_M induced by $M(x)$ satisfies*

$$F_M(\bar\varphi) = \sum_{n=1}^{\infty} \lambda_n\langle a_n, \bar\varphi\rangle.$$

Proof. We shall prove the theorem when $M(x)$ satisfies (3.12), (3.13) and (3.14) with $\sigma \leqslant 1$. The proof for $M(x)$ satisfying (3.14), (3.15) and (3.16) with $\sigma > 1$ can be found in [9].

Let b_1 and b_2 be the two constants given in (2.1) and let $b > b_2/b_1$. Let $B_{-1} = \emptyset$ and $B_n = B(x_0, b^n\sigma)$ if n is a non-negative integer. We denote by D_n the set $B_n \sim B_{n-1}$ and by M_n the mean value of $M(x)$ on D_n. Then

we can write

$$M(x) = \sum_{n=0}^{\infty} (M(x)-M_n)\chi_{D_n}(x) + \sum_{n=0}^{\infty} M_n\chi_{D_n}(x). \tag{3.20}$$

Denoting $\alpha_n(x) = (M(x)-M_n)\chi_{D_n}(x)$ and $\gamma_n = \|\alpha\|_2\mu(B_n)^{1/p-1/2}$, we can see that the functions $a_n^*(x)$ defined by $a_n^*(x) = \gamma_n^{-1}\alpha_n(x)$ are $(p, 2)$-atoms with support contained in the balls B_n. By Minkowski's inequality and (3.19) we get that for all $n \geqslant 0$,

$$\|\alpha_n\|_2 \leqslant c\cdot b^{-ns/2}\sigma^{1/q-1/p-(s/2)(1-\varrho)}.$$

Then we have $\sum_{n=0}^{\infty} |\gamma_n|^p \leqslant C$, which shows part (i) of the theorem for $\sum_{n=0}^{\infty}(M(x)-M_n)\chi_{D_n}(x)$. In order to show that $\sum_{n=0}^{\infty} M_n\chi_{D_n}(x)$ also satisfies (i), let the sequence $\{t_n\}$ be defined by

$$t_n = \int_{X\sim B_{n-1}} M(x)\,d\mu(x).$$

We can see that

$$t_n - t_{n+1} = \mu(D_n)M_n \tag{3.21}$$

and since $t_0 = 0$, we can write

$$\sum_{n=0}^{\infty} M_n\chi_{D_n}(x) = \sum_{n=1}^{\infty} t_n\big(\mu(D_n)^{-1}\chi_{D_n}(x) - \mu(D_{n-1})^{-1}\chi_{D_{n-1}}(x)\big). \tag{3.22}$$

Denoting

$$\beta_n(x) = t_n\big(\mu(D_n)^{-1}\chi_{D_n}(x) - \mu(D_{n-1})^{-1}\chi_{D_{n-1}}(x)\big)$$

and

$$\tilde\gamma_n = \|\beta_n\|_2\mu(B_n)^{1/p-1/2}$$

we can see that the functions $a_n^{**}(x)$ defined by $a_n^{**}(x) = \tilde\gamma_n^{-1}\beta_n(x)$ are $(p, 2)$-atoms with support contained in the balls B_n. To finish the proof of part (i) we only need to show that there exists constant C independent of $M(x)$ such that $\sum_{n=0}^{\infty} |\gamma_n|^p \leqslant C$. By Minkowski's inequality it follows that

$$\|\|\beta_n\|\|_2 \leqslant c|t_n|\,\mu(B_n)^{-1/2}.$$

Then, from (3.12) we get

$$\|\beta_n\|_2 \leqslant c\cdot b^{-ns/2}\sigma^{1/q-1/p-(s/2)(1-\varrho)}.$$

Therefore, as above we obtain $\sum_{n=1}^{\infty} |\tilde{\gamma}_n|^p \leqslant c$.

Let us prove (ii). First, we observe that the series $\sum_{n=0}^{\infty} \lambda_n F_{a_n}(\bar{\varphi})$ is finite for every $\bar{\varphi} \in \mathrm{Lip}(1/p-1)$. From (3.20) and (3.22) we have

$$\sum_{n=0}^{r} a_n(x) + \sum_{n=0}^{r} \beta_n(x) = M(x)\chi_{B_r}(x) + t_{r+1}\mu(D_r)^{-1}\chi_{D_r}(x). \tag{3.23}$$

Multiplyiting by $\varphi(x)$ and integrating on X we obtain

$$\sum_{n=0}^{\infty} [\gamma_n \langle a_n^*, \bar{\varphi}\rangle + \tilde{\gamma}_n \langle a_n^{**}, \bar{\varphi}\rangle]$$

$$= \int_{B_r} M(x)\varphi(x)\,d\mu(x) + t_{r+1}\mu(D_r)^{-1} \int_{D_r} \varphi(x)\,d\mu(x).$$

Since by Theorem 2 $M(x)\varphi(x)$ is absolutely integrable, we have

$$\sum_{n=0}^{\infty} [\gamma_n \langle a_n^*, \bar{\varphi}\rangle + \tilde{\gamma}_n \langle a_n^{**}, \bar{\varphi}\rangle]$$

$$= \int_X M(x)\varphi(x)\,d\mu(x) + \lim_{r\to+\infty}\Big(t_{r+1}\mu(D_r)^{-1} \int_{D_r} \varphi(x)\,d\mu(x)\Big).$$

On the other hand,

$$\Big|t_{r+1}\mu(D_r)^{-1} \int_{D_r} \varphi(x)\,d\mu(x)\Big| \leqslant c\cdot |t_{r+1}|\big(\|\varphi\|^*_{1/p-1}(b^r\sigma)^{1/p-1} + |m_r(\varphi)|\big).$$

Without loss of generality we assume that $m_0(\varphi) = 0$. Then from (3.21), Lemma 2 and Lemma 1 we get

$$|t_{r+1}\mu(D_r)^{-1} \int_{D_r} \varphi(x)\,d\mu(x)| \leqslant c\cdot \|\varphi\|^*_{1/p-1} b^{r(1/p-1/2-s/2)}.$$

That implies

$$\lim_{r\to\infty}\Big(t_{r+1}\mu(D_r)^{-1} \int_{Dr} \varphi(x)\,d\mu(x)\Big) = 0$$

completing the proof of part (ii).

As a consequence of Theorem 2 and Theorem 3, we have that Ka induces a bounded linear functional $F_{Ka}(\bar{\varphi})$ on $\mathrm{Lip}(1/p-1)$ which belongs to H^p and satisfies $\|F_{Ka}\|_{H^p} \leqslant c$.

Next, we will show that if $f = \sum_{i=1}^{\infty} \lambda_i a_i$ belongs to H^p, then $Kf = \sum_{i=1}^{\infty} \lambda_i K a_i$ is well defined. With this purpose we shall study the action of the dual operator of K on the spaces $\mathrm{Lip}(1/p-1)$.

DEFINITION 2. Let $k(x, y)$ be a weakly strongly singular integral kernel and let $\varphi \in \mathrm{Lip}(1/p-1)$, where $0 < p \leqslant 1$ satisfies $1/p < (1/q-\varrho/2)/(1-\varrho)$. Let $x_0 \in X$, $\sigma \geqslant 1$ and $B = B(x_0, \sigma)$. We define $K_B^{\#}(\varphi)(y)$ as

$$K_B^{\#}(\varphi)(y) = \lim_{\delta\to 0} \int_{B(x_0, 2A\sigma)} k_\delta(x, y)\varphi(x)\,d\mu(x),$$

where the limitis the weak-$L^{q'}$ limit on B and $k_\delta(x, y)$ is the function given in Definition 1.

The following lemma allows us to extend this definition to the whole space X.

LEMMA 3. *Let $B^1 = B^1(\xi_1, \tau_1)$ and $B^2(\xi_2, \tau_2)$ be two balls in X such that $\xi_2 > \tau_1 > 1$ and $B^1 \subset B^2$. Then for every $\varphi \in \mathrm{Lip}(1/p-1)$ we have $K_{B^1}^{\#}(\varphi)(y) = K_{B^2}^{\#}(\varphi)(y)$ almost everywhere in B^1.*

Proof. We have

$$K_{B^2}^{\#}(\varphi)(y) - K_{B^1}^{\#}(\varphi)(y) = \lim_{\delta\to 0} \int_{\tilde{B}^2\sim\tilde{B}^1} k_\delta(x, y)\varphi(x)\,d\mu(x),$$

where $\tilde{B}^1 = B(\xi_1, 2A\tau_1)$ and $\tilde{B}_2 = B(\xi_2, 2A\tau_2)$.

We observe that if $y \in B^1$ and $x \in \tilde{B}^2 - \tilde{B}^1$, then $k(x, y)$ vanishes. Therefore, we have the statement of the lemma.

This lemma shows that if $B^n = B(x_0, n)$, then $K_{B^n}^{\#}(\varphi) = K_{B^{n+1}}^{\#}(\varphi)$ almost everywhere in B^n. Then we define $K^{\#}(\varphi)$ by the condition $K^{\#}(\varphi) = K_{B^n}^{\#}(\varphi)$ almost everywhere in B^n.

LEMMA 4. *Let $k(x, y)$ be a weakly strongly singular kernel and let p satisfy (3.9). Then, for every function $\varphi \in \mathrm{Lip}(1/p-1)$, if $y \in B(x_0, \tau)$, $0 < \tau \leqslant 1$, we have the estimate:*

$$\int_{X\sim B(x_0, 2A\tau^\varrho)} |k(x, y) - k(x, x_0)|\; |\varphi(x)|\,d\mu(x)$$

$$\leqslant c\cdot d(x, y_0)^s \Big(\|\varphi\|^*_{1/p-1}(2A\tau^\varrho)^{1/p-1-s/(1-\theta)} + |m_{B(x_0, 2A\tau^\varrho)}(\varphi)|(2A\tau^\varrho)^{-s/(1-\theta)}\Big),$$

where c is a finite constant independent of φ, τ, x_0 and y.

Proof. Let $B_n = B(x_0, 2Ab^n\tau^\varrho)$, $b > 1$ Let $y \in B(x_0, \tau)$ and let $x \notin B_0$.

Then we have $d(x, x_0) > 2d(y, x_0)^{1-\theta}$. Therefore from (3.5) we get

$$(3.24)\quad \int_{X\sim B_0} |k(x, y)-k(x, x_0)|\ |\varphi(x)|\, d\mu(x)$$
$$\leqslant c\cdot d(y,x_0)^{\varepsilon} \sum_{n=0}^{\infty} (2Ab^n\tau^{\varrho})^{-1-\varepsilon/(1-\theta)}\big(\|\varphi\|^*_{1/p-1}(2Ab^{n+1}\tau^{\varrho})^{1/p}+|m_{n+1}(\varphi)|(2Ab^{n+1}\tau^{\varrho})\big).$$

The proof of lemma follows from Lemma 2 and Lemma 1.

Lemma 5. *Let $\varphi \in \mathrm{Lip}(1/p-1)$ and let $B(x_0, \tau)$ be for $0<\tau\leqslant 1$. Then for $y \in B(x_0, \tau)$ we have*

$$(3.25)\quad K^{\#}(\varphi)(y) = \lim_{\delta\to 0} \int_{B(x_0,2A\tau)} k_\delta(x, y)\varphi(x)\, d\mu(x)+$$
$$+\int_{X\sim B(x_0,2A\tau^{\varrho})} \big(k(x, y)-k(x, x_0)\big)\varphi(x)\, d\mu(x)+C,$$

where the limit is the weak-$L^{q'}$ limit on $B(x_0, \tau)$ and C is a finite positive constant independent of y.

Proof. Let $x \in B_0 = B(x_0, 2A\sigma)$, $\sigma\geqslant 1$ and $y \in B(x_0, \tau)$. Then we have $d(x, x_0)>1$ and $d(x, y)>2\sigma-\tau>1$. Therefore we get

$$(3.26)\quad K^{\#}(\varphi)(y) = \lim_{\delta\to 0} \int_{B_0} k_\delta(x, y)\varphi(x)\, d\mu(x)+\int_{X\sim B_0} \big(k(x, y)-$$
$$-k(x, x_0)\big)\varphi(x)\, d\mu(x).$$

Since $y \in B(x_0, \tau)$, $\tau\leqslant 1$, it follows that

$$\int_{B_0} k_\delta(x, y)\varphi(x)\, d\mu(x) = \int_{B(x_0,2A\tau^{\varrho})} k_\delta(x, y)\varphi(x)\, d\mu(x)+$$
$$+\int_{B_0\sim B(x_0,2A\tau^{\varrho})} k(x, y)\varphi(x)\, d\mu(x).$$

Substituting this equality in (3.26) we have the statement of lemma.

Lemma 6. *$K^{\#}(1) = $ constant.*

Proof. Let $B = B(x_0, \sigma)$, $\sigma\geqslant 1$ and let $B(R) = B(x_0, R\sigma)$ for $R>2A$. We have from Definition 2

$$K^{\#}(1)(y) = \lim_{\delta\to 0} \int_{B(R)} k_\delta(x, y)\, d\mu(x)$$

for $y \in B$. Let $g \in L^2(B, \mu)$ such that $\int_B g(y)\, d\mu(y) = 0$. Taking into account that $1<q<2$ we have $g \in L^q(B, \mu)$. Therefore, we get

$$\int_X K^{\#}(1)(y)g(y)\, d\mu(y) = \lim_{R\to+\infty} (\chi_{B(R)}, Kg)$$

and hence

$$\int_X K^{\#}(1)(y)g(y)\, d\mu(y) = \lim_{R\to+\infty} \big(K^*(\chi_{B(R)}), g\big).$$

From condition (3.6) we have

$$\int_X K^{\#}(1)(y)g(y)\, d\mu(y) = c\cdot\int g(y)\, d\mu(y) = 0,$$

and the lemma is proved.

Theorem 4. *Let p satisfy (3.9) and let $\varphi \in \mathrm{Lip}(1/p-1)$. Then there exists a finite positive constant c, independent of φ, such that*

$$\|K^{\#}(\varphi)\|^*_{1/p-1}\leqslant c\cdot \|\varphi\|^*_{1/p-1}.$$

Proof. Let B be a ball in X with radius $\sigma\geqslant 1$ and let B^1 be the ball with the same center and radius $2A\sigma$. We observe that

$$K^{\#}(\varphi) = K^{\#}\big(\varphi-m_{B^1}(\varphi)\big)+K^{\#}\big(m_{B^1}(\varphi)\big).$$

Since $K^{\#}\big(m_{B^1}(\varphi)\big)$ does not give any contribution to the estimate of $\|K^{\#}(\varphi)\|^*_{1/p-1}$, we can consider $\varphi \in \mathrm{Lip}(1/p-1)$ such that $m_{B^1}(\varphi) = 0$. We have,

$$(3.27)\quad \Big(\mu(B)^{-1} \int_B |K^{\#}(\varphi)(y)-m_B\big(K^{\#}(\varphi)\big)|^2\, d\mu(y)\Big)^{1/2}$$
$$\leqslant 2\Big(\mu(B)^{-1} \int_B |K^{\#}(\varphi)(y)|^2 d\mu(y)\Big)^{1/2}.$$

Let us estimate the integral on the right hand side of (3.27). Let $h \in L^2(B, \mu)$ such that $\|h\|_2 = 1$. Then we get

$$\Big(\mu(B)^{-1} \int_B |K^{\#}(\varphi)(y)|^2 d\mu(y)\Big)^{1/2}$$
$$= \mu(B)^{-1/2}\lim_{\delta\to 0} \int_X h(y) \Big(\int_{B^1} k_\delta(x, y)\varphi(x)\, d\mu(x)\Big)\, d\mu(y).$$

Changing the order of integration we can see that the second member of (3.27) is majorized by

$$c\cdot \mu(B)^{-1/2}\|Kh\|_2\mu(B^1)^{1/p-1/2}\|\varphi\|^*_{1/p-1}.$$

From (3.3) and since $\mu(B^1)\leqslant c\cdot\mu(B)$, it follows that

$$\Big(\mu(B)^{-1} \int_B |K^{\#}(\varphi)(y)|^2 d\mu(y)\Big)^{1/2}\leqslant c\cdot \|\varphi\|^*_{1/p-1}\mu(B)^{1/p-1}.$$

Then we have

$$(3.28)\qquad \Big(\mu(B)^{-1}\int_B |K^{\#}(\varphi)(y)-m_B(K^{\#}(\varphi))|^2 d\mu(y)\Big)^{1/2} \leqslant c\cdot\|\varphi\|^*_{1/p-1}\,\mu(B)^{1/p-1}$$

for every ball B with radius greater or equal to one.

We shall show that (3.28) also holds when the radius of B does not exceed one. For this purpose let B be a ball with radius $\sigma \leqslant 1$ and let B^2 be the a ball with same center and radius $2A\sigma^{\varrho}$. Let $\varphi \in \mathrm{Lip}(1/p-1)$ such that $m_{B^2}(\varphi) = 0$. We proceed as the first part of the proof and we obtain,

$$(3.29)\qquad \Big(\mu(B)^{-1}\int_B |K^{\#}(\varphi)(y)-m_B(K^{\#}(\varphi))|^2 d\mu(y)\Big) \leqslant 2\cdot\Big(\mu(B)^{-1}\int_B |K^{\#}(\varphi)(y)|^{q'} d\mu(y)\Big)^{1/q'},$$

where $1/q+1/q' = 1$. Let $g \in L^q(B,\mu)$ be such that $\|g\|_q = 1$. We get that

$$\Big(\int_B |K^{\#}(\varphi)(y)|^{q'} d\mu(y)\Big)^{1/q'} = \lim_{\delta\to 0}\int_B g(y)\Big(\int_{B^2} k_\delta(x,y)\varphi(x)\,d\mu(x)\Big)d\mu(y) + \int_B g(y)\Big(\int_{X\sim B^2}\big(k(x,y)-k(x,x_0)\big)\varphi(x)\,d\mu(x)\Big)d\mu(y) = I_1+I_2.$$

We have that

$$I_1 = \int_{B^2} Kg(x)\big(\varphi(x)-m_{B^2}(\varphi)\big)\,d\mu(x).$$

Then by Schwarz's inequality and from (3.2) we obtain

$$I_1 \leqslant c\|\varphi\|^*_{1/p-1}\mu(B^2)^{1/p-1/2}.$$

Let us estimate I_2. From Lemma 4 and Hölder's inequality we also obtain

$$I_2 \leqslant c\|\varphi\|^*_{1/p-1}\mu(B^2)^{1/p-1/2}.$$

Since $1/p < (1/q-\varrho/2)/(1-\varrho)$, we get $\sigma^{1/p-1/q} > \sigma^{\varrho(1/p-1/2)}$. Then from (2.1) we have

$$\mu(B^2)^{1/p-1/2} \leqslant c\cdot\sigma^{\varrho(1/p-1/2)} \leqslant c\cdot\sigma^{1/p-1/q} \leqslant c\cdot\mu(B)^{1/p-1/q}$$

and we conclude that I_1 and I_2 are majorized by $c\cdot\|\varphi\|^*_{1/p-1}\mu(B)^{1/p-1/q}$. Replacing this estimate in (3.29), it follows the proof of theorem.

Let $\bar\varphi \in \mathrm{Lip}(1/p-1)$. We define $K^{\#}(\bar\varphi)$ as the class of all the functions on X which differ from $K^{\#}(\varphi)$ in a constant.

By Lemma 3, the class $K^{\#}(\bar\varphi)$ is not empty and by Lemma 6, this definition does not depend of the representative of $\bar\varphi$ chosen. Then under the conditions of Theorem 4 we have

THEOREM 5. *If $\bar\varphi \in \mathrm{Lip}(1/p-1)$, then $K^{\#}(\bar\varphi) \in \mathrm{Lip}(1/p-1)$. Moreover, there exists a finite constant c, independent of φ, such that*

$$|||K^{\#}(\bar\varphi)|||^*_{1/p-1} \leqslant c\cdot|||\bar\varphi|||^*_{1/p-1}.$$

LEMMA 7. *Let $a(x)$ be a function in $L^2(X,\mu)$ with support contained in a ball $B(x_0,\tau)$, $\tau \geqslant 1$, such that*

$$\int_X a(x)\,d\mu(x) = 0.$$

Then, for every $\bar\varphi \in \mathrm{Lip}(1/p-1)$ we have

$$\langle Ka, \bar\varphi\rangle = \langle a, K^{\#}(\bar\varphi)\rangle.$$

Proof. From the linearity of the operator K and by Theorem 2, it follows that $Ka(x)\varphi(x)$ is integrable. If $x \notin B(x_0, 2A\tau)$ and $y \in B(x_0,\tau)$, we have $d(x,y) > 1$. Therefore, we get

$$\int_X Ka(x)\varphi(x)\,d\mu(x) = \lim_{\delta\to 0}\int_{B(x_0,2A\tau)} a(y)\Big(\int_X k_\delta(x,y)\varphi(x)\,d\mu(x)\,d\mu(y)\Big).$$

Since $a(y) \in L^q(B,\mu)$, $q < 2$, we have from (3.4) that

$$\lim_{\delta\to 0}\int_{B(x_0,2A\sigma)} k_\delta(x,y)\varphi(x)\,d\mu(x)$$

exists weakly in L^2 on the ball $B(x_0,\tau)$. This proves the lemma.

We can now prove the following theorem:

THEOREM 6. *Let $k(x,y)$ be a weakly strongly singular integral kernel and let p satisfy (3.9). Then if $f = \sum_{i=1}^{\infty}\lambda_i a_i$ is an element of H^p, the operator $Kf = \sum_{i=1}^{\infty}\lambda_i Ka_i$ is well defined. Moreover, K is linear and there is a finite constant C independent of f such that*

$$\|Kf\|_{H^p} \leqslant c\cdot\|f\|_{H^p}.$$

Proof. Let $f = \sum_{i=1}^{\infty}\lambda_i a_i$ in H^p. By Theorem 3, for every i, $Ka_i \in H^p$ and $\|Ka_i\|_{H^p} \leqslant c$. Let m and n be such that $m < n$. We get

$$\Big\|\sum_{i=1}^{n}\lambda_i Ka_i - \sum_{i=1}^{m}\lambda_i Ka_i\Big\|^p_{H^p} \leqslant c^p\cdot\sum_{i=m+1}^{n}|\lambda_i|^p.$$

It follows that the series $\sum_{i=1}^{\infty}\lambda_i Ka_i$ converges to an element $h \in H^p$. By Theorem 3 we have

$$\langle h, \bar{\varphi}\rangle = \sum_{i=1}^{\infty} \lambda_i \langle K a_i, \bar{\varphi}\rangle$$

and by Lemma 7, we obtain $\langle h, \bar{\varphi}\rangle = \langle f, K^{\#}(\bar{\varphi})\rangle$.

Therefore h depends only on f and does not depend on its r espresentation as a series of multiples of $(p, 2)$-atoms. Then $Kf = h$ is well defined, and it is easily seen that K is a linear operator from H^p into H^p. Moreover,

$$\|Kf\|_{H^p} \leqslant c \cdot \Big(\sum_{i=1}^{\infty} |\lambda_i|^p\Big)^{1/p}$$

that is to say,

$$\|Kf\|_{H^p} \leqslant c \cdot \|f\|_{H^p},$$

which proves the theorem.

Theorem 6 can be applied to obtain boundedness results on $H^p(\boldsymbol{R}^n)$ for the operators T_λ considered in [2] and [6]. For this purpose let $X = \boldsymbol{R}^n$ be endowed with the Lebesgue measure and the quasi-distance $d(x, y) = |x-y|^n$, where $|x-y|$ is the usual euclidean distance. We shall need the following lemmas:

Lemma 8. *Let k be an integrable function on $\boldsymbol{R}^n$ such that $|\hat{k}(x)| \leqslant c(1+|x|^\beta)^{-1}$ for $0 < \beta < n$, where $\hat{k}$ is the Fourier transform of the function k and c is a finite constant. Let φ be a $C^\infty(\boldsymbol{R}^n)$ function which vanishes near zero and is equal to 1 outside a bounded set. Then $k_\delta(x) = \varphi(\delta^{-1}x)k(x)$ satisfies*

$$|\hat{k}_\delta(x)| \leqslant c(1+|x|^\beta)^{-1}.$$

Proof. We shall prove the lemma for $n \geqslant 2$. The case $n = 1$ is simple and will be omitted. Let ψ be a C^∞ bounded support function on $\boldsymbol{R}^n$ such that $\psi+\varphi = 1$. Then, to prove the lemma it is enough to estimate $|\psi(\delta^{-1}x)k(x))\hat{}\,|$ that is to say we shall need to estimate $|\hat{k}(x) * \delta^n \hat{\psi}(\delta x)|$. Let $|B(x, r)|$ be the Lebesgue measure of $B(x, r)$ and let

$$M(\hat{k})(x) = \sup |B(x, r)|^{-1} \int_{B(x,r)} |\hat{k}(y)|\, dy$$

the maximal function of $\hat{k}$.

Then there exists a finite constant c such that

$$|\hat{k}(x) * \delta^n \hat{\psi}(\delta x)| \leqslant c M(\hat{k})(x).$$

Let us to estimate $M(\hat{k})(x)$. Let $B(x, r)$ be such that $|x| > 2r$. We have

$$M(\hat{k})(x) \leqslant \sup |B(x, r)|^{-1} \int_{B(x,r)} |y|^{-\beta}\, dy.$$

Passing to the polar coordinates we obtain

$$M(\hat{k})(x) \leqslant c r^{-1}\big((|x|+r)^{n-\beta} - (|x|-r)^{n-\beta}\big)(n-\beta)^{-1}$$

and by the mean value theorem it follows:

$$M(\hat{k})(x) \leqslant c|x|^{-\beta}. \tag{3.30}$$

On the other hand if $|x| \leqslant 2r$, we get

$$M(\hat{k})(x) \leqslant c \cdot r^{-n} \int_{B(0,3r)} |y|^{-\beta}\, dy$$

which gives us the same estimate as (3.30). From the boundedness of $\hat{k}$ we conclude that $M(\hat{k})(x) \leqslant c(1+|x|^\beta)^{-1}$ and the lemma is proved.

The proof of next lemma follows by a straightforward computation.

Lemma 9. *Let $0 < \theta < 1$, $a' < 0$ and $\lambda = (1-\theta)^{-1} - 1 + a'$. Let ψ be a C^∞ function such that $0 < \psi(t) < 1$, $\psi(t) = 1$ if $0 < t \leqslant 1/2$ and $\psi(t) = 0$ if $t \geqslant 1$. Let*

$$\tilde{k}(x, y) = [\exp(\mathrm{id}(x, y)^{a'/n})]\, d(x, x_0)^{-1-\lambda/n}.$$

Then the function $k(x, y) = \tilde{k}(x, y)\psi\big(d(x, y)\big)$ satisfies

$$|k(x, y) - k(x, x_0)| \leqslant c \cdot d(y, x_0)^{1/n} d(x, x_0)^{-1-1/n-(1-\theta)^{-1}}$$

whenever $d(x, x_0) > 2d(y, x_0)^{1-\theta}$ if $d(y, x_0) \leqslant 1$.

As a consequence of these lemmas and Theorem 6 we have

Proposition. *Let $\lambda = (1-\theta)^{-1}\theta - a(1-a)^{-1}$,*

$$\gamma = (a-1)\lambda/n + a/2 \quad \text{and} \quad a = \big(n(1-\theta)\big)^{-1} - n,$$

where $0 < \theta < a < 1$ and n is the dimension of $\boldsymbol{R}^n$. Then if $n+\lambda > 0$, the operator

$$T_\lambda f(x) = \lim_{\delta \to 0} \int_{\delta < |x-y| \leqslant 1} f(y)|x-y|^{-n-\lambda} \cdot e^{i|x-y|^{a'}}\, dy,$$

where $1/a + 1/a' = 1$, satisfies

$$\|T_\lambda f\|_{H^p(\boldsymbol{R}^n)} \leqslant c\|f\|_{H^p(\boldsymbol{R}^n)}$$

for those p that satisfies

$$n/n+1 < p \leqslant 1 \quad \text{and} \quad 1/p - 1/2 < \gamma(1/2+a+n)/(a+\gamma+n-1/n).$$

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