où la sommation est effectuée pour tous les $s' \in \{0, 1\}^{q-n}$. On a donc, d'après (4.16) et puisque $s \in B$

$$\|x(s_1) - z_n(s)\| \leqslant 4\varepsilon_{p+1} + b_n(s) \leqslant 4\varepsilon_{p+1} + 10^{-2}\delta.$$

De même on a

$$\|x(s_2) - z_{n-1}(s)\| \leqslant 4\varepsilon_{p+1} + 10^{-2}\delta.$$

D'où enfin, grâce à (4.18) on a

$$\|x(s_1) - x(s_2)\| \leqslant \delta/10 + \delta/50 + 8\varepsilon_{p+1} < \delta;$$

une contradiction. Le résultat est démontré.

Remarque. Un résultat de J. Bourgain [1] implique alors que tout opérateur de L^1 dans E possède la propriété de Dunford–Pettis.

Bibliographie

[1] J. Bourgain, *Dunford–Pettis operators on L^1 and the* RNP, Israel J. Math. 37 (1980), 34–47.

[2] J. Bourgain, H. Rosenthal, *Martingales valued in certain subspaces of L^1*, ibid. 37 (1980), 54–75.

[3] W. Roberts, *Compact convex sets with no extreme points in the spaces $L^p([0, 1])$, $0 \leqslant p < 1$*, à paraître.

[4] J. Diestel, *Geometry of Banach spaces, selected topics*, Lecture Notes in Math. nº 485, Springer-Verlag, 1975.

[5] M. Talagrand, *Sur la propriété de Radon–Nikodym dans les espaces de Banach réticulés*, C. R. Acad. Sci. Paris 288 (1979), 907–910.

UNIVERSITÉ PARIS VI, EQUIPE D'ANALYSE
Tour 46-0 4ème Etage
4, Place Jussieu, 75230-Paris Cedex 05

Received June 4, 1982 (1766)

STUDIA MATHEMATICA, T. LXXVI. (1983)

Unconditional decompositions and local unconditional structures in some subspaces of L_p, $1 \leqslant p < 2$

by

ANDRZEJ BORZYSZKOWSKI* (Gdańsk)

Abstract. For every integer $k \geqslant 2$ and for every p, $1 \leqslant p < 2$, there exists a subspace of l_p which is an unconditional sum of a sequence of k-dimensional subspaces and k is the least integer with this property. The method of proof involves the notion of local unconditional structure of order $\leqslant k$. In fact, we prove a stronger claim, thus improving upon the results of T. Ketonen [9].

1. Introduction and main results. Let X be a Banach space and let k be a positive integer. We say that X has *local unconditional structure of order* $\leqslant k$ (in short $\mathscr{U}_k(X) < \infty$) if there is a constant C such that for every finite dimensional subspace E of X there are operators T_j: $E \to X$, $j = 1, \ldots, n$, for which rank $T_j \leqslant k$, $i_E = \sum_{j \leqslant N} T_j$ and

$$\sup_{\varepsilon_j = \pm 1} \Big\| \sum_{j \leqslant N} \varepsilon_j T_j \Big\| \leqslant C.$$

Here i_E stands for the inclusion map, i_E: $E \hookrightarrow X$. The infimum of all numbers C with that property is denoted by $\mathscr{U}_k(X)$.

It is well known (cf. proof of Lemma 4.1 below) that $\mathscr{U}_k(X) < C < \infty$ is equivalent to the following property.

Given a finite dimensional subspace E of X, there are a space V and operators A: $E \to V$, B: $V \to X$ such that $i_E = B \circ A$, V has a Schauder decomposition $\{V_j\}_{j \leqslant N}$, $\dim V_j \leqslant k$ for $j \leqslant N$ and $\|A\| \cdot \|B\| \operatorname{unc}\{V_j\}_{j \leqslant N} \leqslant C$. Here $\operatorname{unc}\{V_j\}_{j \leqslant N}$ stands for the unconditional constant of the decomposition $\{V_j\}_{j \leqslant N}$.

This property in the case $k = 1$ was introduced by Gordon and Lewis [5]. They gave the first examples of Banach spaces for which $\mathscr{U}_1(X) = \infty$. The method of [5] depends on properties of a certain parameter which is now denoted by $\operatorname{gl}(X)$ (cf. Proposition 1.3 below). Other methods were developed in [7] and [6]. The case where X is a subspace of L_1 presented

* The present paper is a part of the author's Ph.D. thesis prepared under the supervision of Professor Tadeusz Figiel at the Institute of Mathematics of Polish Academy of Sciences.

some difficulties which were overcome only by T. Ketonen [9]. He constructed a subspace X of L_1 which has an unconditional decomposition into 2-dimensional subspaces (hence $\mathscr{U}_2(X) < \infty$) but $\mathscr{U}_1(X) = \infty$, thus answering the question posed by Pisier [13].

In the present paper we extend the results of [9] and also simplify some of the arguments. The main results are Theorem 1.1 and Theorem 1.2.

Before stating our results we need some definitions.

Let $\{X_\gamma\}_{\gamma\in\Gamma}$ be a family of linear subspaces of a Banach space X. The *unconditional constant* of $\{X_\gamma\}_{\gamma\in\Gamma}$, denoted by $\operatorname{unc}\{X_\gamma\}_{\gamma\in\Gamma}$, is the infimum of all numbers $C \geqslant 0$ such that for all choices of elements $x_\gamma \in X_\gamma$ fulfilling $\sum_{\gamma\in\Gamma}\|x_\gamma\| < \infty$ and all choices of signs $\{\varepsilon_\gamma\}_{\gamma\in\Gamma}$ the following inequality holds:

$$\Big\|\sum_{\gamma\in\Gamma}\varepsilon_\gamma x_\gamma\Big\| \leqslant C\Big\|\sum_{\gamma\in\Gamma}x_\gamma\Big\|.$$

If $\operatorname{unc}\{X_\gamma\}_\gamma < \infty$ then we say that $\{X_\gamma\}_{\gamma\in\Gamma}$ *forms an unconditional decomposition* of the closed linear subspace Y spanned by $\bigcup_{\gamma\in\Gamma} X_\gamma$. If this is the case then we write $Y = [X_\gamma]_{\gamma\in\Gamma}$.

An unconditional decomposition $\{X_\gamma\}_{\gamma\in\Gamma}$ is said to be *finite dimensional* (resp. *k-dimensional*) provided $\dim X_\gamma < \infty$ for each $\gamma\in\Gamma$ (resp. $\dim X_\gamma \leqslant k$ for each $\gamma\in\Gamma$). If, for every $\gamma\in\Gamma$, X_γ is spanned by a single vector x_γ then we often write $\operatorname{unc}\{x_\gamma\}_{\gamma\in\Gamma}$ instead of $\operatorname{unc}\{X_\gamma\}_{\gamma\in\Gamma}$.

In the sequel the index set Γ will be either a set of integers or a set of pairs of integers.

The standard ordering of a sequence of k-tuples $\{x_n^{(1)}, \ldots, x_n^{(k)}\}_{n=1}^m$, $1 \leqslant k < \infty$, $1 \leqslant m \leqslant \infty$ is $\{y_j\}_{j=1}^{k\cdot m}$ where $y_{k(n-1)+r} = x_n^{(r)}$ for $1 \leqslant r \leqslant k$, $n = 1, \ldots$

Consider a Banach space X with a basis $\{x_n^{(1)}, \ldots, x_n^{(k)}\}_{n=1}^\infty$ and put $X_n = \operatorname{span}\{x_n^{(1)}, \ldots, x_n^{(k)}\}$, $n = 1, \ldots$ We say that this sequence of k-tuples is *symmetric*, or that the decomposition $\{X_n\}_{n=1}^\infty$ is symmetric, if for every permutation σ of the positive integers the sequences $\{x_{\sigma(n)}^{(1)}, \ldots, x_{\sigma(n)}^{(k)}\}_{n=1}^\infty$ and $\{x_n^{(1)}, \ldots, x_n^{(k)}\}_{n=1}^\infty$ are equivalent bases. It is well known that a symmetric decomposition is unconditional.

THEOREM 1.1. *For every p, $1 \leqslant p < 2$, and every integer $k \geqslant 2$ there exists a subspace X of L_p with a basis $\{x_n^{(1)}, \ldots, x_n^{(k)}\}_{n=1}^\infty$ such that*

(i) *the sequence of k-tuples is symmetric i.e. X has a symmetric k-dimensional decomposition,*

(ii) $\mathscr{U}_{k-1}(X) = \infty$.

THEOREM 1.2. *For every p, $1 \leqslant p < 2$, there exists a subspace Y of l_p such that Y has a finite-dimensional unconditional decomposition but, for every integer $k \geqslant 1$, $\mathscr{U}_k(X) = \infty$.*

Theorem 1.1 in the case $k = 2$ is due to Ketonen (cf. [9], Theorem 4.4). Our proof of Theorem 1.1 in the general case is a modification of his argument. Theorem 1.2 is an easy consequence of Theorem 1.1.

The *Gordon–Lewis constant* of a Banach space X is defined by

$$\operatorname{gl}(X) = \sup\{\gamma_1(T);\ T\colon X\to l_2,\ \pi_1(T) \leqslant 1\}.$$

Here $\gamma_1(T)$ denotes the L_1-factorization norm of T and $\pi_1(T)$ the 1-absolutely summing norm of T (cf. [5], [13] or [2]). It is well known that if X is a subspace of L_1 then $\operatorname{gl}(X) = 1$.

Let $a = \{a(n)\}_{n=1}^\infty$ be a non-decreasing sequence of reals such that $a(1) = 1$ and $\lim a(n) = \infty$. Given a Banach space X, we put

$$l_a(X) = \sup_E \inf_{\{T_i\}} \sup_{\varepsilon_i=\pm1}\Big\|\sum_i \varepsilon_i a(\operatorname{rank} T_i)T_i\Big\|.$$

Here E ranges over all finite-dimensional subspaces of X and the infimum is over all finite decompositions $i_E = \sum_i T_i$ of the inclusion map $i_E\colon E \hookrightarrow X$. If $a(n) = n^{1/2}$ then $l(X) = l_a(X)$ is Gordon's parameter (cf. [4], [2]).

PROPOSITION 1.3. *For every integer $k \geqslant 1$,*

$$\operatorname{gl}(X) \leqslant l(X) \leqslant k^{1/2}\cdot\mathscr{U}_k(X).$$

The lower estimate is proved in [2] while the upper estimate follows directly from the definitions.

The following observation is due to T. Figiel (personally communicated to the author).

PROPOSITION 1.4. *Let X be a Banach space such that $l_a(X) < \infty$ for some sequence $a = \{a(n)\}$. Suppose there is a net $\{P_\delta\}_{\delta\in\Delta}$ of finite rank projections on X and there is $\lambda < \infty$ such that $\|P_\delta\| \leqslant \lambda$ for $\delta\in\Delta$ and $\|P_\delta x - x\| \to 0$ for every $x\in X$. Then $\mathscr{U}_k(X) \leqslant 2\cdot l_a(X)$ for some finite k depending only on a, $l_a(X)$ and λ.*

COROLLARY 1.5. *If $Y \subset l_p$ is the space constructed in Theorem 1.2 then $l_a(Y) = \infty$ for any sequence $a(n) \nearrow \infty$.*

The following proposition taken from [9] will be used in the construction of the space X in Theorem 1.1.

PROPOSITION 1.6. *For every p, $1 \leqslant p < \infty$, $p \neq 2$, there exists a subspace Z of L_p with a normalized basis $\{f_n, g_n\}_{n=1}^\infty$ such that*

(i) *the decomposition $\{Z_n\}_{n=1}^\infty$, where $Z_n = \operatorname{span}\{f_n, g_n\}$, $n = 1, \ldots$, is unconditional,*

(ii) *the projection $\sum_n (\alpha_n f_n + \beta_n g_n) \mapsto \sum_n \alpha_n f_n$ is not a bounded operator.*

The proof of Proposition 1.6 given in [9] depends on a special construction of a conditional basis in L_p, $1 < p < \infty$, $p \neq 2$ (cf. [9], Proposition 2.2). Here is a direct proof which works also if $p = 1$.

Proof of Proposition 1.6. Let $\{e_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ be the standard bases in l_p and l_2, respectively. Put $Z = l_p \oplus l_2, f_n = e_n$ and $g_n = 2^{-1/p}(e_n + d_n)$ for $n = 1, \ldots$ It is easily seen that (i) holds and that (ii) is satisfied if $1 \leqslant p < 2$. In the case $p > 2$ one simply takes $f_n = d_n$, $g_n = 2^{-1/p}(e_n + d_n)$ for $n = 1, \ldots$ ∎

A similar argument can be applied if $Z = l_p$, $1 < p < \infty$, $p \neq 2$.

Our notation is basically the same as in [10] and [11]. Let us explain that L_p, $1 \leqslant p < \infty$, denotes the real Banach space of p-integrable real valued functions on the unit interval $[0, 1]$.

If X is a Banach space then $L_2(X)$ denotes the space of strongly measurable functions $f\colon [0,1] \to X$ such that $\int_0^1 \|f(t)\|_X^2 dt < \infty$.

The *Rademacher system* is defined by $r_n(t) = \operatorname{sgn}\sin 2^n \pi t$, $n = 1, \ldots$ By $\mathrm{Rad}_{(n)}(X)$ we denote the subspace of $L_2(X)$ consisting of the functions of the form $f(t) = \sum_{i \leqslant n} r_i(t) x_i$ where $x_i \in X$ if $i \leqslant n$. Finally, $\mathrm{Rad}(X)$ is the closure of $\bigcup_n \mathrm{Rad}_{(n)}(X)$ in $L_2(X)$.

2. Proofs of the main results. The proof of Theorem 1.1 is based on two propositions.

Proposition A. *Suppose that X is a Banach space of finite cotype and that X has an unconditional decomposition $\{X_i\}_{i=1}^{\infty}$ where $\dim X_i = k$ for each i.*

If $\mathscr{U}_{k-1}(X) < \infty$ then there are a constant $D < \infty$ and operators $T_n\colon X \to X$, $n = 1, \ldots,$ such that

(i) *$T_n(X_i) \subset X_i$ for each i,*

(ii) *$\|T_n\| \leqslant D$,*

(iii) *$\|T_n \circ \pi_i - \lambda \pi_i\| \geqslant (2 \cdot k^2)^{-1}$ for every scalar $\lambda \in \boldsymbol{R}$ and for $i = 1, \ldots, n$.*

Here π_i stands for the natural projection onto X_i.

Proposition B. *Let $1 < p < 2$, $k \geqslant 2$. Then there exists a subspace X of L_p with a normalized basis $\{x_n^{(1)}, \ldots, x_n^{(k)}\}_{n=1}^{\infty}$ such that*

(i) *the sequence of k-tuples is symmetric,*

(ii) *if $A = \{a_{ij}\}$, $1 \leqslant i, j \leqslant k$, is a matrix such that there is a bounded operator $T_A\colon X \to X$ which satisfies*

$$T_A x_n^{(i)} = \sum_{j \leqslant k} a_{ij} x_n^{(j)} \quad \textit{for} \quad i = 1, \ldots, k \textit{ and } n = 1, \ldots,$$

then A must be a multiple of the identity matrix.

Remark. The results of Kadec and Pełczyński (cf. [8], Corollary 6) show that a symmetric basic sequence in L_p, $p \geqslant 2$, is equivalent to the standard basis of either l_p or l_2. Hence Proposition B fails in the case $p \geqslant 2$. It remains true for $p = 1$ but the method of proof works only for $p > 1$ (cf. Lemma 3.3).

Proof of Theorem 1.1. Suppose first that $p > 1$. We shall show that the space X constructed in Proposition B has the properties stated in Theorem 1.1. Suppose, on the contrary, that $\mathscr{U}_{k-1}(X) < \infty$. Then Proposition A yields a sequence of operators T_n such that, using the compactness argument and utilizing condition (i) in Proposition B we can produce a non-scalar $k \times k$ matrix A such that T_A is a bounded operator (see [9], pp. 23–24, for the detailed proof of this claim). This, however, contradicts (ii) in Proposition B. Hence Theorem 1.1 in the case $p > 1$ is proved. The case $p = 1$ is also valid because L_1 contains an isometric copy of every L_p space, $1 < p < 2$ (cf. [11], 2.f.5).

Proof of Theorem 1.2. Fix $k \geqslant 2$ and choose a subspace X of L_p with the properties formulated in Theorem 1.1. Put $\tilde{X} = \left(\sum_{n \geqslant 1} \oplus E_n\right)_p$ where $E_n = [X_i]_{i=1}^n$. Clearly, $\mathscr{U}_{k-1}(\tilde{X}) = \infty$ and $\tilde{X}$ is isometric to a subspace of L_p. For every n choose a subspace F_n of L_p spanned by finitely many disjointly supported functions and an operator $T_n\colon E_n \to L_p$ such that $T_n(E_n) \subset F_n$ and $\|T_n - i_{E_n}\| \leqslant 4^{-n}$ (cf. [12], Theorem 2.1). Since F_n is isometric to some l_p^N, we obtain an isomorphic embedding $T\colon \tilde{X} \to \left(\sum_{n=1} \oplus F_n\right)_p \simeq l_p$. We put $Y^{(k)} = T(\tilde{X}) \subset l_p$. One can easily see that the space $Y = \left(\sum_{k=2}^{\infty} \oplus Y^{(k)}\right)_p$ has the required properties. ∎

3. Proofs of Propositions A, B and 1.4.

Proposition 3.1. *Suppose that X is a Banach space of finite cotype. Then the following conditions are equivalent:*

(i) $\mathscr{U}_k(X) < \infty$.

(ii) $\mathscr{U}_k(\mathrm{Rad}(X)) < \infty$.

(iii) *There is $C < \infty$ such that given $E \subset X$, $\dim E < \infty$, there are a space $V = [V_j]_{j=1}^N$ and operators $A\colon E \to V$, $B\colon V \to X$ such that $\dim V_j \leqslant k$ for $j \leqslant N$, $i_E = B \circ A$, $\|A\| \cdot \|B\| \leqslant C$, and for every integer n $\operatorname{unc}\{r_j V_j\}_{\substack{i \leqslant n \\ j \leqslant N}} \leqslant C$.*

Remark. Condition (iii) shows that local unconditional structure of order $\leqslant k$ of the space $\mathrm{Rad}(X)$ has a tensor product nature.

The proof of Proposition 3.1 is given in Section 4.

Lemma 3.2. *Suppose that B is a Banach space, $\dim B < \infty$, $0 < \alpha < 1$ and $\{x_j\}_{j \leqslant N}$ is a sequence of vectors in B such that, putting $e_0 = \sum_{j \leqslant N} x_j$, we have*

$$\inf_{\lambda \in \boldsymbol{R}} \|x_j - \lambda e_0\| \geqslant \alpha \|x_j\| \quad \textit{for} \quad j = 1, \ldots, N.$$

Then there is a sequence $\{\varepsilon_j\}_{j\leqslant N}$ of signs such that

$$\inf_{\lambda\in\boldsymbol{R}}\Big\|\sum_{j\leqslant N}\varepsilon_j x_j-\lambda e_0\Big\|\geqslant a\cdot(\dim B)^{-1}\|e_0\|.$$

Proof of Lemma 3.2. Put $B_0=\operatorname{span}\{e_0\}$ and consider the quotient space $E=B/B_0$. Let y_j be the image of x_j under the quotient map, $j=1,\dots,N$. Observe that

$$\sum_{j\leqslant N}\|y_j\|_E\geqslant\sum_{j\leqslant N}a\|x_j\|\geqslant a\|e_0\|.$$

Now put $\mu=\max\big\|\sum_{j\leqslant n}\varepsilon_j y_j\big\|_E$ where the maximum is taken over all choices of signs, $\varepsilon_j=\pm1$ for $j=1,\dots,N$. It suffices to show that $\sum_{j\leqslant N}\|y_j\|_E\leqslant(\dim E)\cdot\mu$. For every functional $e^*\in E^*$, setting $\varepsilon_j=\operatorname{sgn}e^*(y_j)$, we have that $\sum_{j\leqslant N}|e^*(y_j)|=e^*\big(\sum_{j\leqslant N}\varepsilon_j y_j\big)\leqslant\|e^*\|\mu$. If $(e_i,e_i^*)_{i=1}^{\dim E}$ is an Auerbach system for the space E then

$$\sum_{j\leqslant N}\|y_j\|_E\leqslant\sum_{j\leqslant N}\sum_i|e_i^*(y_j)|=\sum_i\sum_{j\leqslant N}|e_i^*(y_j)|\leqslant(\dim E)\mu.\ \blacksquare$$

Proof of Proposition A. Fix $n\geqslant 1$.

Claim 1. *There are operators $R_{ij}\colon X\to X$, $i\leqslant n$, $j\leqslant N$, such that $R_{ij}=\pi_i\circ R_{ij}\circ\pi_i$, $\operatorname{rank}R_{ij}\leqslant k-1$ and*

(i) $\pi_i=\sum_{j\leqslant N}R_{ij}$, $i=1,\dots,n$,

(ii) $\big\|\sum_{i\leqslant n}\sum_{j\leqslant N}\varepsilon_{ij}R_{ij}\big\|\leqslant D$ *for any double sequence* $\varepsilon_{ij}=\pm1$.

Here D does not depend on n.

In order to prove this we use Proposition 3.1 with k replaced by $k-1$ and $E=[X_i]_{i=1}^n$. In fact, since $\pi_E=\sum_{i\leqslant n}\pi_i$ is a projection onto E with $\|\pi_E\|\leqslant\operatorname{unc}\{X_i\}_{i=1}^\infty$, we may obtain a factorization of π_E such that $\pi_E=B\circ A$, $A\colon X\to V$, $B\colon V\to X$, $\|A\|\cdot\|B\|\leqslant C\cdot\operatorname{unc}\{X_i\}_{i=1}^\infty$ and $\dim V_j\leqslant k-1$ for each $j=1,\dots,N$. Here C is the constant appearing in Proposition 3.1. Consider the space $\operatorname{Rad}(X)$. Tong's diagonal argument (cf. [10], l.c.8) shows that the "diagonal subspace" $[r_iX_i]_{i=1}^\infty$ is complemented in $\operatorname{Rad}(X)$ and the norm of the respective operator $Q\colon\operatorname{Rad}X\to[r_iX_i]_{i=1}^\infty$ fulfills $\|Q\|\leqslant\operatorname{unc}\{X_i\}_{i=1}^\infty$.

Now, using the natural isomorphism $[r_iX_i]_{i=1}^\infty\simeq X$, we have the following factorization of π_E:

$$\pi_E\colon X\simeq[r_iX_i]_{i=1}^\infty\to\operatorname{Rad}(X)\xrightarrow{\tilde A}\operatorname{Rad}(V)\xrightarrow{\tilde B}\operatorname{Rad}(X)\xrightarrow{Q}[r_iX_i]_{i=1}^\infty\simeq X.$$

Denote by $\pi_{ij}\colon\operatorname{Rad}(V)\to\operatorname{Rad}(V)$ the projection onto the subspace r_iV_j, and define R_{ij} to be the composition

$$R_{ij}\colon X\to\operatorname{Rad}(V)\xrightarrow{\pi_{ij}}\operatorname{Rad}(V)\to X,\qquad i=1,\dots,n,\ j=1,\dots,N.$$

It follows from the definition that $R_{ij}=\pi_i\circ R_{ij}\circ\pi_i$, $\operatorname{rank}R_{ij}\leqslant k-1$ for $i\leqslant n$, $j\leqslant N$ and (i) holds. Since $\operatorname{unc}\{r_iV_j\}_{i\leqslant n,\,j\leqslant N}\leqslant C$, we see that (ii) holds with a constant D depending only on C and $\operatorname{unc}\{X_i\}_{i=1}^\infty$.

Now fix i such that $1\leqslant i\leqslant n$.

Claim 2. *In the setting of Claim 1 there is a sequence of signs $\{\varepsilon_{ij}\}_{j\leqslant N}$ such that $S_i=\sum_{j\leqslant N}\varepsilon_{ij}R_{ij}$ satisfies*

$$\inf_{\lambda\in\boldsymbol{R}}\|S_i-\lambda\pi_i\|\geqslant(2k^2)^{-1}.$$

In order to prove this claim let B_i be the space of all operators $R\colon X\to X$ fulfilling $R=\pi_i\circ R\circ\pi_i$, i.e. the space of operators which actually act in the space X_i. Clearly, $\dim B_i=k^2$. We shall use Lemma 3.2 with $B=B_i$ and $x_j=R_{ij}$ for $j=1,\dots,N$ and $a=(1+\|\pi_i\|)^{-1}$.

It is seen from the triangle inequality that $\|R_{ij}-\lambda\pi_i\|\geqslant\|R_{ij}\|-|\lambda|\cdot\|\pi_i\|$. On the other hand, since $\operatorname{rank}R_{ij}\leqslant k-1$, there is a vector $x\in X_i$ such that $\|x\|=1$, $R_{ij}x=0$. Hence we obtain $\|R_{ij}-\lambda\pi_i\|\geqslant\|R_{ij}x-\lambda x\|\geqslant|\lambda|$. Thus for each scalar $\lambda\in\boldsymbol{R}$ we have

$$\|R_{ij}-\lambda\pi_i\|\geqslant\max\{|\lambda|,\|R_{ij}\|-|\lambda|\,\|\pi_i\|\}\geqslant(1+\|\pi_i\|)^{-1}\|R_{ij}\|.$$

Now Lemma 3.2 yields a sequence $\{\varepsilon_{ij}\}_{j\leqslant N}$ of signs such that for every scalar $\lambda\in\boldsymbol{R}$

$$\Big\|\sum_{j\leqslant N}\varepsilon_{ij}R_{ij}-\lambda\pi_i\Big\|\geqslant\|\pi_i\|(1+\|\pi_i\|)^{-1}(\dim B_i)^{-1}\geqslant(2k^2)^{-1}.$$

Clearly, the operators $T_n=\sum_{i\leqslant n}S_i=\sum_{i\leqslant n}\sum_{j\leqslant N}\varepsilon_{ij}R_{ij}$, $n=1,\dots$, have the properties required in Proposition A. $\blacksquare$

The space X whose existence is claimed in Proposition B is not constructed directly but with the use of a symmetrization procedure in the setting of L_p spaces. The following lemma shows that it will suffice if we prove a weaker fact.

Lemma 3.3. *Let $\{y_n^{(1)},\dots,y_n^{(k)}\}_{n=1}^\infty$ be a normalized basic sequence in L_p, $1<p<2$. Suppose that the spaces $Y_n=\operatorname{span}\{y_n^{(1)},\dots,y_n^{(k)}\}$, $n=1,\dots$, form an unconditional decomposition of $Y=[Y_n]_{n=1}^\infty$. Then there exists another basic sequence $\{x_n^{(1)},\dots,x_n^{(k)}\}_{n=1}^\infty$ in L_p which is a symmetric sequence of k-tuples and has the following property:*

There is a sequence $\{N_j\}_{j=1}^{\infty}$ of sets of integers fulfilling $\max N_j < \min N_{j+1}$ *for every j, and there is a sequence $\{a_j\}_{j=1}^{\infty}$ of reals such that the sequences $\{y_j^{(1)}, \ldots, y_j^{(k)}\}_{j=1}^{\infty}$ and*

$$\Big\{a_j \sum_{i\in N_j} x_i^{(1)}, \ldots, a_j \sum_{i\in N_j} x_i^{(k)}\Big\}_{j=1}^{\infty}$$

are equivalent bases.

Moreover, if $\{y_n^{(1)}, \ldots, y_n^{(k)}\}_{n=1}^{\infty}$ satisfies (ii) *in Proposition* B *then so does $\{x_n^{(1)}, \ldots, x_n^{(k)}\}_{n=1}^{\infty}$.*

The proof of Lemma 3.3 uses the interpolation technique and the results of H. P. Rosenthal [14] on subspaces of L_p, $p>2$. We omit this proof, because it is an obvious modification of the proof of Lemma 4.3 in [9].

Proof of Proposition B. We shall define a basis $\{y_n^{(1)}, \ldots, y_n^{(k)}\}_{n=1}^{\infty}$ of the space $l_p \oplus l_2$ which satisfies the assumption of Lemma 3.3. Let us replace $l_p \oplus l_2$ by its isomorphic copy

$$Y = Z^{(1)} \oplus \ldots \oplus Z^{(k-1)} \oplus l_2 \oplus l_p \oplus l_2$$

where each $Z^{(r)}$ is a copy of $Z = l_p \oplus l_2$ taken with the basis $\{f_n^{(r)}, g_n^{(r)}\}_{n=1}^{\infty}$ we described in Proposition 1.6.

The $y_n^{(i)}$'s where $n = (k+2)m+j$, $j=0$ or $j=-1$, are by definition the elements of the bases in the $Z^{(r)}$'s arranged as the following table shows:

i	1	2	3	4	5	...	k
$n = (k+2)m-1$	$*$	$f_m^{(2)}$	$g_m^{(2)}$	$f_m^{(4)}$	$g_m^{(4)}$	...	$*$
$n = (k+2)m$	$f_m^{(1)}$	$g_m^{(1)}$	$f_m^{(3)}$	$g_m^{(3)}$	$f_m^{(5)}$	...	$*$

We put $y_{(k+2)m}^{(k)} = g_m^{(k)}$ if k is an even integer while for odd k, $y_{(k+2)m-1}^{(k)} = g_m^{(k)}$. One copy of l_2 is reserved just to fill up remaining spare places in the above table.

Remaining vectors are defined to be the consecutive standard basis vectors of l_p or l_2, namely, if $n \equiv i \mod (k+2)$, $1 \leqslant i \leqslant k$, then $y_n^{(i)}$ is taken from l_2, otherwise from l_p.

Suppose now that $A = \{a_{ij}\}_{i,j\leqslant k}$ is a matrix such that T_A is a bounded operator. Comparing a typical vector $y = \sum_{n\equiv s} a_n y_n^{(s)}$, $1 \leqslant s \leqslant k$, with its image under T_A,

$$T_A(y) = \sum_{n\equiv s} a_n \sum_{j\leqslant k} a_{sj} y_n^{(j)},$$

we see that $a_{sj} = 0$ for $j \neq s$, $1 \leqslant j \leqslant k$. In order to see that $a_{ss} = a_{s+1,s+1}$ we compare a typical vector

$$y = \sum_{n\equiv 0} \alpha_n y_n^{(s)} + \beta_n y_n^{(s+1)}$$

with

$$T_A(y) = \sum_{n\equiv 0} a_{ss}\alpha_n y_n^{(s)} + a_{s+1,s+1}\beta_n y_n^{(s+1)}.$$

This gives the desired result if s is an odd integer. If s is even, one should consider the sum over $n \equiv -1$.

Now one can conclude the proof by applying Lemma 3.3. ∎

Proof of Proposition 1.4. Let $E \subset X$ be a finite-dimensional subspace. The conditions imposed on X allow us to choose a finite-dimensional subspace F containing E which is the range of a projection $Q\colon X \to F$ of norm $\leqslant \lambda$. Put $\eta = 1/3$ and choose operators $T_i\colon F \to X$ such that $i_F = \sum_i T_i$ and

$$\sup_{\varepsilon_i=\pm 1} \Big\| \sum_i \varepsilon_i a(\operatorname{rank} T_i) T_i \Big\| \leqslant (1+\eta) l_a(X).$$

Fix an integer k such that $a(k) \geqslant (1+\eta^{-1}) \cdot \lambda \cdot l_a(X)$ and put $A = \{i \mid \operatorname{rank} T_i \leqslant k\}$ and $B = \{i \mid \operatorname{rank} T_i > k\}$. Now, setting $U = \sum_{i\in B} T_i$, we have

$$\|U \circ Q\| \leqslant (1+\eta) l_a(X) \lambda [a(k)]^{-1} \leqslant \eta.$$

Hence $S = i_X - UQ$ is an isomorphism, $\|S^{-1}\| \leqslant (1-\eta)^{-1}$, and $S|_F = \sum_{i\in A} T_i$. Set $R_i = S^{-1} \circ T_i$ for $i \in A$. These operators form an unconditional decomposition of the inclusion map i_F and

$$\begin{aligned}\sup_{\varepsilon_i=\pm 1} \Big\| \sum_{i\in A} \varepsilon_i R_i \Big\| &\leqslant \|S^{-1}\| \sup_{\varepsilon_i=\pm 1} \Big\| \sum_{i\in A} \varepsilon_i T_i \Big\| \\ &\leqslant \|S^{-1}\| \sup_{\varepsilon_i=\pm 1} \Big\| \sum_i \varepsilon_i a(\operatorname{rank} T_i) T_i \Big\| \\ &\leqslant (1+\eta)(1-\eta)^{-1} \cdot l_a(X) = 2 \cdot l_a(X).\end{aligned}$$

Of course, $\operatorname{rank} R_i \leqslant k$ for $i \in A$. Thus $\mathscr{U}_k(X) \leqslant 2 l_a(X)$. ∎

4. Proof of Proposition 3.1. The proof is based on the renorming technique analogous to that used in [2], Proposition 2.6, and on some results of Pisier [13] concerning unconditional structures in tensor products. The renorming technique we refer to is an essence of the following lemma (here and throughout this section $C_q(X)$ stands for the cotype q constant of a Banach space X).

LEMMA 4.1. *Suppose that $\mathscr{U}_k(X) < D < \infty$ and $C_q(X) < \infty$. Fix $q_0 > q$. Then for any $E \subset X$, $\dim E < \infty$ there are a space $V = [V_j]_{j\leqslant N}$ and operators $A\colon E \to V$, $B\colon V \to X$ such that $i_E = B\circ A$, $\dim V_j \leqslant k$ for $j = 1, \ldots, N$, $\operatorname{unc}\{V_j\}_{j\leqslant N} = 1$, $\|A\|\,\|B\| \leqslant D$ and $C_{q_0}(V) \leqslant M$ where M depends on q, q_0 and $C_q(X)$ only.*

In the proof of Lemma 4.1 we shall make use of the following proposition due to Enflo, Lindenstrauss and Pisier [1].

PROPOSITION 4.2. *There is a function $C_2 = C_2(q_1, q_2, C_1)$ where $2 \leqslant q_1 < q_2 < \infty$ and $C_1 < \infty$ such that if Y is a subspace of a Banach space Z and $C_{q_1}(Y) \leqslant C_1$, $C_{q_1}(Z/Y) \leqslant C_1$ then $C_{q_2}(Z) \leqslant C_2$.*

Proof of Lemma 4.1. Fix a finite-dimensional subspace E of X and choose a sequence of operators $T_j\colon E \to X$, $j = 1, \ldots, N$ such that $i_E = \sum_{j\leqslant N} T_j$, $\operatorname{rank} T_j \leqslant k$ and $\|\sum_{j\leqslant N} \varepsilon_j T_j\| \leqslant D$ for any $\varepsilon_j = \pm 1$.

Put $V_j = T_j(E) \subset X$ and let V be the linear space of all sequences $d = \{d_j\}_{j\leqslant N}$ where $d_j \in V_j$ for $j = 1, \ldots, N$. We define a norm $\|\cdot\|_V$ on V by

$$\|d\|_V = \sup_{\varepsilon_j = \pm 1} \Big\| \sum_{j\leqslant N} \varepsilon_j d_j \Big\|_X.$$

The operators $A\colon E \to V$, $B\colon V \to X$ are defined in a natural way, i.e. $A(e) = \{T_j e\}_{j\leqslant N}$ if $e \in E$ and $B(d) = \sum_{j\leqslant N} d_j$ if $d = \{d_j\}_{j\leqslant N} \in V$. Clearly, the definition yields $\operatorname{unc}\{V_j\}_{j\leqslant N} = 1$, $\|A\| \leqslant D$ and $\|B\| \leqslant 1$.

CLAIM. *V has a lower q-estimate with the constant $C_q(X)$.*

In order to see it consider any sequence of disjoint subsets $A_i \subset \{1, \ldots, N\}$, $i = 1, \ldots, m$. Let $\chi_i\colon V \to V$ be the operators defined by $\chi_i d = \{\chi_{A_i}(j) d_j\}_{j\leqslant N}$ where $d = \{d_j\}_{j\leqslant N} \in V$ and χ_{A_i} stands for the characteristic function of the set A_i, $i = 1, \ldots, m$. Fix $y = \{y_j\}_{j\leqslant N}$ and write $\tilde{y} = \sum_{i\leqslant m} \chi_i y$. Now, fix any sequence $\{\varepsilon_j\}_{j\leqslant N}$ of signs and put

$$z^{(i)} = \varepsilon\chi_i y = \{\varepsilon_j \chi_{A_i}(j) y_j\}_{j\leqslant N}.$$

For every sequence $\{\tau_i\}_{i\leqslant m}$ of signs we have

$$\Big\| \sum_{i\leqslant m} \tau_i B z^{(i)} \Big\|_X = \Big\| \sum_{i\leqslant m} \tau_i \sum_{j\in A_i} \varepsilon_j y_j \Big\|_X \leqslant \|\tilde{y}\|_V.$$

Since X is of cotype q, we have

$$C_q(X)^{-1} \Big(\sum_{i\leqslant m} \|Bz^{(i)}\|_X^q \Big)^{1/q} \leqslant \max_{\tau_i = \pm 1} \Big\| \sum_{i\leqslant m} \tau_i B z^{(i)} \Big\|_X \leqslant \|\tilde{y}\|_V.$$

Taking supremum over all choices of $\{z^{(i)}\}_{i\leqslant m}$, i.e. all choices of $\{\varepsilon_j\}_{j\leqslant N}$, we obtain

$$C_q(X)^{-1} \Big(\sum_{i\leqslant m} \|\chi_i y\|_V^q \Big)^{1/q} \leqslant \|\tilde{y}\|_V.$$

Since y and $\{A_i\}_{i\leqslant m}$ were arbitrary, our claim is proved.

It is well known that if a space Z with an unconditional basis has a lower q-estimate, $q \geqslant 2$, then, for every $q_1 > q$, Z is of cotype q_1 (cf. [11], p. 88). Using induction with respect to k one can extend the latter result to spaces which have a k-dimensional unconditional decomposition.

For choose a subspace Y of V such that $Y = [Y_j]_{j\leqslant N}$, $Y_j \subset V_j$ and, if $\dim V_j = k$, then $1 \leqslant \dim Y_j \leqslant k-1$, $j = 1, \ldots, N$. Then both Y and V/Y have a $(k-1)$-dimensional unconditional decomposition and the induction works in view of Proposition 4.2. ∎

The following proposition is a special case of the results of Pisier [13].

PROPOSITION 4.3. *Let X be a Banach space such that $\operatorname{gl}(X) < \infty$ and for some q, $C_q(X) < \infty$. Suppose that a double sequence $\{x_{ij}\}_{i,j}$ of vectors in X satisfies*

$$\Big\| \sum_{i,j} a_{ij} x_{ij} \Big\| = \Big\| \sum_{i,j} \varepsilon_i' \varepsilon_j'' a_{ij} x_{ij} \Big\|$$

for any finite matrix $\{a_{ij}\}_{i,j}$ of reals and any choice of signs, $\varepsilon_i' = \pm 1$, $\varepsilon_j'' = \pm 1$. Then $\operatorname{unc}\{x_{ij}\}_{i,j} \leqslant Q$ where Q depends only on q, $C_q(X)$ and $\operatorname{gl}(X)$.

Proof of Lemma 3.1.

(i) $\Rightarrow$ (iii). We claim that the space V and the factorization $i_E = A\circ B$, $A\colon E \to V$, $B\colon V \to X$ constructed in Lemma 4.1 fulfils the requirements of (iii) in Lemma 3.1. In order to see this we fix n and choose a double sequence $\{v_{ij}\}_{\substack{i\leqslant n\\ j\leqslant N}}$ of elements of $L_2(V)$ such that $v_{ij} \in r_i V_j$ for $i \leqslant n$, $j \leqslant N$. This sequence is contained in a subspace of $L_2(V)$ denoted by $L_2^{2^n}(V)$ and consisting of the functions which are constant on the dyadic intervals of length 2^{-n}. It is clear that

$$\operatorname{gl}\big(L_2^{2^n}(V)\big) \leqslant k^{1/2} \cdot \operatorname{unc}\{V_j\}_{j\leqslant N} = k^{1/2} \quad \text{and} \quad C_{q_0}\big(L_2^{2^n}(V)\big) = C_{q_0}(V) \leqslant M.$$

The double sequence $\{v_{ij}\}_{\substack{i\leqslant n\\ j\leqslant N}}$ fulfils the assumptions of Proposition 4.3, hence $\operatorname{unc}\{v_{ij}\}_{i,j} \leqslant Q$. This implies that $\operatorname{unc}\{r_i V_j\}_{\substack{i\leqslant N\\ j\leqslant N}} \leqslant Q$ and we may see that Q depends on k, q, q_0 and $C_q(X)$. Setting $C = \max\{Q, D\}$ we see that the requirements of (iii) in Proposition 3.1 are fulfilled.

(iii) $\Rightarrow$ (ii). Let $F \subset \operatorname{Rad}(X)$ be a finite-dimensional subspace. By the standard perturbation argument it suffices to consider the case where

$F = \mathrm{Rad}_{(n)}(E)$ for some $E \subset X$ and a positive integer n. Then the factorization of i_E given in (iii) yields a factorization of i_F, $i_F = B \circ A$ such that $\|A\| \cdot \|B\| \operatorname{unc}\{r_i' V_j\}_{\substack{i \leqslant n \\ j \leqslant N}} \leqslant C^2$.

(ii) $\Rightarrow$ (i). This implication is obvious. ■

Acknowledgment. I should like to thank my supervisor Professor Tadeusz Figiel for introducing me to the subject and for his generous help in preparing this paper. I also thank Dr. Timo Ketonen for a helpful conversation.

References

[1] P. Enflo, J. Lindenstrauss and G. Pisier, *On the "three space problem"* Math. Scand. 36 (1975), 199–210.

[2] T. Figiel and W. B. Johnson, *Large subspaces of l_∞^n and estimates of the Gordon–Lewis constant,* Israel J. Math. 37 (1980), 92–112.

[3] T. Figiel, W. B. Johnson and L. Tzafriri, *On Banach lattices and spaces having local unconditional structure with applications to Lorentz function spaces,* J. Approximation Theory 13 (1975), 395–412.

[4] Y. Gordon, *Unconditional Schauder decompositions of normed ideals of operators between some l_p-spaces,* Pacific J. Math. 60 (1975), 71–82.

[5] Y. Gordon and D. R. Lewis, *Absolutely summing operators and local unconditional structures,* Acta Math. 133 (1974), 27–48.

[6] W. B. Johnson, J. Lindenstrauss and G. Schechtman, *On the relations between several notions of unconditional structure,* Israel J. Math. 37 (1980), 120–129.

[7] W. B. Johnson and L. Tzafriri, *Some more Banach spaces which do not have local unconditional structure,* Houston J. Math. 3 (1977), 55–60.

[8] M. I. Kadec and A. Pełczyński, *Bases, lacunary sequences and complemented subspaces in the space L_p,* Studia Math. 21 (1962), 161–176.

[9] T. Ketonen, *On unconditionality in L_p spaces,* Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes 35.

[10] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I, Sequence spaces,* Springer-Verlag, 1977.

[11] —, — *Clasical Banach Spaces II, Function spaces,* Springer-Verlag, 1979.

[12] A. Pełczyński and H. P. Rosenthal, *Localization techniques in L_p spaces,* Studia Math. 52 (1975), 263–289.

[13] G. Pisier, *Some results on Banach spaces without local unconditional structure,* Compositio Math. 37 (1978), 3–19.

[14] H. P. Rosenthal, *On the subspaces of L_p ($p > 2$) spanned by sequences of independent random variables,* Israel J. Math. 8 (1970), 273–303.

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES

Received June 14, 1982 (1769)

STUDIA MATHEMATICA, T. LXXVI. (1983)

On impulsive control with long run average cost criterion

by

ŁUKASZ STETTNER (Warszawa)

Abstract. Discrete and continuous time impulsive control problems with long run average cost criterion are considered. The paper generalizes the results obtained by M. Robin in [9]. The methods of the proofs are different from those of [9].

Introduction. Impulsive control, introduced first by Bensoussan and Lions in [1], is the one of the most applicable types of stochastic control. This control consists in shifting current states of a Markov process (x_t) to new random states ξ_i, $i = 1, 2, \ldots$, at moment τ_i, respectively. With each strategy $V = (\tau_i, \xi_i)_{i \in N}$ is associated the long run average cost functional $J(V)$ consisting of the "holding cost" $f(x)$ and "replacement cost" $k(x_{\tau_i}, \xi_i) = c(x_{\tau_i}) + d(\xi_i)$ per unit time

$$J_x(V) = \liminf_{t \uparrow \infty} (1/t) E_x^V \Big\{ \int_0^t f(y_s)\, ds + \sum_{i=1}^{\infty} \chi_{\tau_i \leqslant t} [c(x_{\tau_i}) + d(\xi_i)] \Big\}.$$

The studies of impulsive control problems with such functional were originated by M. Robin in [9] for Markov processes having nice ergodic properties. This paper generalizes his results. We complete and extend his results to Fellerian Markov processes with general state space E. In particular, we show that the value function is constant and find optimal or ε-optimal strategies. We also prove that the use of general stopping times τ_i, instead of those of the form $\tau_i = \tau_{i-1} + \sigma_i \cdot \Theta_{\tau_{i-1}}$ as in the paper [9], does not change the optimal value of the functional.

We start with the discrete time impulsive control. Due to the special form of the controlled system we obtain results more general and complete than those which follow from the existing theory of the long run average cost, see [3]. Next we consider continuous time impulsive control. Methods of some of the proofs are similar to the martingale ones introduced by P. Mandl in the context of adaptive control ([4], [5]).

1. Discrete time case. Let $\Omega = E^N$ be the space of all sequences with values in E, where (E, B) denotes a measurable state space. Suppose that for any $\omega \in \Omega$, $x_n(\omega) = \omega(n)$ and $F_n = \sigma\{x_m, m \leqslant n\}$, $F = F_\infty$.