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Bounded variation and invariant measures

by

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Abstract. In this paper we study the existence and properties of invariant measures absolutely continuous with respect to the Lebesgue measure for piecewise continuous expanding maps of an interval with finitely or infinitely many pieces of monotonicity. The technique of bounded variation is used. The Bernoulli property and some limit theorems are proved for resulting dynamical systems.

Introduction. We study the existence and ergodic properties of invariant measures for various maps. The Bernoulli property and the central limit theorem are also discussed. Our formalism is similar to that of Hofbauer and Keller [1]. Our paper allows us to extend the results of [1], so that the maps with infinitely many pieces of monotonicity are included in the general theory, in particular, the maps considered by Walters [2].

The way we have chosen is different from that of Hofbauer and Keller. Their method relies on the theorem of Tulcea-Romanescu and Marinescu. Instead of that we apply the uniform ergodic theory. Of course, slightly modifying our method we can follow [1]. On the other hand, key estimations in their paper cannot be used in our case. Besides, we present simplified proofs of many facts, in particular the Bernoulli property and limit theorems. We have also explained the cyclic structure of the Perron–Frobenius operator.

Our reference list is not complete. The reader can find a lot of literature in [1].

§1. Let X be a totally ordered order-complete set. Open intervals constitute a base of a compact topology in X, making X into a topological space. If X is separable, then X is homeomorphic with a closed subset of an interval. By $\mathscr{B}$ we denote the σ-algebra of all Borel subsets of X. We fix a regular, Borel probabilistic measure m on X.

Three functional spaces will appear below. Two of them are well known: $L_\infty(m)$ and $L_1(m)$. We often write L_∞ and L_1. Norms in these spaces are denoted by $\|\cdot\|_\infty$ and $\|\cdot\|_1$, respectively. Given $f\colon X \to R$ we

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define variation of f on a subset $C \subset X$:

(1) $$\operatorname{Var}_C f = \sup\left\{\sum_{i=1}^{p} |f(x_i) - f(x_{i-1})|\right\},$$

where the supremum is over all sequences $(x_0, \ldots, x_p)$, $x_0, \ldots, x_p \in C$ and $x_0 \leqslant x_1 \leqslant \ldots \leqslant x_p$. In place of $\operatorname{Var}_X f$ we write $\operatorname{Var} f$.

Let

(2) $$\mathrm{BV} = \{f \in L_\infty\colon f \text{ has a version of bounded variation}\}.$$

BV is a Banach space with the norm

(3) $$\|f\| = \max(\|f\|_1, \inf\{\operatorname{Var}\tilde{f}\colon \tilde{f} \text{ is a version of } f\}).$$

Remark 1. Every $f \in \mathrm{BV}$ has a version $\tilde{f}$ with minimal variation. This holds iff for every $x_0 \in X$

(4) $$\tilde{f}(x_0) \in [\lim_{x\to x_0(-)} \tilde{f}, \lim_{x\to x_0(+)} \tilde{f}].$$

One-sided limits always exist for $\tilde{f}$. If we do not talk about versions of of f explicitly, we assume that we have chosen a version satisfying (4). We notice that $\operatorname{Var}\tilde{f}$ does not depend on the choice of $\tilde{f}$ with this property.

Let $T\colon U \to X$ be a continuous map, $U \subset X$ is open and dense, and $m(U) = 1$. Let $S = X \setminus U$. We assume that:

There exists a countable family β of closed intervals with disjoint interiors such that $\bigcup\beta \supset U$ and for any $B \in \beta$ the set $B \cap S$ consists exactly of the endpoints of B.

For any $B \in \beta$, $T|_{B\cap U}$ admits an extension to a homeomorphism of B with some interval in X.

A function $g\colon X \to R_+$ is given, $\|g\|_\infty < 1$, $\operatorname{Var} g < +\infty$, $g|_S = 0$ and the operator $P\colon L_1 \to L_1$ defined by

(5) $$Pf(x) = \sum_{y\in T^{-1}(x)} g(y) f(y)$$

preserves m, which means $m(Pf) = m(f)$ for each $f \in L_1$.

All subsets of X that occur below are measurable.

Remark 2. P is the Perron–Frobenius operator for T.

The proof of this fact will be divided into steps (a), (b) and (c).

(a) *T is non-singular.*

Proof. For any $f, h \in L_1$ such that $f\cdot h \in L_1$, $P(f\circ T\cdot h) = f\cdot Ph$. We put $f = \chi_A$, $A \subset X$, and $h = \chi_B$, $B \in \beta$. So

$$m(T^{-1}A \cap B) = m(\chi_A \circ T\chi_B) = m\big(P(\chi_A\circ T\cdot\chi_B)\big) = m(\chi_A\cdot P\chi_B) \leqslant m(A \cap TB)\|g\|_\infty$$

since $P\chi_B|_{X\setminus TB} = 0$ and $\|P\chi_B\|_\infty \leqslant \|g\|_\infty$. Thus, $m(A) = 0$ implies

$$m(T^{-1}A) \leqslant \sum_{B\in\beta} m(T^{-1}A \cap B) = 0. \blacksquare$$

(b) *$g \neq 0$ a.e.*

Proof. $m(g^{-1}(0)) = m(P\chi_{g^{-1}(0)}) = 0$, because $P(\chi_{g^{-1}(0)}) = 0$.

(c) *Put $J = 1/g$ where it exists. Then J is the jacobian of T.*

Proof. For every $B \in \beta$ and $A \subset X$:

$$m(TB \cap A) = m(\chi_A\cdot P(\chi_B\cdot J)) = m\big(P(\chi_A\circ T\cdot\chi_B\cdot J)\big) = m(\chi_A\circ T\cdot\chi_B\cdot J) = \int_{T^{-1}A\cap B} J\,dm$$

since $P(\chi_B\cdot J) = \chi_{TB}$ a.e. $\blacksquare$

We are going to check that iterations of T satisfy our conditions. For our purposes, a partition will mean a countable family of closed intervals such that each two of them can have only an endpoint in common and exhausting X up to a set of measure 0. So β is a partition.

Put $S_N = \bigcup_{k=0}^{N-1} T^{-k}(S)$ and $U_N = X \setminus S_N$ for $N \geqslant 1$. T^N is well defined on U_N. Denote by $\beta^N = \bigvee_{k=0}^{N-1} T^{-k}(\beta)$ the family of all sets of the form $B_0 \cap T^{-1}B_1 \cap \ldots \cap T^{-(N-1)}B_{N-1}$, where $B_0, \ldots, B_{N-1} \in \beta$.

Lemma 1. *For every $f \in \mathrm{BV}$*

(6) $$\operatorname{Var}(f\cdot g) = \sum_{B\in\beta} \operatorname{Var}_B(f\cdot g).$$

Proof. This is an easy consequence of the fact that β is a partition and $g|_S = 0$. $\blacksquare$

We define g_N by $g_N|_{S_N} = 0$ and $g_N|_{U_N} = g\cdot g\circ T \ldots g\circ T^{N-1}$. We will see that T^N, U_N, S_N, g_N, β^N satisfy our conditions for $N \geqslant 1$. The only non-trivial thing to verify is $\operatorname{Var} g_N < +\infty$. But this is a consequence of the following lemma:

Lemma 2. $\operatorname{Var} g_N \leqslant 2^{N-1}(\operatorname{Var} g)^N$, $N \geqslant 1$.

Proof. Case $N = 1$ is obvious. If it is true for some $N \geqslant 1$, then by Lemma 1:

$$\operatorname{Var} g_{N+1} = \operatorname{Var}(g_N\circ T\cdot g) = \sum_{B\in\beta} \operatorname{Var}_B(g_N\circ T\cdot g).$$

But

$$\operatorname{Var}_B(g_N\circ T\cdot g) \leqslant \|g_N\|_\infty \cdot \operatorname{Var}_B g + \operatorname{Var}_B g_N\circ T\cdot \|g\cdot\chi_B\|_\infty \leqslant \operatorname{Var} g_N \operatorname{Var}_B g + \operatorname{Var} g_N \operatorname{Var}_B g = 2\operatorname{Var} g_N \operatorname{Var}_B g.$$

So $\operatorname{Var} g_{N+1} \leqslant 2\operatorname{Var} g_N \operatorname{Var} g$. $\blacksquare$

COROLLARY 1. *If we replace the hypothesis* $\|g\|_\infty < 1$ *by* $\|g_N\|_\infty < 1$ *for some* $N \geqslant 1$, *then* T^N *satisfies our conditions.*

Remark 3. We will see later that

$$\sup_N \operatorname{Var} g_N < +\infty. \tag{7}$$

LEMMA 3. *For every* $B \in \beta^N$, $N \geqslant 1$,

$$m(B) \leqslant \|g_N\|_\infty \leqslant \|g\|_\infty^N. \tag{8}$$

COROLLARY 2. *Let* C *be a closed interval and assume that for every* $N \geqslant 1$ *there exists* $B \in \beta^N$ *such that* $B \supset C$. *Then* $m(C) = 0$. *So* m *has no atoms. Moreover, collapsing all maximal closed intervals of measure* 0, *we can assume that* β *is a generator.*

(We do it from now on.)

LEMMA 4. *For every* $f \in \mathrm{BV}$,

$$\sum_{B \in \beta} \operatorname{Var} P(f \cdot \chi_B) = \operatorname{Var}(f \cdot g). \tag{9}$$

Proof. We notice that $P(f \cdot \chi_B) \circ T|_B = f \cdot g \cdot \chi_B$. So we have $\operatorname{Var} P(f \cdot \chi_B) = \operatorname{Var}_B(f \cdot g \cdot \chi_B) = \operatorname{Var}_B(f \cdot g)$ and we apply Lemma 1. ∎

LEMMA 5. *Let* a *be a finite partition. Then*

$$\operatorname{Var}(f \cdot g) \leqslant \lambda \operatorname{Var} f + D \sum_{A \in a} \Big| \int_A f\, dm \Big|, \tag{10}$$

where $D = \max_{A \in a} \operatorname{Var}_A g / m(A)$ *and* $\lambda = \|g\|_\infty + \max_{A \in a} \operatorname{Var}_A g$.

Proof. We have $\operatorname{Var}_A(f \cdot g) \leqslant \operatorname{Var}_A f \cdot \|g\|_\infty + \|f \cdot \chi_A\|_\infty \operatorname{Var}_A g$ and $\|f \cdot \chi_A\|_\infty \leqslant 1/m(A) \big| \int_A f\, dm \big| + \operatorname{Var}_A f$. Since $\operatorname{Var}(f \cdot g) = \sum_{A \in a} \operatorname{Var}_A(f \cdot g)$, we get (10). ∎

LEMMA 6. *For every* $\varepsilon > 0$ *there exists a partition* a *which is finite and*

$$\max_{A \in a} \operatorname{Var}_A g \leqslant \|g\|_\infty + \varepsilon. \tag{11}$$

Proof. Jumps of g do not exceed $\|g\|_\infty$. Thus, for every $x \in X$ and some interval containing x, $\operatorname{Var}_{U_x} g < \|g\|_\infty + \varepsilon$. Taking a finer than the cover $\{U_x\}_{x \in X}$ we obtain (11). ∎

COROLLARY 3. *For every* $N \geqslant 1$ *and* $\lambda > 2\|g_N\|_\infty$ *there exists* $D > 0$ *such that for every* $f \in \mathrm{BV}$:

$$\operatorname{Var} P^N f \leqslant \sum_{B \in \beta^N} \operatorname{Var} P^N(f \cdot \chi_B) \leqslant \lambda \operatorname{Var} f + D \|f\|_1. \tag{12}$$

Proof. We notice that $P^N f = \sum_{B \in \beta^N} P^N(f \cdot \chi_B)$. We also have $\sum_{A \in a} \big| \int_A f\, dm \big| \leqslant \|f\|_1$. So Lemmas 4, 5 and 6 give Corollary 3. ∎

Since the sequence $(\|g_N\|_\infty)_{N=1}^\infty$ is submultiplicative, the limit $\lim_{N \to \infty} \|g_N\|_\infty^{1/N} = \vartheta$ exists.

PROPOSITION 1. *Given* $\varkappa \in (\vartheta, 1)$ *we can find* $F \geqslant 0$ *such that for every* $f \in \mathrm{BV}$,

$$\sum_{B \in \beta^n} \operatorname{Var} P^n(f \cdot \chi_B) \leqslant F(\|f\|_1 + \varkappa^n \operatorname{Var} f) \tag{13}$$

for $n = 1, 2, \ldots$

Proof. First we fix $k, l \geqslant 1$. We will prove the following lemma:

LEMMA 7. *Let* λ_k, λ_l *and* D_k, D_l *be such that* (12) *holds with* $N = k$ *and* $N = l$, *respectively, if one replaces* λ *by* λ_k, λ_l *and* D *by* D_k, D_l, *respectively. Then* (12) *holds with* $N = k + l$ *and* $\lambda = \lambda_k \lambda_l$ *and* $D = D_k + \lambda_k D_l$.

Proof. We notice that for every $B_1 \in \beta^k$ and $B_2 \in \beta^l$:

$$P^k\big(\chi_{B_1} \cdot P^l(\chi_{B_2} \cdot f)\big) = P^{k+l}(\chi_B \cdot f), \tag{14}$$

where $B = T^{-l} B_1 \cap B_2 \in \beta^{k+l}$. We use (12) for $P^l(f \cdot \chi_{B_2})$ instead of f:

$$\sum_{B_1 \in \beta^k} \operatorname{Var} P^{k+l}(\chi_{B_2 \cap T^{-l} B_1} \cdot f) \leqslant \lambda_k \operatorname{Var} P^l(\chi_{B_2} \cdot f) + D_k \|P^l(\chi_{B_2} \cdot f)\|_1. \tag{15}$$

We see that $\|P^l(\chi_{B_2} \cdot f)\|_1 \leqslant \int_{B_2} |f|\, dm$. Since if B_1 runs through β^k and B_2 runs through β^l, then $B_2 \cap T^{-l} B_1$ runs through β^{k+l}, we have

$$\begin{aligned} \sum_{B \in \beta^{k+l}} \operatorname{Var} P^{k+l}(\chi_B \cdot f) &\leqslant \lambda_k \sum_{B_2 \in \beta^l} \operatorname{Var} P^l(\chi_{B_2} \cdot f) + D_k \|f\|_1 \\ &\leqslant \lambda_k(\lambda_l \operatorname{Var} f + D_l \|f\|_1) + D_k \|f\|_1 \\ &= \lambda_k \lambda_l \operatorname{Var} f + (D_k + \lambda_k D_l) \|f\|_1. \end{aligned} \tag{16}$$

So, we proved the lemma. ∎

Now we fix M such that $\varkappa > \|2 g_M\|_\infty^{1/M}$. We put $\lambda_M = \varkappa^M$. We also fix some λ_0 and D_0 such that (12) holds with $\lambda = \lambda_0$ and $D = D_0$ for $N = 1, 2, \ldots, M$. For any given n we can find k, l such that $n = k \cdot M + l$ and $0 \leqslant l \leqslant M - 1$. For $k \geqslant 1$ we put $D_n = D_M + \lambda_M D_{n-M}$ and $\lambda_n = \lambda_M \lambda_{n-M}$. So, by induction argument we see that for every $N \geqslant 1$ and $f \in \mathrm{BV}$ (12) holds with D_N, λ_N instead of D and λ. On the other hand,

$$\lambda_n \leqslant \lambda_0 \lambda_M^k = \lambda_0 / \varkappa^l \cdot \varkappa^n, \qquad D_n \leqslant D_0(1 + \lambda_M + \ldots + \lambda_M^{k-1}). \tag{17}$$

We put $F = \max\big(D_0/(1 - \lambda_M),\ \lambda_0/\varkappa^{M-1}\big)$. It is easy to see that (13) holds. ∎

COROLLARY 4. *For every $N \geqslant 1$ and $f \in \mathrm{BV}$,*

$$\sum_{B \in \beta^N} \|P^N(f\cdot\chi_B)\| \leqslant (2F+1)\|f\|. \tag{18}$$

Therefore, if P is meant as an operator on BV, *then*

$$\sup_N \|P^N\| \leqslant 2F+1. \tag{19}$$

Proof. For terms $P^N(f\cdot\chi_B)$ with the property

$$\mathrm{Var} P^N(f\cdot\chi_B) > \|P^N(f\cdot\chi_B)\|_1$$

we use (18). Other terms give at most $\|f\|_1 \leqslant \|f\|$. ∎

LEMMA 8. *For any $f \in \mathrm{BV}$ and a finite partition α,*

$$\mathrm{Var} E_m(f\,|\,\alpha) \leqslant \mathrm{Var} f, \tag{20}$$

where $E_m(f\,|\,\alpha)$ is the conditional expectation of f with respect to α. ∎

PROPOSITION 2. *There exist $N \geqslant 1$ and a finite-dimensional operator K on* BV *such that $\|P^N - K\| < 1$.*

Proof. We choose N and a partition α such that Lemma 5 holds for g_N with some $\lambda < 1/2$. Let $Ef = E_m(f\,|\,\alpha)$ for every $f \in \mathrm{BV}$ and $K = P^N E$. We will prove that this choice is good.

Let us fix $f \in \mathrm{BV}$ and take $h = f - E_m(f\,|\,\alpha) = (I-E)f$. For every $A \in \alpha$, $\int_A h\,dm = 0$, so $\mathrm{Var} P^N h \leqslant \lambda \mathrm{Var} h \leqslant 2\lambda \mathrm{Var} f$ (Lemma 8) and $\|P^N h\|_1 \leqslant \|h\|_1 = \sum_{B\in\beta^N} \int_B |h|\,dm$. We see that $\|h\cdot\chi_B\|_\infty \leqslant 2\|f\cdot\chi_B\|_\infty \leqslant 2\mathrm{Var}_B f$ and $m(B) \leqslant \|g_N\|_\infty \leqslant \lambda$ for $B \in \beta^N$, so $\|P^N h\|_1 \leqslant 2\lambda\|f\|$. So $\|P^N h\| \leqslant 2\lambda\|f\|$. We recall that $2\lambda < 1$. ∎

§ 2. In this section we give a description of some spectral properties of P.

THEOREM 1. *If P is meant as operator on* BV, *then:*

(a) *$\sigma(P)\cap S^1$ consists of a finite number of simple poles of the resolvent of P. Moreover, $\sigma(P)\cap S^1$ is a union of full cyclic groups.*

(b) *Other points of $\sigma(P)$ are contained within a circle of radius $r \in (0,1)$.*

(c) *If $\sigma(P)\cap S^1 = \{\xi_1, \ldots, \xi_L\}$, we denote by Q_j the projector on the corresponding eigenspace, $j = 1, \ldots, L$; then P admits the representation*

$$P = \sum_{j=1}^{L} \xi_j Q_j + R,$$

where $R\colon \mathrm{BV}\to\mathrm{BV}$ and $\varrho(R) = \inf_N \|R^N\|^{1/N} < r$. Operators $Q_1, \ldots, Q_L, R$ commute and, moreover, $Q_iQ_j = 0$ for $i \neq j$ and $Q_iR = RQ_i = 0$ for $i, j = 1, 2, \ldots, L$.

Proof. All these facts are implied by Proposition 2 and (19), by using the uniform ergodic theorem (see [3]), except for the possibility to represent $\sigma(P)\cap S^1$ as a union of full cyclic groups. It is a consequence of the theory of positive operators (see [4]). We would like to quote one theorem which refers to our situation.

Let Y be a Banach lattice. We assume that the absolute value on Y allows an extension on the complexification of Y. Let $U\colon Y\to Y$ be a positive operator such that there exists a strictly positive functional on Y which is an eigenvector of U^* pertaining the eigenvalue $\varrho(U)$. Then the peripheral point spectrum of U is cyclic ($\xi \in \sigma_P(U)$, $|\xi| = \varrho(U)$ $\Rightarrow \xi(\xi/|\xi|)^n \in \sigma_P(U)$, $n = 1, 2, \ldots$).

THEOREM 2. *Operators $Q_1, \ldots, Q_L, R$ have unique extensions to operators on L_1. Moreover, $Q_j(L_1) \subset \mathrm{BV}$ and Q_j is bounded as an operator from L_1 to* BV, *$\|Q_j\|_1 \leqslant 1$, $j = 1, \ldots, L$, and $\sup_N \|R^N\|_1 < +\infty$. For every $f \in L_1$ $\lim_{N\to\infty} R^N f = 0$.*

Proof. We notice that for $j = 1, \ldots, L$ and $\xi \in S^1$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} (P/\xi)^k = \begin{cases} 0 & \text{as} \quad \xi \notin \{\xi_1, \ldots, \xi_L\}, \\ Q_j & \text{as} \quad \xi = \xi_j, \end{cases} \tag{21}$$

where the convergence is in BV. But $\|P/\xi\|_1 = 1$, so Q_j can be defined on L_1 by (29), because BV is dense in L_1, and the extension is unique. Using Lemma 8 and Corollary 5 we easily prove other points. ∎

Now we are ready to analyse the situation when $\sigma(P)\cap S^1$ consists of 1 only. This allows us to describe the eigenvalues of P in the general case.

Let $P = Q + R$ be the decomposition from Theorem 1. By $f_1\wedge f_2$ we denote $\min(f_1, f_2)$. We write Tf instead of $f\circ T$.

THEOREM 3. *There exist nonnegative functions $\varphi_1, \ldots, \varphi_s \in \mathrm{BV}$ and $\psi_1, \ldots, \psi_s \in L_\infty$ such that:*

(a) *For every $f \in L_1$,*

$$Qf = \sum_{i=1}^{s} m(\psi_i\cdot f)\varphi_i. \tag{22}$$

(b) *$P\varphi_i = \varphi_i$, $T\psi_i = \psi_i$ for $i = 1, \ldots, s$.*

(c) *$m(\varphi_i\psi_j) = \delta_{ij}$, $\psi_i\wedge\psi_j = 0 = \varphi_i\wedge\varphi_j$ as $i \neq j$; $m(\varphi_i) = 1$ for $i = 1, \ldots, s$.*

(d) *There exist measurable sets $C_1, \ldots, C_s \subset X$ such that $\psi_i = \chi_{C_i}$ a.e. for $i = 1, \ldots, s$ and $X = \bigcup_{i=1}^{s} C_i$ a.e.*

(e) $\bigcap_{N=1}^{\infty} T^N(L_1) = \bigcap_{N=1}^{\infty} T^N(L_\infty) = \langle \psi_1, \ldots, \psi_s \rangle$ *(vector space spanned by* $\psi_1, \ldots, \psi_s$).

(f) *For every* $f \in L_1$, $T^N f \to Q^* f$ *in* $\sigma(L_1, \mathrm{BV})$*-topology as* $N \to +\infty$. *Similarly, for every* $f \in L_\infty$, $T^N f \to Q^* f$ *in* $\sigma(L_\infty, L_1)$*-topology and*

$$Q^* f = \sum_{i=1}^{s} m(\varphi_i \cdot f)\psi_i.$$

Proof. Q is a positive operator by (21). Let $Z = \mathscr{R}(Q) = \ker(I-P)$. Then Z is a Banach sublattice of BV and L_1. Indeed, let $f_1, f_2 \in Z$. Then $Q(f_1 \wedge f_2) \leqslant Qf_1 \wedge Qf_2 = f_1 \wedge f_2$. But Q preserves m and we have $m(Q(f_1\wedge f_2)) = m(f_1 \wedge f_2)$. So $Q(f_1\wedge f_2) = f_1 \wedge f_2$ a.e.

Let $\Delta = \{\varphi \in Z\colon m(\varphi) = 1 \text{ and } \varphi \geqslant 0\}$. We notice that if φ, φ' are two distinct extreme points of Δ, then $\varphi \wedge \varphi' = 0$. So extreme points of Δ are linearly independent and there is only a finite number of them. We denote them by $\varphi_1, \ldots, \varphi_s$. Hence $s \leqslant \dim Z$. On the other hand, $s \geqslant \dim Z$, since $\varphi_1, \ldots, \varphi_s$ span Δ, by the Krein–Millman theorem, and Δ spans Z, because $\Delta = Q(\{f \in L_1\colon f \geqslant 0 \text{ and } m(f) = 1\})$. So $s = \dim Z$. For every $f \in Z$,

$$f = \sum_i \varphi_i^*(f)\varphi_i, \tag{23}$$

where $(\varphi_i^*)_{i=1}^{s}$ is the basis dual to $(\varphi_i)_{i=1}^{s}$. So for every $f \in L_1$,

$$Qf = \sum_i \varphi_i^*(Qf)\varphi_i = \sum_i \mu_i(f)\varphi_i, \tag{24}$$

where $\mu_i = Q^*\varphi_i^*$ is a positive functional on L_1. As such, μ_i can be represented as

$$\mu_i(f) = m(\psi_i f), \quad f \in L_1, \tag{25}$$

where $\psi_1, \ldots, \psi_s \in L_\infty$. We obtained (a). We notice that $P^* = T$. Using $PQf = Qf = QPf = \sum_i m(\psi_i Pf)\varphi_i$, we infer that $m(P^*\psi_i \cdot f) = m(\psi_i \cdot Pf) = m(\psi_i f)$ for every $f \in L_1$ and $i = 1, \ldots, s$. So $P^*(\psi_i) = \psi_i = T\psi_i$.

Now we use $Q^2 = Q$. We have $\sum_i m(\psi_i f)\varphi_i = Qf = Q^2 f = Q\big(\sum_i m(\psi_i f)\varphi_i\big) = \sum_{i,j} m(\psi_j\varphi_i) m(\psi_i f)\varphi_j$. This gives $m(\psi_i\varphi_j) = \delta_{ij}$. Now we check (f). It is easy to see that $m(Qf\cdot h) = m(f\cdot Q^*h) = \sum_i m(\psi_i f)m(\varphi_i h)$ and (f) follows by

$$m(T^N f\cdot h) = m(f\cdot P^N h) \to m(f\cdot Qh) = m(Q^*f\cdot h) \quad \text{as} \quad N \to +\infty.$$

We notice that $T^N\varphi_i \to \sum_j m(\varphi_i\varphi_j)\psi_j = m(\varphi_i^2)\psi_i$ in $\sigma(L_\infty, L_1)$-topology as $N\to\infty$, $i = 1, \ldots, s$. We also have (for $i \neq j$) $\psi_i \wedge T^N\varphi_j = (\psi_i \wedge \varphi_j)\circ T^N = 0$, $N = 1, 2, \ldots$ We get $0 = m(\psi_i T^N\varphi_j) \to m(\psi_i\psi_j)m(\varphi_j^2)$ as $N\to\infty$. Since $m(\varphi_j^2) > 0$, $m(\psi_i\psi_j) = 0$ for $i \neq j$. So, $\psi_i \wedge \psi_j = 0$ for $i \neq j$.

We completed (a), (b), (c) and (f).

Since $T1 = 1$, so $Q^*1 = 1$ and

$$1 = \sum_{i=1}^{s} \psi_i. \tag{26}$$

This gives (d).

Let $h \in \bigcap_{N=1}^{\infty} T^N(L_\infty)$. We choose a sequence $h_N \in L_\infty$ such that $h = h_N \circ T^N$ and $\|h_N\|_\infty \leqslant \|h\|_\infty$. Since $(Q^*h_N)_{N=1}^{\infty}$ is bounded in $\langle\psi_1, \ldots, \psi_s\rangle$, we can choose a subsequence $N_k \to \infty$ such that $Q^*h_{N_k} \to h_0$ as $k\to\infty$. We will see that $h = h_0$ a.e. In fact, for every $f \in L_1$,

$$\begin{aligned} m(f\cdot h) &= m(f\cdot h_{N_k}\circ T^{N_k}) = m(P^{N_k}f\cdot h_{N_k}) \\ &= m\big((P^{N_k}-Q)f\cdot h_{N_k}\big) + m(Qf\cdot h_{N_k}). \end{aligned} \tag{27}$$

Since the first term tends to 0 and $m(Qf\cdot h_{N_k}) = m(f\cdot Q^* h_{N_k})$, we obtain $m(f\cdot h) = m(f\cdot h_0)$ for every f. So, $h = h_0$ a.e. and $h \in \langle\psi_1, \ldots, \psi_s\rangle$. By approximation argument,

$$\bigcap_{N=1}^{\infty} T^N(L_1) = \bigcap_{N=1}^{\infty} T^N(L_\infty). \ \blacksquare$$

COROLLARY 5. *Let* $\mathscr{B}_\infty = \bigcap_{N=0}^{\infty} T^{-N}(\mathscr{B})$. *Then* $\mathscr{B}_\infty$ *is generated by decomposition* $(C_1, \ldots, C_s)$. *The eigenspace of* T *for the eigenvalue* 1 *is exactly* $\langle\psi_1, \ldots, \psi_s\rangle$.

Proof. $\bigcap_{N=1}^{\infty} T^N(L_\infty) = \{f \in L_\infty\colon f \text{ is } \mathscr{B}_\infty\text{-measurable}\}$. $\blacksquare$

Remark 4. (a) $\psi_1, \ldots, \psi_s$ determine extreme rays of the positive cone in $\mathscr{R}(Q^*)$.

(b) Dynamical systems (T_i, ν_i), where $T_i = T|_{C_i}$ and $\nu_i = \varphi_i\cdot m$, $i = 1, \ldots, s$, are exact and ν_i is the only invariant measure for T_i absolutely continuous with respect to $m|_{C_i}$.

THEOREM 4. *Let* T *be as in* § 1. *Fix some* M *verifying* $\sigma(P^M)\cap S^1 = \{1\}$. *Let* $\varphi_1, \ldots, \varphi_s$, $\psi_1 = \chi_{C_1}, \ldots, \psi_s = \chi_{C_s}$ *be constructed by Theorem* 3 *applied to* P^M. *Then there exists a permutation* π *of the set* $\{1, \ldots, s\}$ *such that*

$$P\varphi_i = \varphi_{\pi(i)}, \quad T\psi_{\pi(i)} = \psi_i \quad \textit{for} \quad i = 1, \ldots, s. \tag{28}$$

Let $l_1, \ldots, l_q$ be the lengths of the independent cycles of π. Then $\sigma(P)\cap S^1$ is a union of cyclic groups of ranks $l_1, \ldots, l_q$. Let $l \in \{l, \ldots, l_q\}$

and $\xi^l = 1$. The eigenspace for ξ is generated by functions

$$\frac{1}{l}\sum_{k=0}^{l-1}\xi^{-k}P^k\varphi_i = \frac{1}{l}\sum_{k=0}^{l-1}\xi^{-k}\varphi_{\pi^k(i)}, \tag{29}$$

where i belongs to some cycle of length l. Similarly, the eigenfunctions for T are given by

$$\frac{1}{l}\sum_{k=0}^{l-1}\xi^k\psi_{\pi^k(i)}, \qquad i = 1, \ldots, s. \tag{30}$$

Remark 5. It is easy to prove that

$$s \leqslant \operatorname{Card}\alpha, \tag{31}$$

where α is chosen as in Proposition 2.

§ 3. We are going to discuss in detail the Bernoulli property for the systems (T_i, ν_i) (see Remark 4(b)). The restriction to such dynamical systems is equivalent to the assumption that 1 is the only eigenvalue of P on the unit circle and there exists only one $\varphi \in L_1$, $P\varphi = \varphi$ and $m(\varphi) = 1$ $(\varphi \geqslant 0)$. The system (T, μ), where $\mu = \varphi m$, is exact by § 2.

Let us introduce the following notation. Given partitions ξ, ζ, we define

$$\begin{aligned} d(\xi,\zeta) &= \sum_{A\in\xi, B\in\zeta} |\mu(A\cap B) - \mu(A)\mu(B)| \\ &= \sum_{A\in\xi}\sum_{B\in\zeta} |\mu(A\,|\,B) - \mu(A)|\,\mu(B). \end{aligned} \tag{32}$$

By the Ornstein theory [5], the Bernoulli property is implied by

$$\sup_{t,l\geqslant 1} d(\beta^t, \beta_{t+n}^{t+n+l}) \to 0 \quad \text{as} \quad n\to\infty. \tag{33}$$

We denote by $\mathscr{F}_{t_1}^{t_2}$ $(0 \leqslant t_1 \leqslant t_2 \leqslant +\infty)$ the σ-algebra generated by the partition $\beta_{t_1}^{t_2} = \bigvee_{t=t_1}^{t_2} T^{-t}(\beta)$. Let

$$b(n) = \sup_{t\geqslant 1} E_\mu\Big(\sup_{A\in\mathscr{F}_{t+n}^{\infty}} |\mu(A\,|\,\mathscr{F}_0^t) - \mu(A)|\Big). \tag{34}$$

Let us notice that $2b(n)$ is equal to the left-hand side of (33) (cf. [1]). So, we want to prove $b(n)\to 0$ as $n\to\infty$.

THEOREM 5. *There exist $K \geqslant 0$ and $r \in (0, 1)$ such that*

$$b(n) \leqslant Kr^n, \quad n = 1, 2, \ldots \tag{35}$$

Proof. Take $A \in \mathscr{F}_{t+n}^{\infty}$, $t, n \geqslant 1$. There exists $\tilde{A} \in \mathscr{F}_0^{\infty} = \mathscr{B}$ such that $A = T^{-(t+n)}\tilde{A}$. We have

$$\mu(A\,|\,\mathscr{F}_0^t) = \frac{1}{\mu(B)}\int_A \chi_B\varphi\,dm = \frac{1}{\mu(B)}\int_{\tilde{A}} P^{t+n}(\chi_B\varphi)\,dm$$

on $B \in \beta^t$. Of course, $\mu(A) = \mu(\tilde{A}) = \int_{\tilde{A}}\varphi\,dm$. This gives

$$\begin{aligned} |\mu(A\,|\,\mathscr{F}_0^t) - \mu(A)| &\leqslant \int_{\tilde{A}} |P^{t+n}(\chi_B\varphi)/\mu(B) - \varphi|\,dm \\ &\leqslant \|P^{t+n}(\chi_B\varphi)/\mu(B) - \varphi\|_1. \end{aligned} \tag{36}$$

This implies

$$b(n) \leqslant \sup_{t\geqslant 1}\sum_{B\in\beta^t} \|P^{t+n}(\chi_B\varphi) - \mu(B)\varphi\|_1 \quad \text{for} \quad n \geqslant 1. \tag{37}$$

But

$$\begin{aligned} \|P^{t+n}(\chi_B\varphi) - \mu(B)\varphi\|_1 &= \big\|P^n\big(P^t(\chi_B\varphi)\big) - \mu(B)\varphi\big\|_1 \\ &\leqslant K_1\cdot r^n\big(\|P^t(\chi_B\varphi)\| + \mu(B)\|\varphi\|\big) \end{aligned}$$

by Theorem 1 (c) since in our case $\sigma(P)\cap S^1 = \{1\}$ and, by Theorem 2, the projection of $P^t(\chi_B\varphi)$ on the eigenspace equals $m\big(P^t(\chi_B\varphi)\big)\,\varphi = \mu(B)\varphi$, Using Corollary 4 we obtain

$$b(n) \leqslant K_1 r^n\Big(\sum_{B\in\beta^t}\|P^t(\chi_B\varphi)\| + \|\varphi\|\Big) \leqslant K_1(2F+2)\|\varphi\|\,r^n. \tag{38}$$

So, we can put $K = K_1(2F+2)\|\varphi\|$. ■

COROLLARY 6. *The natural extension of the dynamical system (T, ν) is isomorphic with some Bernoulli shift.*

(b) *Limit theorems hold in the form proposed by Hofbauer and Keller in* [1].

§ 4. Many examples can be found in [1]. We give one example that cannot be verified there.

Put $X = [0, 1]$ and let m be the Lebesgue measure on $[0, 1]$. Let us choose a family of open disjoint intervals $(I_j)_{j=1}$ for which $m(I\setminus\bigcup_{j=1}^{\infty} I_j) = 0$. We consider $T\colon \bigcup_{j=1}^{\infty} I_j \to I$ such that, for any j, $T|_{I_j}$ is linear with slope k_i. Our conditions in this case are equivalent to

$$\inf_{1\leqslant i<+\infty} k_i > 1, \qquad \sum_{i=1}^{\infty} k_i^{-1} < +\infty. \tag{39}$$

The following example shows that the second condition cannot be omitted:

$$I_j = (1/2^j, 1/2^{j-1}), \quad j = 1, 2, \ldots, \quad (T/I_j)(x) = 2(x - 1/2^j). \tag{40}$$

T has no measures absolutely continuous with respect to m, because almost every point of $[0, 1]$ tends to 0 under iterations of T. This example shows that our method is very effective for such T.

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References

[1] F. Hofbauer, G. Keller, *Ergodic properties of invariant measures for piecewise monotonic transformations*, preprint.

[2] P. Walters, *Invariant measures and equilibrium states for some mappings which extend distances*, Trans. Amer. Math. Soc. 236 (1978), 121–153.

[3] N. Dunford, J. T. Schwartz, *Linear operators, Part I. General theory*, Wiley, New York 1979.

[4] H. H. Schaefer, *Banach lattices and positive operators*, Springer-Verlag, Berlin, Heidelberg, New York 1979.

[5] P. S. Ornstein, *Ergodic theory, randomness and dynamical systems*, Yale University Press, New Haven and London 1974.

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