*elements in $A$ possessing small powers. Or equivalently, for the classes of locally convex or $B_0$-algebras, the permanent radical of an algebra is the set of all its elements on which operate all formal power series.*

Remark. The above characterization shows that the concept of permanent radical in the classes $\mathscr{LC}$ or $\mathscr{B}_0$ has an absolute character (cf. Definition 1.2). This is not true for such classes as $\mathscr{M}$, $\mathscr{M}_0$, or $\mathscr{B}$. In fact, in these classes we have $\operatorname{rad}_{\mathscr{M}} A = \operatorname{rad} A$, what follows from the fact that for $A \in \mathscr{M}$ its radical is given by

$$\operatorname{rad} A = \bigcap_a \{x \in A : \lim_n \|x^n\|_a^{1/n} = 0\},$$

where the intersection is taken with respect to all continuous seminorms on $A$ satisfying relation (4).

3.5. Corollary. *If $A$ is a Banach algebra then its $\mathscr{LC}$-permanent radical, or $\mathscr{B}_0$-permanent radical coincides with the set of all its nilpotent elements, and equals to $\operatorname{rad}_{\mathscr{T}} A$.*

Let us remark (cf. remarks at the end of Section 2), that if $A \in \mathscr{B}$ and $\operatorname{rad} A$ contains elements of arbitrarily high orders, then the set $\operatorname{rad}_{\mathscr{B}_0} A$ is a non-closed ideal in $A$. This was, in fact, known to Rolewicz, who used it in [1] to the construction of a $B_0$-algebra possessing a non-closed radical.

We do not know whether Theorem 3.4 is true for the class of all topological algebras.

Problem. Let $A$ be a topological algebra. Does the ideal $I_s(A)$ of all elements of $A$ possessing small powers coincide with the $\mathscr{T}$-permanent radical $\operatorname{rad}_{\mathscr{T}} A$ of $A$?

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# Holomorphic functional calculus and quotient Banach algebras

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**Abstract.** Let $\mathscr{A}$ be a commutative associative Banach algebra with unit, and $\alpha$ an ideal of $\mathscr{A}$ with a Banach norm stronger than the norm induced by that of $\mathscr{A}$. Let $\bar{a}_1, \ldots, \bar{a}_n$ be elements of $\mathscr{A}/\alpha$. We define $\operatorname{sp}(\bar{a}_1, \ldots, \bar{a}_n)$. We construct a homomorphism $\mathscr{O}(\operatorname{sp}\bar{a}, \mathscr{A}/\alpha) \to \mathscr{A}/\alpha$, mapping $z_i$ onto $\bar{a}_i$, and unit on unit. This relative holomorphic functional calculus ($\mathscr{A} \bmod \alpha$) generalizes classical holomorphic functional calculus (where $\alpha = 0$).


Let $\mathscr{A}$ be a Banach algebra, which is commutative, associative, and with unit. Let $\alpha$ be a Banach ideal. Let $\bar{a}_1, \ldots, \bar{a}_n$ be elements of $\mathscr{A}/\alpha$, or, if you prefer $a_1, \ldots, a_n$ elements of $\mathscr{A}$, where of course $a_i \in \bar{a}_i$. The spectrum $\operatorname{sp}(\bar{a}_1, \ldots, \bar{a}_n)$, i.e. $\operatorname{sp}_\alpha(a_1, \ldots, a_n)$ is the set of $(s_1, \ldots, s_n) \in C^n$ such that

$$\sum_1^n (\bar{a}_i - s_i)\mathscr{A}/\alpha \neq \mathscr{A}/\alpha,$$

i.e.

$$\sum_1^n (a_i - s_i)\mathscr{A} + \alpha \neq \mathscr{A}.$$

Let now $U \subseteq C^n$ be open in $C^n$, $U \supseteq \operatorname{sp}(\bar{a}_1, \ldots, \bar{a}_n)$, let

$$\mathscr{O}(U, \mathscr{A}/\alpha) = \mathscr{O}(U, \mathscr{A})/\mathscr{O}(U, \alpha).$$

Call 1 the constant function on $U$, equal to 1, and $z_i$ the holomorphic mapping $z_i: (s_1, \ldots, s_n) \to s_i$. We shall construct a homomorphism

$$\mathscr{O}(U, \mathscr{A}/\alpha) \to \mathscr{A}/\alpha$$

which maps $z_i$ on $\bar{a}_i$, 1 onto 1. This homomorphism is induced by a continuous linear mapping $\mathscr{O}(U, \mathscr{A}) \to \mathscr{A}$ which maps $\mathscr{O}(U, \alpha)$ into $\alpha$. If $U$ is a schlicht domain of holomorphy, the homomorphism above is unique.

This is the first of two papers. In the second paper, we shall prove that every ideal of a quasi-Banach algebra has at least one quasi-Banach structure. The results above can be stretched, give homomorphisms $\mathcal{O}(U, \mathcal{A}/a) \to \mathcal{A}/a$ mapping $z_i$ on $\bar{a}_i$, and 1 on 1.

As a corrollary, a Šilov theorem can be proved, if $a$ is an ideal (non-closed), whose hull,

$$\operatorname{Hull} a = X_1 \cup X_2$$

is not connected, $X_1$ and $X_2$ compact, disjoint, then

$$a = a_1 \cap a_2 \quad \text{with} \quad \operatorname{Hull} a_1 = X_1, \quad \operatorname{Hull} a_2 = X_2.$$

And an Arens and Calderón theorem can be proved (where $a$ is any ideal, and $\mathcal{A}$ is pseudo-Banach).

The main results of this paper were proved a long time ago, they were published in a memoir [1]. The setting there is much more general that it is here. The memoir is difficult to read, and now out of print.

The reader knows that the unit of $C$, of $\mathcal{A}$, of $\mathcal{A}/a$, the constant function equal to 1, to the unit of $\mathcal{A}$, of $\mathcal{A}/a$, etc. should all have different notations. And the reader knows that this author denotes all these objects by 1.

**1. Quotient Banach spaces, algebras, and some cohomology.** Before beginning the main dish of this paper, we give a couple of entrées. Not all textbooks use quotient Banach spaces, or quotient Banach algebras. Nor do they all use homology, or cohomology taking values in a Banach space modulo a Banach subspace, or a Banach algebra modulo a Banach ideal. The results here are elementary.

Let $(E, \|\ \|_E)$ be a Banach space. Let $F$ be a subspace of $E$, with a norm $\|\ \|_F$ which is a Banach norm on $F$ and is stronger on $F$ than $\|\ \|_E$, i.e.

$$\exists \varepsilon > 0 \colon \forall x \in F \ \|x\|_F \geqslant \varepsilon \|x\|_E.$$

Observe that two Banach norms on $F$, both stronger than $\|\ \|_E$, are equivalent because of the closed graph theorem. *$(F, \|\ \|_F)$ is a Banach subspace of $(E, \|\ \|_E)$; $E/F$ is a quotient Banach space* ([2], [3], [4]).

Let $(\mathcal{A}, \|\ \|_{\mathcal{A}})$ be an associative, commutative Banach algebra with a unit. (These conditions can be weakened, but in this paper, we shall only consider such algebras.) Let $a$ be an ideal of $\mathcal{A}$, and a Banach subspace of $(\mathcal{A}, \|\ \|_{\mathcal{A}})$. Call $n_a$ the norm of $a$. Multiplication $(\mathcal{A} \times a) \to a$ has a separately closed graph, i.e. is separately continuous, i.e. is joint continuous $\mathcal{A} \times a \to a$. Let

$$\|x\|_a = \sup\{n_a(a \cdot x) \mid \|a\|_{\mathcal{A}} \leqslant 1\}.$$

Then $\|\ \|_a$ is equivalent to $n_a$ and $\|a \cdot x\|_a \leqslant \|a\|_{\mathcal{A}} \|x\|_a$; *$(a, \|\ \|_a)$ is a Banach ideal of $(\mathcal{A}, \|\ \|_{\mathcal{A}})$; $\mathcal{A}/a$ is a quotient Banach algebra.*



Let now $E$ be a Banach space. Let $U \subseteq C^n$ be open. Let $\Omega(U, E)$ be the alternating forms of $d\bar{z}_1, \ldots, d\bar{z}_n$ with coefficients of class $C_\infty$ on $U$ taking its values in $E$, and $\Omega_c(U, E)$ be the set of elements of $\Omega(U, E)$ whose coefficients have compact supports. Of course

$$\Omega(U, E) = \bigoplus_1^n \Omega^i(U, E) \quad \text{and} \quad \Omega_c(U, E) = \bigoplus_1^n \Omega_c^i(U, E),$$

where $\Omega^i(U, E)$ (resp. $\Omega_c^i(U, E)$) have for elements the homogeneous forms of degree $i$ and belong to $\Omega(U, E)$ (resp. $\Omega_c(U, E)$).

Let $F$ be a Banach subspace of $E$; $\Omega(U, F) \subseteq \Omega(U, E)$ and $\Omega_c(U, F) \subseteq \Omega_c(U, E)$. And

$$\Omega(U, E/F) = \Omega(U, E)/\Omega(U, F), \qquad \Omega_c(U, E/F) = \Omega_c(U, E)/\Omega_c(U, F).$$

Finally, $\bar{\partial}$ is a linear mapping

$$\Omega(U, E) \to \Omega(U, E), \quad \Omega(U, F) \to \Omega(U, F), \quad \Omega_c(U, E) \to \Omega_c(U, E)$$
$$\text{and} \quad \Omega_c(U, F) \to \Omega_c(U, F).$$

And $\bar{\partial}$ induces linear mappings

$$\Omega(U, E/F) \to \Omega(U, E/F) \quad \text{and} \quad \Omega_c(U, E/F) \to \Omega_c(U, E/F)$$

which we call again $\bar{\partial}$. In all cases, we see that $\bar{\partial}^2 = 0$.

DEFINITION 1.

$$Z(U, E/F) = \operatorname{Ker}\big(\bar{\partial}, \Omega(U, E/F)\big), \qquad Z_c(U, E/F) = \operatorname{Ker}\big(\bar{\partial}, \Omega_c(U, E/F)\big),$$
$$B(U, E/F) = \operatorname{Im}\big(\bar{\partial}, \Omega(U, E/F)\big)$$
$$\text{and} \quad B_c(U, E/F) = \operatorname{Im}\big(\bar{\partial}, \Omega_c(U, E/F)\big).$$

With the above notations,

$$H^*(U, E/F) = Z(U, E/F)/B(U, E/F),$$
$$H_c^*(U, E/F) = Z_c(U, E/F)/B_c(U, E/F).$$

What must I add?

$$\Omega(U, E/F) = \bigoplus_{i=0}^n \Omega^i(U, E/F), \qquad \Omega_c(U, E/F) = \bigoplus_{i=0}^n \Omega_c^i(U, E/F),$$

$\bar{\partial}$ maps $\Omega^i$ into $\Omega^{i+1}$ and $\Omega_c^i$ into $\Omega_c^{i+1}$; $Z^i$, $Z_c^i$ are the kernels of $\bar{\partial}$, or better $\bar{\partial}_i$ on $\Omega^i$, $\Omega_c^i$, while $B^i$, $B_c^i$ are the images of $\Omega^{i-1}$, $\Omega_c^{i-1}$; $H^i = Z^i/B^i$, $H_c^i = Z_c^i/B_c^i$; finally

$$H^*(U, E/F) = \bigoplus H^i(U, E/F), \qquad H_c^*(U, E/F) = \bigoplus H_c^i(U, E/F).$$

It is also clear that $H^*$ and $H^*_c$ are isomorphic to $Z_1/B_1$ and $Z_{1c}/B_{1c}$ where

$$Z_1(U, E \bmod F) = \bar\partial^{-1}\big(\Omega(U, F)\big) \cap \Omega(U, E),$$
$$B_1(U, E \bmod F) = \bar\partial\Omega(U, E) + \Omega(U, F)$$

and similarly for $Z_{1c}(U, E \bmod F)$ and $B_{1c}(U, E \bmod F)$.

Now, look at the mapping $\Omega_c(U, E) \to E$ defined by the integration

$$\omega \to \int_U \omega \wedge ds_1 \wedge \ldots \wedge ds_n.$$

The mapping maps $\Omega_c(U, F)$ into $F$ and $\bar\partial\Omega_c(U, E)$ onto 0.

Definition 2. $I$: $H^n_c(U, E/F) \to E/F$ is induced by the mapping

$$\omega \to \int \omega \wedge ds_1 \wedge \ldots \wedge ds_n, \quad Z^n_{1c}(U, E \bmod F) \to E.$$

A couple of more remarks. Let $\mathscr{A}$ be a Banach algebra and $\alpha$ a (two sided) Banach ideal. Then $\Omega(U, \mathscr{A}/\alpha)$ is an algebra and $\Omega_c(U, \mathscr{A}/\alpha)$ is an ideal.

Proposition 1. *$H^*(U, \mathscr{A}/\alpha)$ is an algebra, and $H^*_c(U, \mathscr{A}/\alpha)$ is a module on $H^*(U, \mathscr{A}/\alpha)$.*

Let $U$, $V$ be open in $C^n$, $C^m$. Direct multiplication $(\omega_1, \omega_2) \to \omega_1 \wedge \omega_2$ maps $\Omega(U, \mathscr{A}/\alpha) \times \Omega(V, \mathscr{A}/\alpha)$ into $\Omega(U \times V, \mathscr{A}/\alpha)$ and $\Omega_c(U, \mathscr{A}/\alpha) \times \Omega_c(V, \mathscr{A}/\alpha)$ into $\Omega_c(U \times V, \mathscr{A}/\alpha)$.

Definition 3. Direct multiplication

$$H^*(U, \mathscr{A}/\alpha) \times H^*(V, \mathscr{A}/\alpha) \to H^*(U \times V, \mathscr{A}/\alpha)$$

and

$$H^*_c(U, \mathscr{A}/\alpha) \times H^*_c(V, \mathscr{A}/\alpha) \to H^*_c(U \times V, \mathscr{A}/\alpha)$$

are the bilinear mappings induced by $(\omega_1, \omega_2) \to \omega_1 \wedge \omega_2$.

Still another definition must be given. Let $U \subseteq C^n$ be open, let $E/F$ be a quotient Banach space and $\mathscr{A}/\alpha$ be a quotient Banach algebra.

Definition 4.

$$\mathcal{O}(U, E/F) = \mathcal{O}(U, E)/\mathcal{O}(U, F) \quad \text{and} \quad \mathcal{O}(U, \mathscr{A}/\alpha) = \mathcal{O}(U, \mathscr{A})/\mathcal{O}(U, \alpha).$$

$\mathcal{O}(U, E/F)$ is a vector space, $\mathcal{O}(U, \mathscr{A}/\alpha)$ is an algebra, both depend functorially on $E/F$ and $\mathscr{A}/\alpha$ [5].

**2.** Let $\mathscr{A}$, $\| \ \|$ be an associative and commutative Banach algebra with a unit. Let $\alpha$, $\| \ \|_\alpha$ be a Banach ideal of $\mathscr{A}$. Let $\bar a_1, \ldots, \bar a_n$ be elements of $\mathscr{A}/\alpha$ and $a_1, \ldots, a_n$ elements of the equivalence classes, $a_i \in \bar a_i$.

Definition 5. The spectrum $\operatorname{sp}(\bar a_1, \ldots, \bar a_n)$ of $(\bar a_1, \ldots, \bar a_n)$, equal to the spectrum of $(a_1, \ldots, a_n)$ modulo $\alpha$, $\operatorname{sp}_\alpha(a_1, \ldots, a_n)$, is the set of $(s_1, \ldots s_n) \in C^n$ such that



$$\sum_1^n (\bar a_i - s_i)\mathscr{A}/\alpha = \mathscr{A}/\alpha, \qquad \sum_1^n (a_i - s_i)\mathscr{A} + \alpha = \mathscr{A}.$$

Of course, $\operatorname{sp}_\alpha(a_1, \ldots, a_n)$ is compact, non empty (if $\alpha$ is a proper ideal). If $\operatorname{Hull}\alpha$ is the set of maximal ideals which contain $\alpha$, if $\hat a \in C(\mathscr{M})$ is the Gelfand transform of $a \in \mathscr{A}$ and $\mathscr{M}$ the structure space of $\mathscr{A}$, then $\operatorname{sp}_\alpha(a_1, \ldots, a_n) = \operatorname{sp}(\bar a_1, \ldots, \bar a_n)$ is the image of $\operatorname{Hull}\alpha$ by $(\hat a_1, \ldots, \hat a_n)$.

Let $U$ be an open subset of $C^n$ which contains $\operatorname{sp}_\alpha(a_1, \ldots, a_n)$.

Proposition 2. *Functions $u_1, \ldots, u_n, v, y$ exist on $C^n$, where $u_1, \ldots, u_n, y$ are of class $C_\infty$ and $\mathscr{A}$-valued, $y$ having compact support in $U$, while $v$ is of class $C_\infty$, $\alpha$-valued, and*

$$1 = \sum_1^n (a_i - s_i)u_i(s) + v(s) + y(s)$$

*for all $s \in C^n$.*

Note that $v$ is $\alpha$-valued, and of class $C_\infty$ as an $\alpha$-valued function.

The proof is nearly classical. Let $(s_1, \ldots, s_n) \in C^n \setminus \operatorname{sp}_\alpha(a_1, \ldots, a_n)$. Elements $u_{1s}, \ldots, u_{ns}, v_s$ of $\mathscr{A}$, resp. $\alpha$ exist such that

$$1 = \sum_1^n (a_i - s_i)u_{is} + v_s.$$

If $s'$ is near to $s$,

$$1 = \sum_1^n (a_i - s'_i)u_{is}(s') + v_s(s')$$

with

$$u_{is}(s') = u_{is} \cdot \Big(1 - \sum_1^n (s'_i - s_i)a_{is}\Big)^{-1}, \qquad v_s(s') = v_s \cdot \Big(1 - \sum_1^n (s'_i - s_i)a_{is}\Big)^{-1}.$$

These functions are of class $C_\infty$, respectively $\mathscr{A}$-valued and $\alpha$-valued.

Let also

$$u_{i,\infty}(s') = -\bar s'_i\Big(|s'|^2 - \sum \bar s'_i a_i\Big)^{-1}$$

and $v_\infty(s') = 0$ when $s'$ is near to infinity.

A partition of unity of class $C_\infty$ yields the functions $u_i(s)$, $v(s)$, $y(s)$ announced in Proposition 2.

Definition 6. Let $a_1, \ldots, a_n$ be elements of $\mathscr{A}$ and $U$ an open subset of $C^n$, $U \supseteq \operatorname{sp}_\alpha(a_1, \ldots, a_n)$. Then $D_U(a_1, \ldots, a_n)$ will be the set of $(u_1, \ldots, u_n, v, y)$ where $u_i$, $y$ are of class $C_\infty$, $\mathscr{A}$-valued, $y$ of compact support in $U$, and $v$ is of class $C_\infty$, $a$-valued, where $1 = \sum_1^n (a_i - s_i) u_i(s) + v(s) + y(s)$.

Proposition 2 shows that $D_U(a_1, \ldots, a_n)$ is not empty.

Proposition 3. *Let* $(u, v, y)$ *and* $(u', v', y')$ *belong to* $D_U(a_1, \ldots, a_n)$. *Then*

$$u_i' = u_i + \sum_j \varphi_{ij}(a_j - s_j) + \psi_i + \eta_i,$$

$$v' = v - \sum_j \psi_j (a_j - s_j) - \eta,$$

$$y' = y - \sum_j \eta_j (a_j - s_j) + \eta,$$

where $\varphi_{ij} = -\varphi_{ji}$ and $\eta_i$ are $\mathscr{A}$-valued of class $C_\infty$; $\psi_i, \eta$ are $a$-valued of class $C_\infty$; $\eta_i, \eta$ have compact supports in $U$. Also, let $(u, v, y) \in D_U(a_1, \ldots, a_n)$, let $\varphi_{ij}, \psi_i, \eta_i, \eta$ be functions such as above and let $(u', v', y')$ determined as above. Then $(u', v', y') \in D_U(a_1, \ldots, a_n)$.

It is clear that $(u', v', y') \in D_U(a_1, \ldots, a_n)$ if $(u, v, y) \in D_U(a_1, \ldots, a_n)$; and $(u', v', y')$ is obtained from $(u, v, y)$ by the relations above. In the other direction, let $(u, v, y)$ and $(u', v', y')$ belong to $D_U(a)$. Then

$$\begin{aligned} u_i' - u_i &= u_i'\Big(\sum (a_j - s_j) u_j + v + y\Big) - \Big(\sum (a_j - s_j) u_j' + v' + y'\Big) u_i \\ &= \sum (a_j - s_j)(u_i' u_j - u_j' u_i) + (u_i' v - v' u_i) + (u_i' y - y' u_i), \\ v' - v &= v' \Big(\sum (a_j - s_j) u_j + v + y\Big) - \Big(\sum (a_j - s_j) u_j' + v' + y'\Big) v \\ &= \sum (a_j - s_j)(v' u_j - u_j' v) + 0 + (v' y - y' v), \\ y' - y &= y' \Big(\sum (a_j - s_j) u_j + v + y\Big) - \Big(\sum (a_j - s_j) u_j' + v' + y'\Big) y \\ &= \sum (a_j - s_j)(y' u_j - u_j' y) + (y' v - v' y) + 0. \end{aligned}$$

We let

$$\varphi_{ij} = u_i' u_j - u_j' u_i, \qquad \psi_i = u_i' v - v' u_i, \qquad \eta_i = u_i' y - y' u_i, \qquad \eta = v' y - y' v$$

and Proposition 3 is proved.

This proposition can be rewritten

Proposition 3′. *Let* $(u, v, y)$ *and* $(u', v', y')$ *be elements of* $D_U(a_1, \ldots, a_n)$. *We can find a finite chain of* $(u^{(r)}, v^{(r)}, y^{(r)})$ *of elements of* $D_U(a)$ *with*

$$(u^{(0)}, v^{(0)}, y^{(0)}) = (u, v, y), \qquad (u^{(N)}, v^{(N)}, y^{(N)}) = (u', v', y'),$$



*and for all* $r$ $(0 \leqslant r < N)$ *we take a step from* $(u^{(r)}, v^{(r)}, y^{(r)})$ *to* $(u^{(r+1)}, v^{(r+1)}, y^{(r+1)})$ *of one of the following forms*

$$\begin{aligned} \text{A:}\quad & u_i^{(r+1)} = u_i^{(r)} + (a_j - s_j)\varphi, \qquad u_j^{(r+1)} = u_j^{(r)} - (a_i - s_i)\varphi, \\ & u_k^{(r+1)} = u_k^{(r)} \quad (k \neq i \text{ and } k \neq j), \qquad v' = v \quad \text{and} \quad y' = y, \\ \text{B:}\quad & u_i^{(r+1)} = u_i^{(r)} + \eta, \qquad y^{(r+1)} = y^{(r)} - (a_i - s_i)\eta, \\ & u_j^{(r+1)} = u_j^{(r)} \quad (j \neq i), \qquad v^{(r+1)} = v^{(r)}, \\ \text{C:}\quad & u_i^{(r+1)} = u_i^{(r)} + \psi_i, \qquad v^{(r+1)} = v^{(r)} - \sum (a_j - s_j)\psi_j - \eta', \qquad y^{(r+1)} = y^{(r)} + \eta', \end{aligned}$$

*where* $\varphi, \eta$ *are of class* $C_\infty$ *as* $\mathscr{A}$*-valued,* $\psi_i, \eta'$ *are of class* $C_\infty$ *as* $a$*-valued, and where* $\eta$ *and* $\eta'$ *have compact support in* $U$.

**3.** Let $a_1, \ldots, a_n$ be elements of $\mathscr{A}$ and $U$ an open subset of $\boldsymbol{C}^n$ containing $\mathrm{sp}_a(a_1, \ldots, a_n)$. Let $(u, v, y) \in D_U(a)$ and let $k \in \boldsymbol{N}$, $k > 0$.

Proposition 4. *The class of* $\bar\partial$*-cohomology of*

$$\frac{(n+k)!}{k!} y^k \bar\partial u_1 \wedge \ldots \wedge \bar\partial u_n,$$

*with compact supports in* $U$ *and values in* $\mathscr{A}$ *modulo* $a$ *does not depend on* $k$, *nor on* $u_1, \ldots, u_n$, *but only on* $a_1, \ldots, a_n$.

This means that

$$\frac{(n+k')!}{k'!} y'^{k'} \bar\partial u_1' \wedge \ldots \wedge \bar\partial u_n' - \frac{(n+k)!}{k!} y^k \bar\partial u_1 \wedge \ldots \wedge \bar\partial u_n = \bar\partial Y + Z,$$

where $Y \in \Omega_c^{n-1}(U, \mathscr{A})$ and $Z \in \Omega_c^n(U, a)$.

Definition 7. $\tau_U(a_1, \ldots, a_n) \in H_c^n(U, \mathscr{A}/a)$ is the class of cohomology of

$$\frac{(n+k)!}{k!} y^k \bar\partial u_1 \wedge \ldots \wedge \bar\partial u_n$$

described in Proposition 4.

(a) Consider first the case where $(u, v, y) = (u', v', y')$ and where $k' = k-1$, $k' > 0$. Let

$$\tilde\omega = \sum (-1)^{i-1} u_i \bar\partial u_1 \wedge \ldots \wedge \bar\partial u_{i-1} \wedge \bar\partial u_{i+1} \wedge \ldots \wedge \bar\partial u_n.$$

It is clear that

$$\bar\partial \tilde\omega = n \bar\partial u_1 \wedge \ldots \wedge \bar\partial u_n.$$

Also

$$1 = \sum (a_i - s_i) u_i + v + y.$$

Differentiating

$$0 = \sum (a_i - s_i) \bar\partial u_i + \bar\partial v + \bar\partial y.$$

Of course

$$(a_i - s_i) \bar\partial u_i \wedge \tilde\omega = \sum (a_i - s_i) u_i \cdot \bar\partial u_1 \wedge \ldots \wedge \bar\partial u_n.$$

In this relation we replace $\sum (a_i - s_i) \bar\partial u_i$ and $\sum (a_i - s_i) u_i$ by $-\bar\partial v - \bar\partial y$ and $1 - v - y$, respectively. Transposing, changing signs, we see that

$$\bar\partial y \wedge \tilde\omega = (y-1) \bar\partial u_1 \wedge \ldots \wedge \bar\partial u_n + \psi_1,$$

where $\psi_1$ is $\alpha$-valued of class $C_\infty$. Therefore

$$\bar\partial (y^k \tilde\omega) = k y^{k-1} \bar\partial y \wedge \tilde\omega + n y^k \bar\partial u_1 \wedge \ldots \wedge \bar\partial u_n$$

$$= (n+k) y^k \bar\partial u_1 \wedge \ldots \wedge \bar\partial u_n - k y^{k-1} \bar\partial u_1 \wedge \ldots \wedge \bar\partial u_n + \psi_2,$$

where again $\psi_2$ is $\alpha$-valued, of class $C_\infty$ and with compact support in $U$. Multiplying this relation by $(n+k-1)!/k!$, we obtain the result when $(u, v, y) = (u', v', y')$.

(b) Assume that $k = k'$. Consider

$$y^k \bar\partial u_1 \wedge \ldots \wedge \bar\partial u_n - y'^k \bar\partial u'_1 \wedge \ldots \wedge \bar\partial u'_n.$$

Proposition 3′ shows that it is sufficient to prove that the expression is equal to $\bar\partial\theta + \psi$ with $\theta, \psi$ forms of class $C_\infty$, respectively $\mathscr{A}$-valued and $\alpha$-valued, both with compact support in $U$, when we go from $(u, v, y)$ to $(u', v', y')$ by one of the steps A, B, C.

Steps C are trivial, $\theta = 0$ even. Steps A are not much more difficult. Assume for example that

$$u'_1 = u_1 + (a_2 - s_2)\varphi, \qquad u'_2 = u_2 - (a_1 - s_1)\varphi,$$

and $u'_3 = u_3, \ldots, u'_n = u_n$, $v' = v$, $y' = y$. Then

$$y'^k \bar\partial u'_1 \wedge \ldots \wedge \bar\partial u'_n - y^k \bar\partial u_1 \wedge \ldots \wedge \bar\partial u_n$$

$$= y^k \big((a_2 - s_2) \bar\partial\varphi \wedge \bar\partial u_2 - \bar\partial u_1 \wedge (a_1 - s_1) \bar\partial \varphi\big) \wedge \bar\partial u_3 \wedge \ldots \wedge \bar\partial u_n$$

$$= \frac{1}{k+1} \bar\partial (y^{k+1}) \wedge \bar\partial\varphi \wedge \bar\partial u_3 \wedge \ldots \wedge \bar\partial u_n.$$



(c) We must still consider a step $B$, say $u'_1 = u_1 - \eta$, $y' = y + (a_1 - s_1)\eta$, $u'_2 = u_2, \ldots, u'_n = u_n$, $v' = v$. We observe that

$$y'^k \bar\partial u'_1 - y^k \bar\partial u_1 = \sum_1^k \binom{k}{k'} (a_1 - s_1)^{k-k'} \eta^{k-k'} y^{k'} \bar\partial u_1 -$$

$$- \sum_0^k \binom{k}{k'} (a_1 - s_1)^{k-k'} \eta^{k-k'} y^{k'} \bar\partial \eta$$

$$= \sum_0^{k-1} \frac{k!}{k'!(k-k'+1)!} (a_1 - s_1)^{k-k'} \varrho_{k'} - (a_1 - s_1)^k \eta^k \bar\partial\eta$$

with

$$\varrho_{k'} = k'(a_1 - s_1) \eta^{k-k'+1} y^{k'-1} \bar\partial u_1 - (k - k' + 1) \eta^{k-k'} y^{k'} \bar\partial\eta.$$

We look at $\varrho_{k'} \wedge \bar\partial u_2 \wedge \ldots \wedge \bar\partial u_n$, keeping in mind the fact that

$$(a_1 - s_1) \bar\partial u_1 \wedge \bar\partial u_2 \wedge \ldots \wedge \bar\partial u_n = -\bar\partial y \wedge \bar\partial u_2 \wedge \ldots \wedge \bar\partial u_n + \psi_1,$$

where $\psi_1$ is an $\alpha$-valued form of class $C_\infty$ with compact support in $U$, and hence

$$\varrho_{k'} \wedge \bar\partial u_2 \wedge \ldots \wedge \bar\partial u_n = -k' \eta^{k-k'+1} y^{k'-1} \bar\partial y \wedge \bar\partial u_2 \wedge \ldots \wedge \bar\partial u_n -$$

$$-(k - k' + 1) \eta^{k-k'} y^{k'} \bar\partial\eta \wedge \bar\partial u_2 \wedge \ldots \wedge \bar\partial u_n + \psi_2$$

$$= -\bar\partial (\eta^{k-k'+1} y^{k'}) \wedge \bar\partial u_2 \wedge \ldots \wedge \bar\partial u_n + \psi_2$$

so

$$y^k \bar\partial u_1 \wedge \ldots \wedge \bar\partial u_n - y'^k \bar\partial u'_1 \wedge \ldots \wedge \bar\partial u'_n = \bar\partial\theta + \psi.$$

**4.** Proposition 5. *Let $\bar a_1, \ldots, \bar a_n, \bar b_1, \ldots, \bar b_m$ be elements of $\mathscr{A}/\alpha$; let $U \supseteq \operatorname{sp}(\bar a_1, \ldots, \bar a_n)$ and $V \supseteq \operatorname{sp}(\bar b_1, \ldots, \bar b_m)$ be open in $C^n$, $C^m$, respectively. Then*

$$\tau_{U \times V}(\bar a_1, \ldots, \bar a_n, \bar b_1, \ldots, \bar b_m) = \tau_U(\bar a_1, \ldots, \bar a_n) \wedge \tau_V(\bar b_1, \ldots, \bar b_m).$$

Look at $a_i \in \bar a_i$, $b_j \in \bar b_j$, then $(u, v, y) \in D_U(a)$, $(u', v', y') \in D_V(b)$, i.e.

$$\sum_1^n (a_i - s_i) u_i(s) + v(s) + y(s) = 1,$$

$$\sum_1^m (b_j - t_j) u'_j(t) + v'(t) + y'(t) = 1.$$

Then $(u, yu', v + yv', yy') \in D_{U\times V}(a, b)$, i.e.

$$\sum (a_i - s_i) u_i(s) + \sum (b_j - t_j) y(s) u'_j(t) + \big(v(s) + y(s) v'(t)\big) + y(s) y'(t) = 1.$$

The class of cohomology $\tau_{U\times V}(a, b)$ contains the form

$$\frac{(n+m+k)!}{k!}(yy')^k\bar\partial u_1\wedge\ldots\wedge\bar\partial u_n\wedge\bar\partial(yu_1')\wedge\ldots\wedge\bar\partial(yu_m')$$
$$=\frac{(n+m+k)!}{k!}y^{m+k}\bar\partial u_1\wedge\ldots\wedge\bar\partial u_n\wedge y'^k\bar\partial u_1'\wedge\ldots\wedge\bar\partial u_m'$$

the equation being valid because $\bar\partial u_1\wedge\ldots\wedge\bar\partial u_n\wedge\bar\partial y=0$ since it is of degree $n+1$ and involves only $d\bar s_1,\ldots,d\bar s_n$.

The proposition is proved.

Proposition 6. *Let again $\bar a_1,\ldots,\bar a_n$ be elements of $\mathscr{A}/a$, and $U\subseteq \boldsymbol{C}^n$, $U\supseteq \operatorname{sp}(\bar a_1,\ldots,\bar a_n)$ Then*

$$\frac{1}{(2\pi i)^n}I\big(\tau_U(\bar a_1,\ldots,\bar a_n)\big)=\bar 1$$

*where $\bar 1$ is the unit of $\mathscr{A}/a$.*

Of course, $I\big(\tau_U(\bar a_1,\ldots,\bar a_n)\big)=I\big(\tau_{\boldsymbol{C}^n}(\bar a_1,\ldots,\bar a_n)\big)$ and this is the product of $n$ factors, each equal to $I\big(\tau_C(a_i)\big)$. Let $u$ be of class $C_\infty$, $\mathscr{A}$-valued, and such that $1-(a_1-s_1)u_1$ has compact support in $\boldsymbol{C}$. Then

$$I\big(\tau_C(\bar a_1)\big)\equiv\frac{(1+k)!}{k!}\int_C y^k\bar\partial u\wedge ds\equiv 2\pi i \mod a.$$

Definition 8. Let $U\supseteq\operatorname{sp}(\bar a_1,\ldots,\bar a_n)$ and $f\in\mathcal{O}(U,\mathscr{A}/a)$. Then

$$f[\bar a]=f[\bar a_1,\ldots,\bar a_n]=\frac{1}{(2\pi i)^n}I\big(f\cdot\tau_U(\bar a)\big).$$

We have already shown that $f[\bar a]\equiv 1 \mod a$ if $f$ is the constant function 1. Let $z_i$: $(s_1,\ldots,s_n)\to s_i$ be the $i$th coordinate.

Proposition 7. *$f[\bar a]=0$ iff $f$ belongs to the ideal generated by $(z_1-\bar a_1,\ldots$ $\ldots,z_n-\bar a_n)$ in $\mathcal{O}(U,\mathscr{A}/a)$.*

It is clear that

$$(z_1-a_1)y^{n+k}\bar\partial u_1\wedge\ldots\wedge\bar\partial u_n$$
$$=-\frac{1}{n+k+1}\bar\partial y^{n+k+1}\bar\partial u_2\wedge\ldots\wedge\bar\partial u_n-\bar\partial v\wedge\bar\partial u_2\wedge\ldots\wedge\bar\partial u_n$$

so $(z_1-\bar a_1)\tau_U(a)=0$ and similarly $(z_i-\bar a_i)\tau_U(\bar a)=0$. Let $f=\sum(z_i-\bar a_i)g_i$ with $g_i\in\mathcal{O}(U,\mathscr{A}/a)$. Then $f\cdot\tau_U(\bar a)=0$ and therefore $f[\bar a]=0$.



Let $U, V$ be open in $\boldsymbol{C}^n$, $\boldsymbol{C}^m$, respectively. Let $f\in\mathcal{O}(U,\mathscr{A}/a)$, $g\in\mathcal{O}(V,\mathscr{A}/a)$. Then the $f\times g$ maps $(s,t)$ onto $\big(f(s),g(t)\big)$.

Lemma 1. *Under the usual hypotheses,*

$$f\times g[\bar a,\bar b]=f[\bar a]g[\bar b].$$

This is clear. We want to prove that

Proposition 8. *Let $U\subseteq\boldsymbol{C}^n$ be open, $U\supseteq\operatorname{sp}(\bar a_1,\ldots,\bar a_n)$. The linear mapping $\mathcal{O}(U,\mathscr{A}/a)\to\mathscr{A}/a$ is multiplicative, i.e. it is a morphism of algebras.*

To prove Proposition 8, we shall use Lemma 1 above, and

Lemma 2. *Let $T$: $\boldsymbol{C}^n\to\boldsymbol{C}^m$ be linear, let $\bar a\in(\mathscr{A}/a)^n$ and $b=Ta\in(\mathscr{A}/a)^m$. Let $U$ be open in $\boldsymbol{C}^n$, $V$ open in $\boldsymbol{C}^m$, with $U\supseteq\operatorname{sp}\bar a$, $V\supseteq\operatorname{sp}\bar b$, $TU\subseteq V$. Let $g\in\mathcal{O}(U)$, then*

$$g[T\bar a]=(g\circ T)[\bar a]$$

Remember that $\operatorname{sp}\bar b=T\operatorname{sp}\bar a$.

Every linear mapping is the composition of an invertible mapping, a projection, an injection, and again an invertible mapping. It is sufficient to prove the lemma in the three following cases:

(a) $T$: $\boldsymbol{C}^n\to\boldsymbol{C}^n$ is invertible.

(b) $T$: $\boldsymbol{C}^n\to\boldsymbol{C}^{n-1}$ maps $(s_1,\ldots,s_n)$ onto $(s_1,\ldots,s_{n-1})$.

(c) $T$: $\boldsymbol{C}^n\to\boldsymbol{C}^{n+1}$ maps $(s_1,\ldots,s_n)$ onto $(s_1,\ldots,s_n,0)$.

So, (a), let $T$ be invertible, let $b_i=(Ta)_i=\sum T_{ij}a_j$, and $S=T^{-1}$. Let also ${}^tS$ be the transpose of $S$. We assume that

$$\sum(a_i-s_i)u_i(s)+v(s)+y(s)=1,$$

i.e.

$$\sum(b_i-Ts_i)({}^tSu)_i(s)+v(s)+y(s)=1$$

or again, with $t_i=(Ts)_i$

$$\sum(b_i-t_i){}^tSu(St)+v(St)+y(St)=1.$$

We let $u'={}^tS(uS)$, $y'=yS$. The kernel $\bar\partial u_1'\wedge\ldots\wedge\bar\partial u_n'$ evaluated at $t=Ts$ will be the product of $\bar\partial u_1\wedge\ldots\wedge\bar\partial u_n$ by

$$\det{}^tS\cdot\overline{\det S}=|\det S|^2.$$

One factor, $\det S$, comes from the replacement of

$$\bar\partial u_1'\wedge\ldots\wedge\bar\partial u_n'\quad\text{by}\quad\bar\partial u_1\wedge\ldots\wedge\bar\partial u_n$$

the other factor $\overline{\det S}$ comes from the change of variables; remember that $\bar\partial$ involves complex conjugation. Apply the formulae for the change of variables under multiple integrals.

Next, (b), $T$ maps $(s_1, \ldots, s_n)$ onto $(s_1, \ldots, s_{n-1})$ and $(\bar{a}_1, \ldots, \bar{a}_n)$ onto $(\bar{a}_1, \ldots, \bar{a}_{n-1})$. We have $T\operatorname{sp}\bar{a} = \operatorname{sp}T\bar{a}$. If $g \in \mathcal{O}(\operatorname{sp}T\bar{a})$, $g\circ T$ is the direct product $g\times 1$ of $g$ by the constant function, equal to 1, and depending only on $s_n$. And $g\circ T[\bar{a}_1, \ldots, \bar{a}_n] = g[\bar{a}_1, \ldots, \bar{a}_{n-1}]\cdot 1[\bar{a}_n] = g[T\bar{a}]$ as required.

Finally, (c), we must consider $T$: $(s_1, \ldots, s_n)\to(s_1, \ldots, s_n, 0)$, and we compare $\bar{a} \in (\mathcal{A}/a)^n$ and $(\bar{a}, 0) \in (\mathcal{A}/a)^{n+1}$. We notice that $\operatorname{sp}(\bar{a}, 0) = \operatorname{sp} a \times \{0\}$.

Let $V \subseteq C^{n+1}$ containing $\operatorname{sp}(\bar{a}, 0)$, and assume first that every connected component of $V\cap(C^n\times\{0\})$ meets $\operatorname{sp}(\bar{a}, 0)$. A function $g \in \mathcal{O}(V)$ can be written

$$g(z_1, \ldots, z_{n+1}) = g(z_1, \ldots, z_n, 0) + z_{n+1}h(z_1, \ldots, z_{n+1}),$$
$$= g\circ T(z)\times 1(z_{n+1}) + z_{n+1}h(z_1, \ldots, z_{n+1})$$

where $1(z_{n+1})$ is a function of one variable, $z_{n+1}$, constant and equal to 1, and

$$h(z_1, \ldots, z_{n+1}) = \frac{g(z_1, \ldots, z_{n+1}) - g(z_1, \ldots, z_n, 0)}{z_{n+1}}.$$

The function $h$ is holomorphic on $V$,

$$g[\bar{a}_1, \ldots, \bar{a}_n, 0] = g\circ T[\bar{a}]\cdot 1 + z_{n+1}(0)\cdot h[\bar{a}] = g\circ T[\bar{a}]$$

($z_{n+1}(0)$ is the value of $z_{n+1}$ at the origin).

If not all connected components of $V\cap C^n\times\{0\}$ meet $\operatorname{sp}(a, 0)$, we let $X$ be the union of the components of $V\cap C^n\times\{0\}$ which do not meet $\operatorname{sp}(\bar{a}, 0)$ and $V_1 = V\setminus X$. Then $V_1$ is open, contained in $V$, contains $\operatorname{sp}(\bar{a}, 0)$, $h$ is holomorphic on $V_1$, we can define $h[\bar{a}, 0]$. As above $g[\bar{a}, 0] = g\circ T[\bar{a}]$.

Proof of Proposition 8 is now easy. Let $f, g$ belong to $\mathcal{O}(U, \mathcal{A}/a)$. Then $f\times g(s, t) = f(s)g(t)$ and $f\times g[\bar{a}, \bar{a}] = f[\bar{a}]g[\bar{a}]$. On the other hand, consider $T$: $s\to(s, s)$, $C^n\to C^{2n}$, then

$$f\times g(s, s) = f\times g(Ts) = f\cdot g(s)$$

hence

$$f[\bar{a}]g[\bar{a}] = f\times g[\bar{a}, \bar{a}] = f\times g[Ta] = f\cdot g[\bar{a}].$$

Let $U$ be open in $C^m$, $U \supseteq \operatorname{sp}(\bar{a}_1, \ldots, \bar{a}_n)$. Let $g_1, \ldots, g_m$ be elements of $\mathcal{O}(U) = \mathcal{O}(U, C)$. Then $g_1[\bar{a}], \ldots, g_m[\bar{a}]$ belong to $\mathcal{A}/a$. The spectral mapping theorem shows that

$$\operatorname{sp}(g[\bar{a}]) = g(\operatorname{sp}\bar{a})$$

and

$$\operatorname{sp}(\bar{a}, g[\bar{a}]) = \{(z, g(z))|\ z \in \operatorname{sp}\bar{a}\}.$$



Proposition 9. *Let $V \subseteq C^m$ be open, $V \supseteq g(U)$ and $f \in \mathcal{O}(V, \mathcal{A}/a)$; then $f\circ g \in \mathcal{O}(U, \mathcal{A}/a)$ and*

$$f[g[\bar{a}]] = f\circ g[\bar{a}].$$

Consider the function $F(z, y) = f\circ g(z) - f(y)$. This function is holomorphic on a neighbourhood of $\operatorname{sp}(\bar{a}, g[\bar{a}])$ and vanishes on $\{(z, u(z))|\ z \in U\}$. Next, the functions

$$G_i(z, y) = \frac{F(z, y_1, \ldots, y_i, g_{i+1}, \ldots, g_m) - F(z, y_1, \ldots, y_{i-1}, g_i, \ldots, g_n)}{y_i - g_i(z)}$$

are holomorphic on a neighbourhood of $\operatorname{sp}(\bar{a}, g[\bar{a}])$, so $G_i[\bar{a}, g[\bar{a}]]$ is defined and

$$f[g[\bar{a}]] - f\circ g[\bar{a}] = F[\bar{a}, g[\bar{a}]] = \sum (g_i[\bar{a}] - g_i[\bar{a}])G_i[\bar{a}, g[\bar{a}]] = 0$$

so Proposition 9 is proved.

**5.** Let $q$: $\mathcal{A}\to\mathcal{A}/a$ be the quotient mapping $a_i\to\bar{a}_i = a_i + a$.

Definition 9. Let $U \subseteq C^n$ be open. A linear mapping $u$: $\mathcal{O}(U)\to\mathcal{A}/a$ *lifts* if a continuous linear mapping $u_1$: $\mathcal{O}(U)\to\mathcal{A}$ exists such that $u = q\circ u_1$.

It is clear that the mapping $f\to f[\bar{a}]$, $\mathcal{O}(U)\to\mathcal{A}/a$ lifts (Definition 8).

Proposition 10. *Let $U \subseteq C^n$ be a (schlicht) domain of holomorphy, $U \supseteq \operatorname{sp}(\bar{a}_1, \ldots, \bar{a}_n)$. Only one homomorphism $\mathcal{O}(U)\to\mathcal{A}/a$ exists, which lifts, and maps $z_i$ on $\bar{a}_i$ and unit on unit.*

Let $\sigma$: $\mathcal{O}(U)\to\mathcal{A}/a$ be the mapping $f\to f[a]$ and $\sigma_1$: $\mathcal{O}(U)\to\mathcal{A}$ a lifting, i.e. $\sigma = q\circ\sigma_1$. Let $\tau$: $\mathcal{O}(U)\to\mathcal{A}/a$ be another homomorphism, which lifts, i.e. $\tau = q\circ\tau_1$, and maps $z_i$ on $\bar{a}_i$ and unit on unit.

$\mathcal{O}(U)\hat{\otimes}\mathcal{O}(U) = \mathcal{O}(U\times U)$, where $\hat{\otimes}$ is the completed projective tensor product. $\sigma_1\otimes\tau_1$ maps $\mathcal{O}(U)\hat{\otimes}\mathcal{O}(U)$ into $\mathcal{A}\hat{\otimes}\mathcal{A}$, multiplications maps $\mathcal{A}\hat{\otimes}\mathcal{A}$ into $\mathcal{A}$, the composition $\mathcal{O}(U\times U)\to\mathcal{A}$ will be called $\varrho_1$, and $\varrho = q\circ\varrho_1$.

Of course, $(f, g)\to\sigma_1(f)\sigma_1(g) - \sigma_1(f\cdot g)$ and $(\varphi, \psi)\to\tau_1(\varphi)\tau_1(\psi) - \tau_1(\varphi\cdot\psi)$ are bilinear and continuous $\mathcal{O}(U)\times\mathcal{O}(U)\to a$ (because of the closed graph and Banach–Steinhaus theorems). Look at $F = \sum\lambda_n f_n\otimes\varphi_n$ and $G = \sum\mu_m g_m\otimes\psi_m$ elements of $\mathcal{O}(U\times U)$. Then

$$\begin{aligned}\varrho_1(F)\varrho_1(G) - \varrho_1(F\cdot G)&\\ &= \sum\lambda_n\mu_m[\sigma_1(f_n)\tau_1(\varphi_n)\sigma_1(g_m)\tau_1(\psi_m) - \sigma_1(f_n\cdot g_m)\tau_1(\varphi_n\cdot\psi_m)]\\ &= \sum\lambda_n\mu_m[(\sigma_1(f_n)\sigma_1(g_m) - \sigma_1(f_n g_m))\tau_1(\varphi_n)\tau_1(\psi_m)] +\\ &\quad + \sum\lambda_n\mu_m\sigma_1(f_n g_m)[\tau_1(\varphi_n)\tau_1(\psi_m) - \tau_1(\varphi_n\cdot\psi_m)] \in a.\end{aligned}$$

So $\varrho = q \circ \varrho_1$: $\mathcal{O}(U \times U) \to \mathcal{A}/\alpha$ is a homomorphism. This mapping maps $z_i \otimes 1 - 1 \otimes z_i$ onto 0.

$U$ is a domain of holomorphy. The restriction to the diagonal $\mathcal{O}(U \times U) \to \mathcal{O}(U)$ is surjective; its kernel is also generated by the $z_i \otimes 1 - 1 \otimes z_i$. The quotient is $\mathcal{O}(U)$, $\varrho(1 \otimes f) = \varrho(f \otimes 1) = \tau(f) = \sigma(f)$. And Proposition 10 is proved.

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# Quasi-Banach algebras, ideals, and holomorphic functional calculus

by

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**Abstract.** Quasi-Banach algebras are unions $\mathcal{A} = \bigcup \mathcal{A}_i$ of Banach algebras. Assume $\mathcal{A}$ to be commutative. All ideals of $\mathcal{A}$ turn out to be unions $\alpha = \bigcup \alpha_{ij}$ of Banach ideals. Results about relative holomorphic functional can be stretched, are applicable to quasi-Banach algebras modulo general ideals.

As application, we have a Šilov result. Let $\mathcal{A}$ be a commutative, associative general quasi-Banach algebra with unit, and $\alpha$ a general ideal. Assume that the hull, Hull $\alpha$ is not connected. Hull $\alpha = X_0 \cup X_1$. An $e \in \mathcal{A}$ then exists, idempotent modulo $\alpha$, $e^2 - e \in \alpha$, whose Gelfand transform $\hat{e}$ vanishes on $X_0$ and is equal to 1 on $X_1$. Or, if you prefer $\alpha = \alpha_0 \cap \alpha_1$ where Hull $\alpha_0 = X_0$, Hull $\alpha_1 = X_1$.

Let $\mathcal{A}$ be a commutative Banach, or quasi-Banach algebra with unit. Every ideal $\alpha$ of $\mathcal{A}$ has at least one quasi-Banach structure. Inductive limits allow us to stretch the holomorphic functional calculus from quotient Banach algebras to quotient quasi-Banach algebras. We define $f[\alpha]$ when $\bar{a}_1, \ldots, \bar{a}_n$ belong to $\mathcal{A}/\alpha$, $U \supseteq \operatorname{sp}(\bar{a}_1, \ldots, \bar{a}_n)$ is open in $C^n$ and $f \in \mathcal{O}(U, \mathcal{A}/\alpha)$.

As application, let $\alpha$ be an ideal of $\mathcal{A}$ whose hull is not connected, Hull $\alpha = X_0 \cup X_1$ with $X_0, X_1$ compact and disjoint. Then $e \in \mathcal{A}$ exists which is idempotent modulo $\alpha$, i.e. $e^2 - e \in \alpha$, and whose Gelfand transform $\hat{e}$ vanishes on $X_0$ and is equal to 1 on $X_1$. Ideals $\alpha_0, \alpha_1$ can be found, with Hull $\alpha_0 = X_0$, Hull $\alpha_1 = X_1$, and $\alpha = \alpha_0 \cap \alpha_1$. The ideal $\alpha$ is not assumed closed, the result generalises Šilov's, which applies when $\alpha$ is closed [4].

This paper originated from a discussion about $\mathcal{A}(D)$, the algebra of continuous functions on $D = \{z \mid |z| \leqslant 1\} \subseteq C$ which are holomorphic on the interior. Primary ideals are those whose hull has a single element. Those whose hull lies in the interior of $D$ are closed. What can we say about ideals whose hull is finite and contained in $\mathring{D}$? We see that these ideals are the intersections of closed primary ideals, so they are closed.

Arens's and Calderón's result [2] can also be generalised. Let $\bar{a}_1, \ldots, \bar{a}_n$ belong to $\mathcal{A}/\alpha$. Let $V \subseteq C^{n+1}$ be open, $F = F(z_1, \ldots, z_n, y) \in \mathcal{O}(V)$, $f \in C(\text{Hull}\,\alpha)$, with for all $m \in \text{Hull}\,\alpha$, $(\hat{a}(m), f(m)) \in V$, and again for all $m \in \text{Hull}\,\alpha$,

$$F(\hat{a}(m), f(m)) = 0, \qquad \frac{\partial F}{\partial y}(\hat{a}(m), f(m)) \neq 0.$$