So $\varrho = q\circ\varrho_1$: $\mathcal{O}(U\times U)\to\mathcal{A}/\mathfrak{a}$ is a homomorphism. This mapping maps $z_i\otimes 1-1\otimes z_i$ onto 0.

U is a domain of holomorphy. The restriction to the diagonal $\mathcal{O}(U\times U)\to\mathcal{O}(U)$ is surjective; its kernel is also generated by the $z_i\otimes 1-1\otimes z_i$. The quotient is $\mathcal{O}(U)$, $\varrho(1\otimes f)=\varrho(f\otimes 1)=\tau(f)=\sigma(f)$. And Proposition 10 is proved.

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Quasi-Banach algebras, ideals, and holomorphic functional calculus

by

L. WAELBROECK (Bruxelles)

Abstract. Quasi-Banach algebras are unions $\mathcal{A}=\bigcup\mathcal{A}_i$ of Banach algebras. Assume $\mathcal{A}$ to be commutative. All ideals of $\mathcal{A}$ turn out to be unions $\mathfrak{a}=\bigcup\mathfrak{a}_{ij}$ of Banach ideals. Results about relative holomorphic functional can be stretched, are applicable to quasi-Banach algebras modulo general ideals.

As application, we have a Šilov result. Let $\mathcal{A}$ be a commutative, associative general quasi-Banach algebra with unit, and $\mathfrak{a}$ a general ideal. Assume that the hull, $\operatorname{Hull}\mathfrak{a}$ is not connected. $\operatorname{Hull}\mathfrak{a}=X_0\cup X_1$. An $e\in\mathcal{A}$ then exists, idempotent modulo $\mathfrak{a}$, $e^2-e\in\mathfrak{a}$, whose Gelfand transform $\hat{e}$ vanishes on X_0 and is equal to 1 on X_1. Or, if you prefer $\mathfrak{a}=\mathfrak{a}_0\cap\mathfrak{a}_1$ where $\operatorname{Hull}\mathfrak{a}_0=X_0$, $\operatorname{Hull}\mathfrak{a}_1=X_1$.

Let $\mathcal{A}$ be a commutative Banach, or quasi-Banach algebra with unit. Every ideal $\mathfrak{a}$ of $\mathcal{A}$ has at least one quasi-Banach structure. Inductive limits allow us to stretch the holomorphic functional calculus from quotient Banach algebras to quotient quasi-Banach algebras. We define $f[\mathfrak{a}]$ when $\bar{a}_1,\ldots,\bar{a}_n$ belong to $\mathcal{A}/\mathfrak{a}$, $U\supseteq\operatorname{sp}(\bar{a}_1,\ldots,\bar{a}_n)$ is open in C^n and $f\in\mathcal{O}(U,\mathcal{A}/\mathfrak{a})$.

As application, let $\mathfrak{a}$ be an ideal of $\mathcal{A}$ whose hull is not connected, $\operatorname{Hull}\mathfrak{a}=X_0\cup X_1$ with X_0, X_1 compact and disjoint. Then $e\in\mathcal{A}$ exists which is idempotent modulo $\mathfrak{a}$, i.e. $e^2-e\in\mathfrak{a}$, and whose Gelfand transform $\hat{e}$ vanishes on X_0 and is equal to 1 on X_1. Ideals $\mathfrak{a}_0$, $\mathfrak{a}_1$ can be found, with $\operatorname{Hull}\mathfrak{a}_0=X_0$, $\operatorname{Hull}\mathfrak{a}_1=X_1$, and $\mathfrak{a}=\mathfrak{a}_0\cap\mathfrak{a}_1$. The ideal $\mathfrak{a}$ is not assumed closed, the result generalises Šilov's, which applies when $\mathfrak{a}$ is closed [4].

This paper originated from a discussion about $\mathcal{A}(D)$, the algebra of continuous functions on $D=\{z \mid |z|\leqslant 1\}\subseteq C$ which are holomorphic on the interior. Primary ideals are those whose hull has a single element. Those whose hull lies in the interior of D are closed. What can we say about ideals whose hull is finite and contained in $\mathring{D}$? We see that these ideals are the intersections of closed primary ideals, so they are closed.

Arens's and Calderón's result [2] can also be generalised. Let $\bar{a}_1,\ldots,\bar{a}_n$ belong to $\mathcal{A}/\mathfrak{a}$. Let $V\subseteq C^{n+1}$ be open, $F=F(z_1,\ldots,z_n,y)\in\mathcal{O}(V)$, $f\in C(\operatorname{Hull}\mathfrak{a})$, with for all $m\in\operatorname{Hull}\mathfrak{a}$, $(\hat{a}(m),f(m))\in V$, and again for all $m\in\operatorname{Hull}\mathfrak{a}$,

$$F\big(\hat{a}(m),f(m)\big)=0,\qquad \frac{\partial F}{\partial y}\big(\hat{a}(m),f(m)\big)\neq 0.$$

Then $\bar{b} \in \mathscr{A}/\alpha$ exists such that $\hat{b}(m) = f(m)$ on $\mathrm{Hull}\,\alpha$, and $F[\bar{a}_1, \ldots, \bar{a}_n, b] = 0$ in $\mathscr{A}/\alpha$.

The above statement is abstract. But consider a holomorphic function $F(z, y)$ on the polydisc $\mathring{D}^{n+1} \subseteq C^{n+1}$, write

$$F(z, y) = \sum F_{k_1 \ldots k_n l} z_1^{k_1} \ldots z_n^{k_n} y^l.$$

Let $a_1, \ldots, a_n \in \mathscr{A}$ with $\|a_i\| < 1$ and $f \in C(\mathrm{Hull}\,\alpha)$, for all $m \in \mathrm{Hull}\,\alpha$, $|f(m)| < 1$. Assume that

$$F\big(\hat{a}(m), f(m)\big) = 0, \qquad \frac{\partial F}{\partial y}\big(\hat{a}(m), f(m)\big) \neq 0;$$

both for all $m \in \mathrm{Hull}\,\alpha$. Then $b \in \mathscr{A}$ exists such that

$$\sum F_{k_1 \ldots k_n} a_1^{k_1} \ldots a_n^{k_n} b^l = 0.$$

For example, if $F(z, y) = e^y - z$, and if $\hat{a}(m)$ has a continuous logarithm on $\mathrm{Hull}\,\alpha$, we find a "logarithm of a modulo α". Or if F is a polynomial; which determines an analytic algebraic function on $\mathrm{Hull}\,\alpha$, we find $b \in \mathscr{A}$ such that $F(a, b) \in \alpha$.

1. Definition 1. A quasi-Banach algebra $\mathscr{A} = \bigcup_{r \in R} \mathscr{A}_r$ is a union of Banach algebras $\mathscr{A}_r$, $r \in R$, where R is a directed set, and $\mathscr{A}_r \subseteq \mathscr{A}_{r'}$, when $r \leqslant r'$, inclusion being a morphism of Banach algebras.

A directed set R is a partially ordered set such for all $r_1, r_2 \in R$, there exists $r_3 \in R$ such that $r_3 \geqslant r_1, r_3 \geqslant r_2$.

Quasi-Banach algebras were considered by Allan, Dales, and McClure [1], with the same terminology as here. Another approach to the same algebras exists. Let $\mathscr{A}$ be an algebra, with an algebra boundedness (bornology), $\mathscr{B}$, in particular $B_1 \cdot B_2 \in \mathscr{B}$, if $B_1 \in \mathscr{B}$, $B_2 \in \mathscr{B}$. Assume that the boundedness is convex, separated, and with a basis of completant subsets B, i.e. which are absolutely convex and such that the normed space $\mathscr{A}_B$ is a Banach space. Then $(\mathscr{A}, \mathscr{B})$ is a b-algebra (or a bornological complete algebra).

A set B is idempotent if $B^2 \subseteq B$. A boundedness is idempotent if for all $B \in \mathscr{B}$, some idempotent set $B_1 \in \mathscr{B}$ exists and some $M \in \boldsymbol{R}_+$ such that $B \subseteq MB_1$. Hogbe-Nlend [3] studies b-algebras with idempotent boundednesses; these are essentially the same as the above quasi-Banach algebras. (See also [5], [6].)

Definition 2. A *quasi-Banach ideal* of $\mathscr{A} = \bigcup_{r \in R} \mathscr{A}_r$ is a union $\alpha = \bigcup_{rs} \alpha_{rs}$ with $r \in R$, $s \in S_r$, each S_r is a directed set, $S_r \subseteq S_{r'}$, when $r \leqslant r'$, each α_{rs} is a Banach ideal of $\mathscr{A}_r$ and $\alpha_{rs} \subseteq \alpha_{r's'}$ when $r \leqslant r'$, $s \leqslant s'$.

Look at $(\mathscr{A}, \mathscr{B})$, a b-algebra. A b-ideal α of $\mathscr{A}$ is an ideal of $\mathscr{A}$, with a b-space boundedness $\mathscr{B}_\alpha$, stronger that $\mathscr{B}$, i.e. $\mathscr{B}_\alpha \subseteq \mathscr{B}$, and such that the multiplication $\mathscr{A} \times \alpha \to \alpha$ is a bounded bilinear mapping, i.e. $B_1 \cdot B_2 \in \mathscr{B}_\alpha$ when $B_1 \in \mathscr{B}$, $B_2 \in \mathscr{B}_\alpha$. (Here α is a left ideal; in this paper $\mathscr{A}$ is commutative, α is therefore two-sided.)

Assume that $(\mathscr{A}, \mathscr{B})$ has an idempotent boundedness, equivalently that $\mathscr{A} = \bigcup_r \mathscr{A}_r$ is quasi-Banach. Let α be a b-ideal of $\mathscr{A}$. For each $r \in R$, let S_r be the set of $B \in \mathscr{B}$ which are bounded in $\mathscr{A}_r$, are completant, and such that $B_r \cdot B \subseteq B$ if B_r is the unit ball of $\mathscr{A}_r$. Write $s = B \in S_r$, and let $\alpha_{rs} = \mathscr{A}_B$. Then α_{rs} is a Banach ideal of $\mathscr{A}_r$ and the pseudo-Banach structure of α is essentially that of $\bigcup_r \alpha_{rs}$. ($\mathscr{A}_B$ is the Banach space absorbed by $B = s \in S_r$, B is completant and $B_r \cdot B \subseteq B$.)

In other words, an algebra $\mathscr{A}$ with an idempotent boundedness $\mathscr{B}$ and a b-ideal α of $\mathscr{A}$ is essentially the same as a quasi Banach algebra $\mathscr{A}$ and a quasi-Banach ideal α of $\mathscr{A}$.

Proposition 1. *Let $\mathscr{A} = \bigcup_r \mathscr{A}_r$ be a commutative quasi-Banach algebra with unit. Let α be an ideal of $\mathscr{A}$. Then α has at least one quasi-Banach structure $\alpha = \bigcup_{rs} \alpha_{rs}$.*

A finitely generated ideal β of a Banach algebra $\mathfrak{B}$ is clearly a Banach ideal of $\mathfrak{B}$. Now look at $\mathscr{A} = \bigcup \mathscr{A}_r$ and an ideal α of $\mathscr{A}$. For each $r \in R$, let S_r be the set of finite sets of $\alpha \cap \mathscr{A}_r$, and for all $r \in S_r$, let α_{rs} be the ideal of $\mathscr{A}_r$ generated by s. Then $\alpha = \bigcup \alpha_{rs}$ is a quasi-Banach ideal of $\mathscr{A}$.

2. Let now $\mathscr{A} = \bigcup \mathscr{A}_r$ be a pseudo-Banach algebra. Let $\alpha = \bigcup \alpha_{rs}$ be a Banach ideal of $\mathscr{A}$. Let $\bar{a}_1, \ldots, \bar{a}_n$ be elements of $\mathscr{A}/\alpha$ and $a_i \in \bar{a}_i$.

Definition 3. The spectrum $\mathrm{sp}(\bar{a}_1, \ldots, \bar{a}_n) = \mathrm{sp}_\alpha(a_1, \ldots, a_n)$, in $\mathscr{A}/\alpha$ or equivalently the spectrum of $(a_1, \ldots, a_n)$ in $\mathscr{A}$ modulo α is the set of $(s_1, \ldots, s_n) \in C^n$ such that

$$1 \notin \sum_1^n (\bar{a}_i - s_i) \mathscr{A}/\alpha$$

or equivalently

$$1 \notin \sum_1^n (a_i - s_i) \mathscr{A} + \alpha.$$

It is clear that $\mathrm{sp}(\bar{a}_1, \ldots, \bar{a}_n)$ is compact in C^n, and if α is proper, it is not empty. Consider $\mathscr{M}$, the set of maximal ideals of $\mathscr{A}$, with its Gelfand topology,

$$\mathrm{Hull}\,\alpha = \{m \in \mathscr{M} \mid m \supseteq \alpha\}$$

is compact, non empty, and if $\hat{a} \in C(\mathscr{M})$ is the Gelfand transform of $a \in \mathscr{A}$, then

$$\operatorname{sp}(\bar{a}_1, \ldots, \bar{a}_n) = \{(\hat{a}_1(m), \ldots, \hat{a}_n(m)) \mid m \in \operatorname{Hull} \mathfrak{a}\}.$$

Let now r be large enough, that for all i, $a_i \in \mathscr{A}_r$. Consider

$$\operatorname{sp}_{\mathfrak{a}_{rs}}(a_1, \ldots, a_n) = X_{rs}.$$

Then $X_{rs} \subseteq C^n$ is compact; if $r' \geqslant r$, $s' \geqslant s$ then $X_{r's'} \subseteq X_{rs}$; also $\bigcap_{rs} X_{rs} = \operatorname{sp}_{\mathfrak{a}}(a_1, \ldots, a_n)$. So by compacity we have

PROPOSITION 2. *Let $U \subseteq C^n$ be open, $U \supseteq \operatorname{sp}(\bar{a}_1, \ldots, \bar{a}_n)$. Then r_0, s_0 exist such that*

$$\operatorname{sp}_{\mathfrak{a}_{rs}}(a_1, \ldots, a_n) \subseteq U$$

when $r \geqslant r_0$, $s \geqslant s_0$.

DEFINITION 4. Let $\mathscr{A} = \bigcup A_r$ be a quasi-Banach algebra and $\mathfrak{a} = \bigcup \mathfrak{a}_{rs}$ a quasi-Banach ideal of $\mathscr{A}$. Let $U \subseteq C^n$ be open. Then $\mathscr{O}(U, \mathscr{A}) = \bigcup \mathscr{O}(U, \mathscr{A}_r)$ and $\mathscr{O}(U, \mathfrak{a}) = \bigcup \mathscr{O}(U, \mathfrak{a}_{rs})$. Further, $\mathscr{O}(U, \mathscr{A}/\mathfrak{a}) = \mathscr{O}(U, \mathscr{A})/ /\mathscr{O}(U, \mathfrak{a})$.

Consider now $\bar{a}_1, \ldots, \bar{a}_n \in \mathscr{A}/\mathfrak{a}$, and $U \subseteq C^n$ open, $U \supseteq \operatorname{sp}(\bar{a}_1, \ldots, \bar{a}_n)$. As usual, let $z_i \in \mathscr{O}(U) \subseteq \mathscr{O}(U, \mathscr{A}/\mathfrak{a})$ be the holomorphic mappings $z_i\colon (s_1, \ldots, s_n) \to s_i$, and call 1 the constant function equal to 1.

PROPOSITION 3. *A morphism $\mathscr{O}(U, \mathscr{A}/\mathfrak{a}) \to \mathscr{A}/\mathfrak{a}$ exists which maps z_i onto $\bar{a}_i$ and 1 onto 1.*

This is clear. Let r, s be large enough, $\operatorname{sp}_{\mathfrak{a}_{rs}}(a_1, \ldots, a_n) \subseteq U$. Proposition 8 [7] gives a morphism $\mathscr{O}(U, \mathscr{A}_r/\mathfrak{a}_{rs}) \to \mathscr{A}_r/\mathfrak{a}_{rs}$. If $r' \geqslant r$, $s' \geqslant s$ you have the obvious commutative diagram

$$\begin{tikzcd}
\mathscr{O}(U, \mathscr{A}_r/\mathfrak{a}_{rs}) \arrow[r] \arrow[d] & \mathscr{A}_r/\mathfrak{a}_{rs} \arrow[d] \\
\mathscr{O}(U, \mathscr{A}_{r'}/\mathfrak{a}_{r's'}) \arrow[r] & \mathscr{A}_{r'}/\mathfrak{a}_{r's'}
\end{tikzcd}$$

The inductive limit of the $\mathscr{A}_r/\mathfrak{a}_{rs}$ is $\mathscr{A}/\mathfrak{a}$. The inductive limit of the $\mathscr{O}(U, \mathfrak{a}_r/\mathscr{A}_{rs})$ is $\mathscr{O}(U, \mathscr{A}/\mathfrak{a})$. And Proposition 3 is proved.

3. PROPOSITION 4. *Let $\mathfrak{a}$ be an ideal $\mathscr{A} = \bigcup \mathscr{A}_r$. Assume that its hull is nonconnected, $\operatorname{Hull} \mathfrak{a} = X_0 \cup X_1$, where X_0 and X_1 are compact and disjoint. Then $e \in \mathscr{A}$ exists, which is idempotent modulo $\mathfrak{a}$, $e^2 - e \in \mathfrak{a}$ and its Gelfand transform $\hat{e}$ vanishes on X_0 and is equal to 1 on X_1.*

The proof is classical. Let $\varphi \in C(\operatorname{Hull} \mathfrak{a})$ be equal to 0 on X_0 and to 1 on X. Stone–Weierstrass shows that φ can be approximated up to ε ($\varepsilon < 1/2$) on $\operatorname{Hull} \mathfrak{a}$ by a polynomial $P(y_1, \ldots, y_n, y_1^*, \ldots, y_n^*)$ of a finite number of $\hat{a}_1, \ldots, \hat{a}_n$, and conjugates of these, where the $\hat{a}_i$ are the Gelfand transforms of $a_i \in \mathscr{A}$. On $\operatorname{sp}(\bar{a}_1, \ldots, \bar{a}_n)$, $P(y_1, \ldots, y_n, y_1^*, \ldots, y_n^*)$ takes values less than ε on $\{(\hat{a}_1(m), \ldots, \hat{a}_n(m)) \mid m \in X_0\}$, while $P-1$ takes values less than ε on $\{(\hat{a}_1(m), \ldots, \hat{a}_n(m)) \mid m \in X_1\}$. The function γ, equal to 0 near to $\hat{a}(X_0)$ and to 1 near to $\hat{a}(X_1)$, is holomorphic on a neighbourhood U of $\operatorname{sp}(\bar{a}_1, \ldots, \bar{a}_n)$. The relative holomorphic functional calculus maps γ onto $\bar{e} \in \mathscr{A}/\mathfrak{a}$ which is idempotent, if $e \in \bar{e}$, $e^2 - e \in \mathfrak{a}$, i.e. e is idempotent modulo $\mathfrak{a}$.

Of course, $\mathfrak{a} + e\mathscr{A} = \mathfrak{a}_0$ and $\mathfrak{a} + (1-e)\mathscr{A} = \mathfrak{a}_1$ are ideals of $\mathscr{A}$; $\operatorname{Hull} \mathfrak{a}_0 = X_0$, $\operatorname{Hull} \mathfrak{a}_1 = X_1$. If $a \in \mathfrak{a}$, then clearly $a \in \mathfrak{a}_0 \cap \mathfrak{a}_1$. Conversely, let $a \in \mathfrak{a}_0 \cap \mathfrak{a}_1$, then

$$a = a' + eb' = a'' + (1-e)b''$$

with $a', a'' \in \mathfrak{a}$, and $b', b'' \in \mathscr{A}$. And

$$a = (1-e)a + ea = (1-e)a' + ea'' + e(1-e)(b' + b'') \in \mathfrak{a},$$

so $\mathfrak{a} = \mathfrak{a}_0 \cap \mathfrak{a}_1$.

COROLLARY. *Let $\mathfrak{a}$ be the ideal considered in Proposition 4. Ideals $\mathfrak{a}_0$, $\mathfrak{a}_1$ exist with $\operatorname{Hull} \mathfrak{a}_0 = X_0$; $\operatorname{Hull} \mathfrak{a}_1 = X_1$, and $\mathfrak{a} = \mathfrak{a}_0 \cap \mathfrak{a}_1$.*

4. Let $\bar{a}_1, \ldots, \bar{a}_n \in \mathscr{A}/\mathfrak{a}$. Let $\varphi \in C(\operatorname{Hull} \mathfrak{a})$. Let V be open in C^{n+1}, for all $m \in \operatorname{Hull} \mathfrak{a}$, $(\hat{a}(m), \varphi(m)) \in V$, let $F \in \mathscr{O}(V)$, and for all $m \in \operatorname{Hull} \mathfrak{a}$, $F(\hat{a}(m), \varphi(m)) = 0$.

PROPOSITION 5. *An element $\bar{b} \in \mathscr{A}/\mathfrak{a}$ exists such that, for all $m \in \operatorname{Hull} \mathfrak{a}$, $\hat{b}(m) = \varphi(m)$ and*

$$F[a_1, \ldots, a_n, b] = 0 \qquad (\text{in } \mathscr{A}/\mathfrak{a}).$$

The proof of this result uses Arens–Calderón's trick. Locally on $\operatorname{Hull} \mathfrak{a}$ φ depends only on $(\hat{a}_1(m), \ldots, \hat{a}_n(m))$. A finite number of elements on $\mathscr{A}/\mathfrak{a}$, say $\bar{c}_1, \ldots, \bar{c}_k$ exist such that globally (on $\operatorname{Hull} \mathfrak{a}$), $\varphi(m) = \varphi(m')$ as soon as $\hat{a}_i(m) = \hat{a}_i(m')$, $c_j(m) = c_j(m')$ $(1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k)$.

An open set $V_1 \subseteq C^{n+k}$ exists such that $V_1 \supseteq \operatorname{sp}(\bar{a}_1, \ldots, \bar{a}_n, \bar{c}_1, \ldots, \bar{c}_k)$, V_1 is projected into V in C^n, and $\varphi_1 \in \mathscr{O}(V_1)$ exists on V_1 such that $F(z, y, \varphi_1(z, y)) = 0$ for all $(z, y) \in V_1$, and for all m, $\bar{\varphi}_1(\hat{a}(m), \hat{c}(m)) = \varphi(m)$.

Then $\varphi_1[\bar{a}, \bar{c}] = \bar{b}$ is the solution of the Arens–Calderón problem.

Added in proof. The discussion mentioned on p. 274 was of course with Y. Domar and happened at the Banach Center Institute, Fall 1977. In the proceedings, Y. Domar mentions the existence of the discussion and give applications, p. 244 (see Y. Domar, *Ideal structure of commutative Banach algebras*, in: *Spectral theory*, Banach Center Publications v. 8, 1982, 241–249).

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Projections onto gradient fields and L^p-estimates for degenerated elliptic operators

by

TADEUSZ IWANIEC (Warszawa)

Abstract. Let $L^m(\boldsymbol{R}^N, \boldsymbol{R}^N)$ be the space of all vector-valued functions $f: \boldsymbol{R}^N \to \boldsymbol{R}^N$, which are integrable with the power $m \geqslant 2$. Consider the subspace $D^m(\boldsymbol{R}^N)$ of all functions which are the gradients of scalar functions on $\boldsymbol{R}^N$. We study the closest point projection $P_m: L^m(\boldsymbol{R}^N, \boldsymbol{R}^N) \to D^m(\boldsymbol{R}^N)$. The main result of the paper is the inequality $\|P_m f\|_p \leqslant A_p \|f\|_p$, for any $p \geqslant m$. In the proof an inequality of Fefferman and Stein is used. As an application of the methods presented we give some regularity results on PDE's and quasiconformal mappings. In particular, we get a stronger version of the Gehring theorem on L^p-integrability of first derivatives of quasiconformal mappings.

Introduction and statement of the results. The main objects of this paper are the Lebesgue spaces $L^m(\boldsymbol{R}^N, \boldsymbol{R}^N)$, $1 \leqslant m < \infty$, of mappings f from $\boldsymbol{R}^N$ to $\boldsymbol{R}^N$ with the standard norm

$$\|f\|_m = \Big(\int |f(x)|^m dx\Big)^{1/m}$$

and their subspaces $D^m(\boldsymbol{R}^N)$ of gradient fields, i.e. of vector-functions of the form $f = \nabla u$, where ∇ is the gradient operator acting on locally integrable functions u for which we can define $f \in L^m(\boldsymbol{R}^N, \boldsymbol{R}^N)$ such that

$$\int \langle f, \varphi \rangle = -\int u \operatorname{div} \varphi$$

for any test function $\varphi \in C_0^\infty(\boldsymbol{R}^N, \boldsymbol{R}^N)$. Hereafter $\langle , \rangle$ is reserved for the scalar product in $\boldsymbol{R}^N$.

Our main results concern the L^p-estimates, $p \geqslant m$, for a projection

$$P_m: L^m(\boldsymbol{R}^N, \boldsymbol{R}^N) \to D^m(\boldsymbol{R}^N), \qquad m \geqslant 2.$$

The interest in bounding such projections is motivated by a number of applications we give to problems of regularity in PDE and quasiconformal mappings.

Let us first consider $m = 2$; then $L^2(\boldsymbol{R}^N, \boldsymbol{R}^N)$ and $D^2(\boldsymbol{R}^N)$ are Hilbert spaces and the orthogonal projection $P: L^2(\boldsymbol{R}^N, \boldsymbol{R}^N) \to D^2(\boldsymbol{R}^N)$ is linear. Therefore, for any $f \in L^2(\boldsymbol{R}^N, \boldsymbol{R}^N)$ we have $Pf = \nabla u$ with an u which mini-