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Norm inequalities relating singular integrals and the maximal function

by

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Abstract. We prove that if the weighted L^p norms $(1<p<\infty)$ of the Riesz transforms are bounded by the weighted L^p norm of the maximal function, then the weight function satisfies the C_p condition of B. Muckenhoupt. Conversely we show that if the weight function satisfies the C_q condition for some $q>p$, then the weighted L^p norm of any standard singular integral is bounded by the weighted L^p norm of the maximal function.

§1. Introduction. We consider the problem of characterizing the non-negative weights w for which $(1<p<\infty)$

$$\int |Tf|^p w \leqslant C\int |Mf|^p w \quad \text{for all appropriate } f \tag{1}$$

where $Tf = K*f$ is a singular integral in R^n with kernel K satisfying the standard conditions

$$\|\hat{K}\|_\infty \leqslant C, \tag{i}$$

$$|K(x)| \leqslant C|x|^{-n}, \tag{ii}$$

$$|K(x)-K(x-y)| \leqslant C|y||x|^{-n-1} \quad \text{for} \quad |y|<|x|/2. \tag{iii}$$

R. Coifman and C. Fefferman have shown ([1]; Theorem III) that (1) holds for $1<p<\infty$ provided the weight w satisfies the A_∞ condition. B. Muckenhoupt has shown ([7]; Theorem 2.1) that in the case when T is the Hilbert transform, inequality (1) does not imply that w satisfies the A_∞ condition. He has derived ([7]; Theorem 1.2) the following necessary condition for (1) (with T the Hilbert transform) which he has conjectured to be sufficient.

(C_p) There are positive constants C, ε such that

$$\int_E w \leqslant C(|E|/|Q|)^\varepsilon \int |M_{\chi_Q}|^p w$$

whenever E is a subset of a cube $Q \subset R^n$.

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Here $|E|$ denotes the Lebesgue measure of E and M is the maximal operator defined by

$$Mf(x) = \sup_{x\in Q \text{ cube}} \frac{1}{|Q|}\int_Q |f|.$$

Our first result is that if (1) holds for the Riesz transforms, then the weight w satisfies the C_p condition. The one dimensional case of this result was obtained by B. Muckenhoupt ([7]; Theorem 1.2). Our second result is that if w satisfies the C_q condition for some $q>p$, then (1) holds. The question of whether or not C_p implies (1) remains open. We now state these results precisely. Throughout this paper Q will denote a cube in R^n with sides parallel to the co-ordinate planes and for $R>0$, RQ denotes the cube concentric with Q having diameter R times that of Q. Finally, the letter C will be used to denote a positive constant not necessarily the same at each occurrence.

THEOREM A. *Let $1<p<\infty$. If the weight w satisfies*

$$(2)\qquad \int |R_j f|^p w \leqslant C\int |Mf|^p w,\qquad 1\leqslant j\leqslant n,\ f \text{ bounded, supp} f \text{ compact}$$

where R_j denotes the j^{th} Riesz transform (formally $\widehat{R_j f}(x) = ix_j|x|^{-1}\hat f(x)$), then w satisfies the C_p condition.

THEOREM B. *Let $1<p<q<\infty$. If w satisfies the C_q condition, then* (1) *holds for all singular integrals with kernel satisfying* (i), (ii), *and* (iii) *above.*

AN APPLICATION. We give sufficient conditions on a pair of weights (w, v) in order that $(1<p<\infty)$

$$(3)\qquad \int |Tf|^p w \leqslant C\int |f|^p v$$

for all singular integrals T as above. Recall that the pair of weights (w, v) satisfies inequality (3) with Tf replaced by the maximal function Mf if and only if ([8])

$$(4)\qquad \int_Q |M(\chi_Q v^{1-p'})|^p w \leqslant C\int_Q v^{1-p'} \quad \text{for all cubes } Q.$$

Thus if the weight pair (w, v) satisfies (4) and if w satisfies the condition $C_{p+\varepsilon}$ for some $\varepsilon>0$, then inequality (3) holds. We remark that C_q weights, unlike A_∞ weights, can vanish on open sets.

§ 2. Proof of Theorem A. We first give an alternate description of the C_p condition due to B. Muckenhoupt ([7]).

LEMMA 1 (Muckenhoupt). *The weight w satisfies the C_p condition if (and trivially only if) there is $C<\infty$ such that*

$$(5)\qquad |E|_w \leqslant \frac{C}{[1+\log(|Q|/|E|)]^p}\int_{R^n} |M\chi_Q|^p w$$

whenever $E\subset Q$ a cube. Here $|E|_w = \int_E w$.

The case $n=1$ of this lemma is contained in [7] and the proof given there extends to $n>1$ with minor modifications which we sketch in an appendix below. In any event one can verify that all arguments using the C_p condition in this paper hold just as well using (5) as the definition.

Proof of Theorem A. The key step here is the observation that $\log Mf$ is in BMO if Mf is finite a.e. ([2]; p. 641). Suppose $E\subset Q$ a cube and set

$$(6)\qquad f = \log^+[(|Q|/|E|)\, M\chi_E].$$

Simple computations show that there is a constant C independent of Q and E such that

$$(7)\qquad f_Q = |Q|^{-1}\int_Q f \leqslant C,$$

$$(8)\qquad \|f\|_{\mathrm{BMO}} = \sup_{\text{cubes } I} |I|^{-1}\int_I |f-f_I| \leqslant C,$$

$$(9)\qquad f = \log(|Q|/|E|) \text{ a.e. on } E.$$

From (8) and the duality of H^1 and BMO ([5]; Theorem 3) we obtain

$$f = f_0 + \sum_{j=1}^n R_j f_j$$

where $\|f_j\|_\infty \leqslant C$, $0\leqslant j\leqslant n$. Let $g_j = \chi_{2Q} f_j$ and $h_j = \chi_{2Q^c} f_j$ for $1\leqslant j\leqslant n$. Here $2Q$ denotes the cube concentric with Q and with twice the side length; $2Q^c$ denotes its complement. Let z be the centre of Q and set $A_j = (R_j h_j)(z)$. Then for $x\in Q$ we have by property (iii)

$$(10)\qquad |R_j h_j(x) - A_j| \leqslant C\int_{2Q^c} |h_j(y)|(|x-z|/|y-z|^{n+1})\,dy \leqslant C \quad (x\in Q)$$

and thus also

(11)

$$\Big|\sum_{j=1}^n A_j\Big| \leqslant C\Big|\frac{1}{|Q|}\int_Q \sum_{j=1}^n R_j h_j\Big| + C \leqslant C\frac{1}{|Q|}\Big[\int_Q f + \int_Q |f_0| + \sum_{j=1}^n \int_Q |R_j g_j|\Big] + C$$

since $f = f_0 + \sum_{j=1}^n R_j g_j + \sum_{j=1}^n R_j h_j$. However,

$$\frac{1}{|Q|}\int_Q |R_j g_j| \leqslant \Big[\frac{1}{|Q|}\int_Q |R_j g_j|^2\Big]^{1/2} \leqslant \Big[\frac{1}{|Q|}\int_Q |f_j|^2\Big]^{1/2} \leqslant C$$

by Hölder's inequality, the L^2 boundedness of the Riesz transforms and the boundedness of the f_j. Combining this with (7), (11) and $\|f_0\|_\infty \leqslant C$ we obtain

$\left|\sum_{j=1}^{n} A_j\right| \leqslant C$ and (10) now yields

$$\left|f-\sum_{j=1}^{n} R_j g_j\right| \leqslant C \quad \text{on } Q.$$

From this and equation (9) we have

$$\sum_{j=1}^{n} |R_j g_j| \geqslant \log(|Q|/|E|)-C \text{ a.e. on } E$$

and from (2) we now obtain

$$|E|_w[\log(|Q|/|E|)-C]^p \leqslant C\sum_{j=1}^{n}\int |R_j g_j|^p w \leqslant C\sum_{j=1}^{n}\int |Mg_j|^p w \leqslant C\int |M\chi_Q|^p w$$

which is (5). Lemma 1 now completes the proof of Theorem A.

§ 3. Proof of Theorem B. We begin with a variant of the Whitney covering lemma used in [3].

WHITNEY COVERING LEMMA. *Given $R \geqslant 1$, there is $C = C(R, n)$ such hat if Ω open $\subset R^n$, then $\Omega = \bigcup_j Q_j$ where the Q_j are disjoint cubes satisfying*

(i) $$5R \leqslant \frac{\operatorname{dist}(Q_j, \Omega^c)}{\operatorname{diam} Q_j} \leqslant 15R,$$

(ii) $$\sum_j \chi_{RQ_j} \leqslant C\chi_\Omega.$$

Proof. Conclusion (ii) is a consequence of (i) and a geometric packing argument ([3]; p. 16). Conclusion (i) in turn can be established easily by standard arguments — see for example [6]; Theorem 2.1.

In attempting to prove Theorem B by the methods of R. Coifman and C. Fefferman in [1], we will be led via the C_q condition to consideration of integrals of the form $\int [\sum_j |M\chi_{Q_j}|^q] w$ where $\{Q_j\}_j$ is a Whitney covering of the open set $\{T^*f > \lambda\}$ (T^* is the maximal operator associated to T — see Lemma 2 below). We thus begin by investigating the operator $M_{p,q}$ defined below in terms of Marcinkiewicz integrals.

DEFINITION. Let $1 < p, q < \infty$ and suppose $f\colon R^n \to [0, \infty]$ is lower semicontinuous. Let $\Omega_k = \{f > 2^k\}$ and define

$$(M_{p,q}f(x))^p = \sum_{k\in Z} 2^{kp} \int_{\Omega_k} \frac{d(y, \Omega_k^c)^{n(q-1)}}{d(y, \Omega_k^c)^{nq}+|x-y|^{nq}}\, dy$$

where $d(y, E)$ denotes the distance from y to the set E.

Fix $R \geqslant 1$ and let $\Omega_k = \bigcup_j Q_j^k$ be as in the Whitney covering lemma. Then

$$M_{p,q}f(x)^p \approx \sum_{k,j} 2^{kp}[M\chi_{Q_j^k}(x)]^q$$

in the sense that the ratio of the right and left sides is bounded between two positive constants depending only on R (and not on x). We use only this latter expression for $M_{p,q}f$ in the sequel.

LEMMA 2. *Suppose $1 < p < q < \infty$ and that w satisfies the C_q condition. Let*

$$T^*f(x) = \sup_{0<\varepsilon<\eta<\infty}\left|\int_{\varepsilon<|y|<\eta} K(y)f(x-y)\,dy\right|$$

where K is a kernel satisfying (i), (ii), *and* (iii) *of* §1. *Then for all f with compact support we have*

(12) $$\int |M_{p,q}(T^*f)|^p w \leqslant C\left[\int |T^*f|^p w + \int |Mf|^p w\right].$$

The proof of Lemma 2 is fairly long and will be postponed to § 4. We remark that Lemma 2 may fail when $p = q$ even for weights w satisfying the A_∞ condition. For example when $p = q = 2$, let f be the characteristic function of the unit interval in R, T the Hilbert transform, and set $w(x) = |x|/(1+(\log|x|)^2)$. Then $M_{2,2}(T^*f)(x) \gtrsim \sqrt{\log|x|}/|x|$ for $|x|$ large and so the left side of (12) is infinite while the right side is finite.

Proof of Theorem B. Suppose first that f is bounded with compact support. Let $\Omega_k = \{T^*f > 2^k\} = \bigcup_j Q_j^k$ be as in the Whitney covering lemma with $R = 1$. By a fundamental inequality of R. Coifman and C. Fefferman ([1]; (8), p. 245) we have

$$|\{x \in Q_j^k;\ T^*f > 2^{k+1},\ Mf \leqslant \gamma 2^k\}| \leqslant C\gamma |Q_j^k|$$

and thus the C_q condition yields

(13) $$\begin{aligned}\int |T^*f|^p w &\leqslant C\sum_k 2^{kp}|\Omega_{k+1}|_w \\ &\leqslant C\sum_k 2^{kp}|\{Mf > \gamma 2^k\}|_w + C\gamma^\varepsilon \sum_{k,j} 2^{kp}\int |M\chi_{Q_j^k}|^q w \\ &\leqslant C_\gamma \int |Mf|^p w + C\gamma^\varepsilon\left[\int |T^*f|^p w + \int |Mf|^p w\right]\end{aligned}$$

by Lemma 2. If we can show $\int |T^*f|^p w < \infty$, then by choosing γ so small that $C\gamma^\varepsilon \leqslant 1/2$, inequality (13) will yield the conclusion of Theorem B for bounded f with compact support. However, if $\operatorname{supp} f \subset Q$ a cube, then

$$\int_{2Q^c} |T^*f|^p w \leqslant C \int_{2Q^c} |Mf|^p w < \infty$$

since property (ii) of the kernel K shows that $T^*f \leqslant CMf$ outside $2Q$. If in addition f is bounded, then ([9]; see 6.2, p. 48) $\int_{2Q} e^{a|T^*f|} < \infty$ for some $a > 0$ and thus $|\{x \in 2Q;\ T^*f > \lambda\}| \leqslant Ce^{-\lambda a}|2Q|$ for $\lambda > 0$. Applying the C_q condition to this latter inequality and integrating we obtain

$$\int_{2Q} |T^*f|^p w \leqslant C \int |M\chi_{2Q}|^q w \leqslant C \int |Mf|^p w < \infty$$

since $q > p$ and $\operatorname{supp} f \subset Q$. Thus (1) holds for bounded f with compact support and a simple limiting argument proves the general case. Indeed, if $\int |Mf|^p w < \infty$ then f is locally integrable and so $T^*f \leqslant \varliminf_{R\to\infty} T^*f_R$ where $f_R(x) = f(x)$ if $|x|, |f(x)| \leqslant R$ and 0 otherwise. An application of Fatou's lemma now completes the proof of Theorem B.

§4. Proof of Lemma 2. We begin with two preliminary lemmas. The first is a variant of Lemma 5.1 in [7].

LEMMA 3. *Suppose w satisfies the C_q condition, $1 < q < \infty$. Then for all $\delta > 0$, there is $C(\delta) < \infty$ such that whenever $\{Q_j\}_j$ is a collection of disjoint subcubes of a cube Q, then*

$$\int_{RQ} \Big[\sum_j |M\chi_{Q_j}|^q\Big] w \leqslant C(\delta)|RQ|_w + \delta \int |M\chi_Q|^q w \tag{14}$$

for all $R \geqslant 2$. Consequently,

$$\int \Big[\sum_j |M\chi_{Q_j}|^q\Big] w \leqslant C \int |M\chi_Q|^q w. \tag{15}$$

Proof. A classical estimate for the Marcinkiewicz integral (see [4]; Theorem 1 (3)) shows that $|E_\lambda| \leqslant Ce^{-a\lambda}|Q|$ for $\lambda > 0$ where a is some positive constant and $E_\lambda = \{\sum_j |M\chi_{Q_j}|^q > \lambda\}$. Since $\sum_j |M\chi_{Q_j}|^q$ is bounded outside $2Q$, the C_q condition implies $|E_\lambda|_w \leqslant Ce^{-\varepsilon a\lambda} \int |M\chi_Q|^q w$ for λ sufficiently large and this in turn yields

$$\int_{RQ\cap E_\lambda} \sum |M\chi_{Q_j}|^q w \leqslant Ce^{-\varepsilon a\lambda} \int |M\chi_Q|^q w.$$

Choosing λ so large that $Ce^{-\varepsilon a\lambda} = \delta$ we obtain the conclusion of Lemma 3 with $C = \lambda$.

LEMMA 4. *Suppose $1 < p < q < \infty$ and that w satisfies the C_q condition. Then for all compactly supported f*

$$\int |M_{p,q}(Mf)|^p w \leqslant C \int |Mf|^p w.$$

Proof. Let $\Omega_k = \{Mf > 2^k\} = \bigcup_j Q_j^k$ be as in the Whitney covering lemma with $R = 10$. Let N be a positive integer (to be chosen later) and fix a Whitney cube Q_i^{k-N}. We now claim

$$|\Omega_k \cap 5Q_i^{k-N}| \leqslant C2^{-N}|Q_i^{k-N}| \tag{16}$$

where C depends only on the dimension n. Indeed, let $g = f\chi_{10Q_i^{k-N}}$ and $h = f - g$. Property (i) of the Whitney covering lemma shows by a standard argument (see e.g. [9]; p. 19) that $Mh(x) \leqslant C2^{k-N}$ for x in $5Q_i^{k-N}$. Now $Mf \leqslant Mg + Mh$ and thus for N so large that $C2^{-N} \leqslant 1/2$, we have

$$\begin{aligned}
|\Omega_k \cap 5Q_i^{k-N}| &\leqslant |\{Mg > (1/2)2^k\}| \\
&\leqslant C\,2^{-k} \int |g| = C\,2^{-k} \int_{10Q_i^{k-N}} |f| \quad \text{since } M \text{ is weak type 1,1} \\
&\leqslant C\,2^{-k}(C\,2^{k-N}|10\,Q_i^{k-N}|) \quad \text{by (i) of the Whitney lemma}
\end{aligned}$$

which proves (16).

Now let $S(k) = 2^{kp} \sum_j \int |M\chi_{Q_j^k}|^q w$ and $S(k; N, i) = 2^{kp} \sum_j \int |M\chi_{Q_j^k}|^q w$ where the latter sum is taken over those j for which $Q_j^k \cap Q_i^{k-N} \neq \emptyset$. Since $Q_j^k \cap Q_i^{k-N} \neq \emptyset$ implies $Q_j^k \subset 5Q_i^{k-N}$ for large N we have

$$\begin{aligned}
S(k; N, i) &\leqslant \int 2^{kp} \sum_{j;\, Q_j^k \subset 5Q_i^{k-N}} |M\chi_{Q_j^k}|^q w \\
&= \int_{10Q_i^{k-N}} + \int_{[10Q_i^{k-N}]^c} = \mathrm{I} + \mathrm{II} \quad \text{for } N \text{ large.}
\end{aligned}$$

By (14) of Lemma 3

$$\mathrm{I} \leqslant C(\delta)2^{kp}|10Q_i^{k-N}|_w + \delta 2^{kp} \int |M\chi_{Q_i^{k-N}}|^q w$$

where $\delta > 0$ is at our disposal. Simple estimates on $M\chi_{Q_j^k}$ show that if x_i^{k-N} denotes the centre of Q_i^{k-N}

$$\begin{aligned}
\mathrm{II} &\leqslant C\,2^{kp} \int_{[10Q_i^{k-N}]^c} \frac{\sum |Q_j^k|^q}{|x - x_i^{k-N}|^{nq}}\, w(x)\,dx \\
&\leqslant C\,2^{kp} \int_{[10Q_i^{k-N}]^c} \left(\frac{C2^{-N}|Q_i^{k-N}|}{|x - x_i^{k-N}|^n}\right)^q w(x)\,dx \quad \text{by (16)} \\
&\leqslant C\,2^{N(p-q)}2^{(k-N)p} \int |M\chi_{Q_i^{k-N}}|^q w.
\end{aligned}$$

Thus for N large

$$S(k)\leqslant \sum_i S(k;N,i)$$
$$\leqslant C(\delta)2^{kp}\int\Big(\sum_i \chi_{10Q_i^{k-N}}\Big)w+[\delta 2^{Np}+C2^{N(p-q)}]S(k-N)$$
$$\leqslant C\ 2^{kp}|\Omega_{k-N}|_w+(1/2)S(k-N)$$

for N sufficiently large and δ sufficiently small upon appealing to property (ii) (with $R=10$) of the Whitney covering lemma. Thus with $S_M=\sum_{k\leqslant M}S(k)$, we have

$$S_M\leqslant(1/2)S_M+C\int|Mf|^p w\quad \text{for all } M. \tag{17}$$

Recall now that f has compact support, say $\operatorname{supp} f\subset Q$ a cube. Let $2^L<|Q|^{-1}\int_Q|f|\leqslant 2^{L+1}$. Then $\Omega_k\subset 2Q$ for $k\geqslant L+1$ and (15) of Lemma 3 shows that

$$\sum_{k=L+1}^{M}\sum_j 2^{kp}\int|M\chi_{Q_j^k}|^q w\leqslant C\int|M\chi_{2Q}|^q w\leqslant C\int|M\chi_Q|^p w<\infty$$

since $q>p$ and $\int|Mf|^p w<\infty$ (otherwise there is nothing to prove). On the other hand if $k\leqslant L$, then $\Omega_k\subset 2^{L-k+2}Q$ and (15) of Lemma 3 yields

$$\sum_{k\leqslant L}\sum_j 2^{kp}\int|M\chi_{Q_j^k}|^q w\leqslant C\sum_{k\leqslant L}2^{kp}\int|M\chi_{2^{L-k}Q}|^q w\leqslant C\ 2^{Lp}\int|M\chi_Q|^p w<\infty$$

since $\sum_{m=0}^{\infty}2^{-mp}|M\chi_{2^mQ}|^q\leqslant C_{p,q}|M\chi_Q|^p$ for $q>p$. Thus $S_M<\infty$ for all M and (17) now yields

$$\int|M_{p,q}(Mf)|^p w\leqslant C\sup_M S_M\leqslant C\int|Mf|^p w$$

and this completes the proof of Lemma 4.

Proof of Lemma 2. Let $\Omega_k=\{T^*f>2^k\}=\bigcup_j Q_j^k$ be as in the Whitney covering lemma with $R=20$. A fundamental inequality of R. Coifman and C. Fefferman states ([1]; (8), p. 245)

$$|\{x\in 10Q_i^{k-1};\ T^*f>2^k\}|\leqslant C\ 2^{-N}|Q_i^{k-1}| \tag{18}$$
$$\text{whenever}\quad 10Q_i^{k-1}\not\subset\{Mf>2^{k-N}\},\qquad N\geqslant 1.$$

Let $\{Mf>2^k\}=\bigcup_j I_j^k$ be as in the Whitney covering lemma with $R=20$.

We observe that for each cube Q_i^{k-1} there are two cases (N will be chosen later).

Case (1). $10Q_i^{k-1}\subset\{Mf>2^{k-N}\}$ in which case $10Q_i^{k-1}\subset C_nI_l^{k-N}$ for some l where $C_n\approx 15Rn^{1/2}=300n^{1/2}$ (choose I_l^{k-N} to contain the centre of Q_i^{k-l}).

Case (2). $10Q_i^{k-1}\not\subset\{Mf>2^{k-N}\}$ in which case (18) implies $\sum_{Q_j^k\subset 10Q_i^{k-1}}|Q_j^k|\leqslant C\,2^{-N}|Q_i^{k-1}|$.

Now let

$$S(k)=\sum_j 2^{kp}\int|M\chi_{Q_j^k}|^q w\quad\text{and}$$

$$S(k;i)=\sum_{j;Q_j^k\cap Q_i^{k-1}\neq\emptyset}2^{kp}\int|M\chi_{Q_j^k}|^q w\leqslant\sum_{j;Q_j^k\subset 10Q_i^{k-1}}2^{kp}\int|M\chi_{Q_j^k}|^q w.$$

The last inequality follows from the fact that $Q_j^k\subset 10Q_i^{k-1}$ whenever $Q_j^k\cap Q_i^{k-1}\neq\emptyset$ (property (i) of the Whitney lemma). Thus

$$S(k;i)\leqslant\int 2^{kp}\sum_{Q_j^k\subset 10Q_i^{k-1}}|M\chi_{Q_j^k}|^q w=\int_{20Q_i^{k-1}}+\int_{[20Q_i^{k-1}]^c}=\mathrm{I}+\mathrm{II}.$$

By (14) of Lemma 3 we have

$$\mathrm{I}\leqslant C(\delta)2^{kp}|20Q_i^{k-1}|_w+\delta 2^{kp}\int|M\chi_{Q_i^{k-1}}|^q w$$

where $\delta>0$ is at our disposal and if x_i^{k-1} denotes the centre of Q_i^{k-1}, then

$$\mathrm{II}\leqslant C2^{kp}\int_{[20Q_i^{k-1}]^c}\frac{\sum|Q_j^k|^q}{|x-x_i^{k-1}|^{nq}}w(x)\,dx$$
$$\leqslant C2^{kp}\int_{[20Q_i^{k-1}]^c}\left(\frac{C2^{-N}|Q_i^{k-1}|}{|x-x_i^{k-1}|^n}\right)^q w(x)\,dx\quad\text{in case (2)}$$
$$\leqslant C2^{kp-Nq}\int|M\chi_{Q_i^{k-1}}|^q w.$$

Combining the estimates for I and II we obtain

$$S(k;i)\leqslant C_\delta 2^{kp}|20Q_i^{k-1}|_w+[\delta+C2^{-Nq}]2^{kp}\int|M\chi_{Q_i^{k-1}}|^q w \tag{19}$$

whenever Q_i^{k-1} is a case (2) cube. Thus

$$S(k)\leqslant\sum_{i;Q_i^{k-1}\text{ is case (1)}}S(k;i)+\sum_{i;Q_i^{k-1}\text{ is case (2)}}S(k;i)=\mathrm{III}+\mathrm{IV}.$$

Now since each Q_j^k intersects at most C of the Q_i^{k-1},

$$\mathrm{III} \leqslant \sum_l C \sum_{Q_j^k \subset C_n I_l^{k-N}} 2^{kp} \int |M\chi_{Q_j^k}|^q w \leqslant C \sum_l 2^{kp} \int |M\chi_{I_l^{k-N}}|^q w$$

by (15) of Lemma 3 and the inequality $M\chi_{2C_nI} \leqslant CM\chi_I$. For the remaining term we have by (19)

$$\mathrm{IV} \leqslant C_\delta 2^{kp} \int \Big(\sum_i \chi_{20Q_i^{k-1}}\Big) w + (\delta + C2^{-Nq}) \sum_i 2^{kp} \int |M\chi_{Q_i^{k-1}}|^q w$$
$$\leqslant C2^{kp} |\Omega_{k-1}|_w + (1/2) S(k-1)$$

by property (ii) of the Whitney covering lemma (with $R = 20$) and upon choosing δ small enough and N large enough. Combining III and IV we have

(20) $$S(k) \leqslant (1/2) S(k-1) + C2^{kp} |\Omega_{k-1}|_w + C2^{kp} \sum_l |M\chi_{I_l^{k-N}}|^q w.$$

Now let $S_M = \sum_{k \leqslant M} S(k)$ and sum inequality (20) over $k \leqslant M$ to obtain

(21) $$S_M \leqslant (1/2) S_M + C \int |T^* f|^p w + C \int |M_{p,q}(Mf)|^p w$$
$$\leqslant (1/2) S_M + C\Big[\int |T^* f|^p w + \int |Mf|^p w\Big]$$

by Lemma 4.

Now the argument used at the end of the proof of Lemma 4 to show that $S_M < \infty$ can also be used here to obtain $S_M < \infty$ for all M (use the fact that $T^* f \leqslant C\ Mf$ outside $2Q$ if supp $f \subset Q$). Thus (21) yields

$$\int |M_{p,q}(T^* f)|^p w \leqslant C \sup_M S_M \leqslant C\Big[\int |T^* f|^p w + \int |Mf|^p w\Big]$$

and this completes the proof of Lemma 2.

Appendix. We sketch a proof of Lemma 1. As already mentioned, the case $n = 1$ is in [7] and the proof given there extends to $n > 1$ with minor modifications. As that proof is fairly long, we limit ourselves here to a brief discussion of the required modifications, assuming that the reader is familiar with Sections 5 and 6 of [7].

Clearly C_p implies (5) so we now assume that (5) holds. Lemma 5.1 of [7] extends to R^n without any essential change in the proof. Thus we can find $0 < \delta < 2^{-n}$ so small that whenever $\{Q_k\}$ is a collection of disjoint subcubes of a cube Q with $\sum_k |Q_k| \leqslant 2^n \delta |Q|$, then

(22) $$\int \Big[\sum_k |M\chi_{Q_k}|^p\Big] w \leqslant (1/2) \int |M\chi_Q|^p w.$$

Now given $E \subset Q$ a cube in R^n, let N be the least integer satisfying $\delta^N |Q| \leqslant |E|$. Define $E_0 = E$ and $E_j = \{M_{dy}\chi_E > \delta^j\}$ for $1 \leqslant j \leqslant N$ where M_{dy} denotes the dyadic maximal operator $M_{dy} f(x) = \sup_{x \in Q \text{ dyadic cube}} |Q|^{-1} \int_Q |f|$. Now $E_j = \bigcup_k Q_k^j$ where the Q_k^j are the maximal dyadic cubes I satisfying $|I|^{-1} \int_I \chi_E > \delta^j$. Thus $\delta^j < |E \cap Q_k^j| / |Q_k^j| \leqslant 2^n \delta^j \leqslant \delta^{j-1}$ and so

(a) each Q_i^{j-1} is strictly contained in some Q_k^j,

(b) $\sum_{Q_i^{j-1} \subset Q_k^j} |Q_i^{j-1}| \leqslant 2^n \delta |Q_k^j|$ for $2 \leqslant j \leqslant N$ and all k.

Using (a), (b) and (22) we obtain

$$\int \Delta_{j-1} w \leqslant (1/2) \int \Delta_j w, \quad 2 \leqslant j \leqslant N$$

where $\Delta_j(x) = \sum_k |M\chi_{Q_k^j}(x)|^p$ and the proof can now be completed by iterating this inequality as in Section 6 of [7].

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