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On non-removable ideals in commutative locally convex algebras

by

W. ŻELAZKO (Warszawa)

To Professor Jan Mikusiński on the occasion of the 70th birthday

Abstract. The main result of this paper gives a characterization of non-removable ideals in the class of commutative locally convex algebras. We give also some sufficient conditions for non-removability of ideals in arbitrary topological algebras and pose several open questions.

§ 1. Introduction. All the algebras considered in this paper are commutative complex algebras having unit elements. The unit element of an algebra A will be denoted by e, or, if it is necessary to indicate the algebra in question, by e_A. The group of all invertible elements of an algebra A is denoted by $G(A)$. The *radical* of an algebra A is the ideal given by

$$\operatorname{rad} A = \{x \in A:\ e - xa \in G(A) \text{ for all } a \text{ in } A\}.$$

By a *topological algebra* we mean a topological linear space together with a jointly continuous associative multiplication making of it an algebra over $\boldsymbol{C}$. We shall be concerned mostly with the class LC of locally convex algebras. The topology of such an algebra can be introduced by means of a family $(\|x\|_\alpha)$, $\alpha \in \mathfrak{A}$, of seminorms. Without loss of generality we can assume the following properties of this family. For each index $\alpha \in \mathfrak{A}$ there exists an index β in $\mathfrak{A}$ such that

(1) $$\|xy\|_\alpha \leqslant \|x\|_\beta \|y\|_\beta$$

for all elements x, y in A. The existence of such a family follows from the joint continuity of the multiplication. We can assume, moreover, that for each finite set $(\alpha_1, \ldots, \alpha_n)$ there is a β in $\mathfrak{A}$ such that

(2) $$\|x\|_{\alpha_1} \leqslant \|x\|_\beta, \quad i = 1, 2, \ldots, n,$$

for all x in A. This makes of the family $(\|x\|_\alpha)$, $\alpha \in \mathfrak{A}$, a partially ordered set, and under condition (2) a seminorm $\|x\|$ on a locally convex algebra A

is continuous if and only if there is an index $\alpha \in \mathfrak{A}$ and a constant C such that

$$\|x\| \leqslant C\|x\|_\alpha$$

for all x in A. In this case we say that the seminorm $\|x\|_\alpha$ *dominates* the seminorm $\|x\|$. Two families of seminorms $(|x|_\alpha)$, $\alpha \in \mathfrak{A}$, and $(\|x\|_\beta)$, $\beta \in \mathfrak{B}$, of an algebra A, are said to be *equivalent* if they give the same topology on A. In the case where both families satisfy condition (2), they are equivalent if and only if each seminorm of one family is dominated by some seminorm of the other family, and conversely.

A locally convex space X is called a B_0*-space* if it is moreover completely metrizable. A B_0*-algebra* is a locally convex algebra which is a B_0-space. The topology of a B_0-algebra can be given by means of a sequence $(\|x\|_i)$ of seminorms which, according to conditions (1) and (2), satisfies the following conditions:

$$\|xy\|_i \leqslant \|x\|_{i+1}\|y\|_{i+1}, \tag{3}$$

for $i = 1, 2, \ldots$ and all elements x and y in A, and

$$\|x\|_i \leqslant \|x\|_{i+1} \tag{4}$$

for $i = 1, 2, \ldots$ and all x in A. For the purpose of this paper we shall denote by B_0 the class of all metrizable locally convex algebras without the customary assumption of completeness. A locally convex algebra A is said to be *locally multiplicatively convex* (shortly *m-convex*) if the system $(\|x\|_\alpha)$, $\alpha \in \mathfrak{A}$, of seminorms giving the topology of A satisfies instead of condition (1) the stronger condition

$$\|xy\|_\alpha \leqslant \|x\|_\alpha\|y\|_\alpha \tag{5}$$

for all α in $\mathfrak{A}$ and all $x, y \in A$. Denote by M the class of all m-convex algebras. For an m-convex algebra with the topology given by means of a system of seminorms $(\|x\|_\alpha)$, $\alpha \in \mathfrak{A}$, satisfying conditions (2) and (5) we let

$$I_\alpha = \{x \in A\colon \|x\|_\alpha = 0\}, \quad \alpha \in \mathfrak{A},$$

which is a closed ideal in A, and denote by $\pi_\alpha(x)$ the natural projection of an element x in A onto the normed algebra $A_\alpha = A/I_\alpha$ equipped with the norm $\|x\|_\alpha$.

We also put $M_0 = B_0 \cap M$, so that the topology of an algebra $A \in M_0$ can be given by means of a sequence of seminorms satisfying conditions (4) and (5).

Denoting respectively by B and T the classes of all (commutative) Banach algebras and topological algebras, we have the following inclusions

$$B \subset M_0 \subset M \subset \mathrm{LC} \subset T$$

and

$$B \subset M_0 \subset B_0 \subset \mathrm{LC} \subset T.$$

Let K be any class of topological algebras and let $A \in K$. We say that a topological algebra B is a *K-extension* of the algebra A, or a superalgebra of the class K for A, if $B \in K$ and there exists a unital topological isomorphism of A into B, i.e., a homeomorphism $\varphi\colon A \to B$, which is an algebra isomorphism and $\varphi(e_A) = e_B$. We can then identify A with $\varphi(A)$ and treat A as a subalgebra of B with the topology inherited from B. In this case we shall write simply $A \subset B$.

The term ideal in the context of a topological algebra A will be used in a purely algebraic sense: it will mean just a proper ideal I (i.e., $I \neq A$) with no assumptions about its topological structure.

1.1. Definition. Let K be any class of topological algebras and let $A \in K$. We say that an ideal $I \subset A$ is *removable in the class K*, or *K-removable*, if there is a K-existension B of A such that I is not contained in any proper ideal of B. In this case we say that the algebra B *removes* the ideal I. Otherwise we say that the ideal I is *K-non-removable*.

Since the smallest ideal J generated in an extension B of A by an ideal $I \subset A$ is of the form

$$J = \Big\{\sum_{i=1}^{n} x_i b_i \in B\colon x_i \in I,\ b_i \in B,\ n \in N\Big\},$$

we immediately see that an extension B of A removes an ideal $I \subset A$ if and only if there are elements $x_1, \ldots, x_n \in I$, $b_1, \ldots, b_n \in B$ such that

$$\sum_{i=1}^{n} x_i b_i = e. \tag{6}$$

1.2. Definition. Let K be any class of topological algebras and let $A \in K$. We say that an n-tuple $(x_1, \ldots, x_n)$ of elements of A is *regular in the class K*, or *K-regular*, if there are a K-extension B of A and elements $b_1, \ldots, b_n \in B$ such that relation (6) holds true. Otherwise we say that the n-tuple $(x_1, \ldots, x_n)$ is *K-singular*.

The proof of the following proposition is straightforward:

1.3. Proposition. *Let K be a class of topological algebras and let $A \in K$. An ideal $I \subset A$ is K-non-removable if and only if for each natural n every n-tuple $(x_1, \ldots, x_n) \subset I$ is K-singular.*

The above proposition reduces the problem of characterization of K-non-removable ideals to the characterization of K-singular or K-regular n-tuples of elements of A.

If K_1 and K_2 are two classes of topological algebras with $K_1 \subset K_2$, then each K_1-removable ideal of an algebra A in K_1 is also K_2-removable, and each K_2-non-removable ideal of A is K_1-non-removable.

1.4. Definition. Let K be a class of topological algebras. We say that the concept of removability of ideals in algebras of the class K is *of absolute character* if each K-non-removable ideal of an algebra A in K is also T-non-removable. Otherwise, we say that the concept of K-removability is of relative character.

This means that if the concept of K-removability of ideals is of relative character, then there is an algebra $A \subset K$ and a K-non-removable ideal $I \subset A$ which is removed by a certain extension B of A outside the class K.

Recently Müller [5] obtained a characterization of B-non-removable ideals; it follows that the concept of B-removability is of absolute character. In paper [12] we reduced the problem of characterization of M-removability to the problem of characterization of B-removability. Thus, both results give the characterization of M-removability and also of M_0-removability. These and related results will be described in § 2, where we shall also give some sufficient conditions for removability of ideals in arbitrary topological algebras.

Our main result—the characterization of LC-non-removable ideals and B_0-non-removable ideals–is given in § 3. We apply here a method used in [15], where we characterized permanently singular elements in the classes LC and B_0.

In § 4 we pose several open questions concerning non-removable ideals in various classes of topological algebras. For more information on topological algebras the reader is referred to [7] and [8].

§ 2. Various types of non-removable ideals in topological algebras.

2.1. Definition. Let A be a topological algebra. We say that a non-void subset $S \subset A$ *consists of joint topological divisors of zero* if there is a neighbourhood U of the origin in A such that for each finite subset $(x_1, \ldots, x_n) \subset S$ and each neighbourhood V of the origin in A there is an element $z \in A \setminus U$ with

$$x_i z \in V \tag{7}$$

for $i = 1, 2, \ldots, n$.

Perhaps a clearer version of this definition can be obtained from the following

2.2. Proposition. *A non-void subset S of a topological algebra A consists of joint topological divisors of zero if and only if there is a net $(z_\alpha) \subset A$ which does not tend to zero but*

$$\lim_\alpha z_\alpha x = 0 \tag{8}$$

for all x in S.

Proof. Suppose that relation (8) holds true. Since the net (z_α) does not tend to zero, there is a neighbourhood U of the origin in A such that for each index α there is an index β, $\beta > \alpha$, with $z_\beta \notin U$. Take any finite n-tuple $(x_1, \ldots, x_n) \subset S$ and any neighbourhood V of the origin in A. Relation (8) implies that for each i, $1 \leqslant i \leqslant n$, there is an index α_i such that

$$z_\alpha x_i \in V \tag{9}$$

for all $\alpha \geqslant \alpha_i$. Take any index γ larger than any α_i, $1 \leqslant i \leqslant n$, and find an index $\beta \geqslant \gamma$ with $z_\beta \notin U$. By (9) we now have

$$z_\beta x_i \in V$$

for $i = 1, 2, \ldots, n$ and relation (7) holds true with $z = z_\beta$. Thus the set S consists of joint topological divisors of zero.

Suppose conversely that a subset $S \subset A$ consists of joint topological divisors of zero. We have to construct a net (z^α), $\alpha \in \mathfrak{A}$, which does not tend to zero in A, such that relation (8) holds true for each x in S. We first construct a directed set $\mathfrak{A}$ of indices. The elements α of $\mathfrak{A}$ are the pairs $\alpha = (F_\alpha, V_\alpha)$, where F_α is a finite subset of S and V_α is a neighbourhood of the origin in A. We take here all finite subsets of S and all neighbourhoods of the origin. Write $\alpha \leqslant \beta$ if $F_\alpha \subset F_\beta$ and $V_\alpha \supset V_\beta$. Let U be a neighbourhood of the origin in A satisfying the condition in Definition 2.1. For any index α we can choose an element $z_\alpha \in A$, $z_\alpha \notin U$, satisfying relation (7) for $V = V_\alpha$ and all x_i in F_α. Obviously the net (z_α) does not tend to zero. It remains to show that relation (8) holds true for all x in S, i.e., we have to show that for an $x \in S$ and a neighbourhood V of zero in A there is an index $\alpha(x, V) \in \mathfrak{A}$ such that

$$z_\alpha x \in V$$

for all $\alpha \geqslant \alpha(x, V)$. But this holds if we take $\alpha(x, V) = (\{x\}, V)$, and we are done.

2.3. Definition. If S is a non-void subset of a topological algebra A, consisting of joint topological divisors of zero, then any net $(z_\alpha) \subset A$ which does not tend to zero and satisfies relation (8) will be called an *annihilating net* for the set S. In this case we shall write $(z_\alpha) \perp S$.

2.4. Proposition. *Let A be a topological algebra and let S be a subset in A consisting of joint topological divisors of zero. The smallest ideal I_S gen-*

erated in A by S is a proper ideal in A and consists also of the joint topological divisors of zero. Moreover, $(z_a)\perp I_S$ if and only if $(z_a)\perp S$.

Proof. We have

$$I_S=\Big\{\sum_{i=1}^n a_ix_i\in A:\ x_i\in S,\ a_i\in A,\ i=1,2,\ldots,n,\ n\in N\Big\}.$$

If $(z_a)\perp S$ then $(z_a)\perp I_S$ since $\lim_a z_a\sum_1^n a_ix_i=\sum_1^n a_i\lim_a z_ax_i=0$.

If $(z_a)\perp I_S$ then $(z_a)\perp S$ since $S\subset I_S$. Also I_S is a proper ideal since the nets (z_a) annihilating S do not tend to zero.

2.5. Proposition. *Let A be a topological algebra and let I be an ideal in A consisting of joint topological divisors of zero. Then I is a T-non-removable ideal in A.*

Proof. Let B be any extension of A. Since I, treated as a subset of B, consists of joint topological divisors of zero, by Proposition 2.4 I generates in B a proper ideal J and $I\subset J$. Thus B does not remove I.

The class of ideals consisting of joint topological divisors of zero in Banach algebras was introduced and studied in paper [10]. Further results concerning these ideals can be found in [5] and [6]. Recently Müller has shown in [5] that for Banach algebras the class of all non-removable ideals coincides with the class of all ideals consisting of joint topological divisors of zero. So an n-tuple $(x_1,\ldots,x_n)\in A\subset B$ is B-regular if and only if t is T-regular and if and only if there is a constant C such that

$$\sum_{i=1}^n\|x_iz\|\geqslant C\|z\| \tag{10}$$

for all z in A. Thus the concept of B-removability of ideals has an absolute character.

Let us note that for a locally convex algebra A, with the topology given by means of the family $(\|x\|_\beta)$, $\beta\in\mathfrak{B}$, a subset $S\subset A$ consists of joint topological divisors of zero if and only if there is a net $(z_a)\subset A$, $a\in\mathfrak{A}$, and a fixed index $\beta_0\in\mathfrak{B}$ such that

$$\|z_a\|_{\beta_0}=1 \tag{11}$$

for all $a\in\mathfrak{A}$, and

$$\lim_a\|z_ax\|_\beta=0 \tag{12}$$

for all x in S and all β in $\mathfrak{B}$.

2.6. Example. Let $A=C(-\infty,\infty)$ be the algebra of all continuous complex-valued functions on the real line. It is an m-convex B_0-algebra in the topology given by the sequence of seminorms

$$\|x\|_i=\max_{|t|\leqslant i}|x(t)|. \tag{13}$$

We shall see that all ideals in A are T-non-removable, whereas not all of them consist of joint topological divisors of zero.

2.7. Proposition. *All finitely generated ideals in the algebra $C(-\infty,\infty)$ consist of joint topological divisor of zero.*

Proof. Let I be an ideal in $C(-\infty,\infty)$ generated by the elements $x_1,\ldots,x_n\in A$, i.e.,

$$I=\Big\{\sum_1^n x_ia_i:\ a_i\in C(-\infty,\infty)\Big\}.$$

The functions $x_1,\ldots,x_n$ must all vanish at a certain point $t_0\in R$; otherwise there would exist continuous functions $y_1,\ldots,y_n$ with $\sum_1^n x_i(t)y_i(t)\equiv 1$, so that the ideal I would be improper. Take any sequence $(z_n(t))$ of positive continuous functions such that the support of z_n is in $[t_0-1/n,t_0+1/n]$, $0\leqslant z_n(t)\leqslant 1$ and $z_n(t_0)=1$. One can easily verify that

$$\lim_n z_nx_i=0,\quad i=1,2,\ldots,n,$$

and $\|z_n\|_k=1$ for $k\geqslant|t_0|$. Thus $(z_n)\perp\{x_1,\ldots,x_n\}$ and by Proposition 2.4 we have $(z_n)\perp I$. The conclusion follows.

2.8. Corollary. *All ideals in $C(-\infty,\infty)$ are T-non-removable.*

Proof. If not, then there is an ideal $I\subset C(-\infty,\infty)$ and elements $x_1,\ldots,x_n\in I$, $b_1,\ldots,b_n\in B$, for some extension B of $C(-\infty,\infty)$ such that

$$\sum_{i=1}^n x_ib_i=e.$$

This means that B removes the ideal in $C(-\infty,\infty)$ generated by the elements $(x_1,\ldots,x_n)$. But it is impossible by Propositions 2.5 and 2.7.

2.9. Remark. There are ideals in $C(-\infty,\infty)$ which do not consist of joint topological divisors of zero.

Proof. Put

$$I=\{x\in A:\ \text{there exists a } t(x)\in R \text{ with } x(t)=0 \text{ for } t\geqslant t(x)\}.$$

Suppose that there is a net (z_a) annihilating I and satisfying relations (11) and (12). Let $t_0 > \beta_0$, where β_0 satisfies relation (12) and take an element x_0 in $C(-\infty, \infty)$ given by

$$x_0(t) = \begin{cases} 1 & \text{for} \quad t \leqslant t_0, \\ \text{continuously decreasing to zero for } t_0 \leqslant t \leqslant t_0+1, \\ 0 & \text{for} \quad t \geqslant t_0+1. \end{cases}$$

We have $x_0 \in I$ and

$$\|z_a x_0\|_{\beta_0} = 1$$

for all a, which is in contradiction with relation (12). The conclusion follows.

The above shows that there is a wider class of T-non-removable ideals than the class of ideals consisting of joint topological divisors of zero. This example suggests the following

2.10. Definition. An ideal I of a topological algebra A is said to *consist locally of joint topological divisors of zero* if each finite subset of I consists of joint topological divisors of zero.

Clearly every ideal consisting of joint topological divisors of zero consists also locally of joint topological divisors of zero. As shown in Remark 2.9, the converse statement fails and the class of ideals given by Definition 2.10 is essentially larger than the class of ideals consisting of joint topological divisors of zero.

Exactly in the same way as in Corollary 2.8 we can prove the following

2.11. Proposition. *All ideals consisting locally of joint topological divisors of zero are T-non-removable.*

Combining the results in [5] and [12] we can give a characterization of M-non-removable (resp. M_0-non-removable) ideals. By the use of Proposition 1.3 we give it in terms of regular n-tuples.

2.12. Theorem. *Let A be an m-convex algebra (resp. an m-convex B_0-algebra). An n-tuple $(x_1, \ldots, x_n)$ of elements of A is M-regular (resp. M_0-regular) if and only if there is a continuous submultiplicative seminorm $\|x\|_0$ on A such that for every submultiplicative continuous seminorm $\|x\|_a$ on A, dominating the seminorm $\|x\|_0$, the n-tuple $(\pi_a(x_1), \ldots, \pi_a(x_n))$ of elements of the normed algebra $A = (A/I_a, \|x\|_a)$ consists of joint topological divisors of zero.*

2.13. Remark. It follows from a result in [9] that the concepts of M and M_0-removability is of relative character. There is an m-convex B_0-algebra A and an M-non-removable ideal in A which can be removed by a certain B_0-extension of A.

Using Theorem 2.12 it is easy to give an example of an m-convex B_0-algebra in which all non-zero ideals are M_0-removable.

2.14. Example. Let E be the algebra of all entire functions of one complex variable. It is an m-convex B_0-algebra with the seminorms of the form (13). Let I be an ideal in E and let a be a non-zero element in I. We shall show that there exists no seminorm $\|x\|_0$ on E satisfying the conditions of Theorem 2.12.

For, if $\|x\|_0$ is such a seminorm, then there is a seminorm $\|x\|_n$ of form (13) and a constant $C > 0$ such that

$$\|x\|_0 \leqslant C\|x\|_n \tag{14}$$

for all x in E. Since a is a non zero entire function, it has at most a countable number of zeros, and there is a number $r > n$ with

$$\inf_{|\zeta|=r} |a(\zeta)| = \delta > 0. \tag{15}$$

By the maximum principle

$$\|x\|_r \geqslant \|x\|_n \tag{16}$$

for all x in E, where

$$\|x\|_r = \max_{|\zeta|=r} |x(\zeta)|.$$

Relation (15) now implies

$$\|az\|_r \geqslant \delta\|z\|_r$$

for all z in E, which implies that $\pi_r(a)$ is not a topological divisor of zero in the normed algebra $(E, \|x\|_r)$. Also, by (14) and (16), $\|x\|_r$ dominates $\|x\|_0$, and we obtain a contradiction proving our assertion.

We now pass to the class T and give yet another type of non-removable ideals. To this end we first give a description of elements having small powers. These elements were introduced in paper [14].

2.15. Definition. Let A be a topological algebra. An element $x \in A$ is said to *have small powers* if for each neighbourhood U of the origin in A there is an integer n such that

$$\lambda x^n \in U \tag{17}$$

for all complex scalars λ.

As shown in [14], the above relation then holds for all exponents n larger than a certain $n(U, x)$. If A is a locally convex algebra, then an element x in A has small powers if and only if for each continuous seminorm $\|x\|_a$ there is an integer $n(a)$ such that

$$\|x^n\|_a = 0 \tag{18}$$

for all $n \geqslant n(a)$. We have shown in [14] that the set $I_s(A)$ of all elements of a topological algebra A which have small powers is an ideal in A contained in the radical of each superalgebra B of A. Also $I_s(A) \subset I_s(B)$ where B is an extension of A. Thus $I_s(A)$ is a T-non-removable ideal for any topological algebra A.

2.16. Example. Let A be the algebra of all power series with complex coefficients and convolution multiplication. It is an m-convex B_0-algebra with seminorms

$$\Big\|\sum_{j=0}^{\infty} a_j t^j\Big\|_n = \sum_{j=0}^{n} |a_j|.$$

Evidently the generator t has small powers, and the only maximal ideal of A consisting of elements of the form $\sum_{j=1}^{\infty} a_j t^j$ coincides with $I_s(A)$. Moreover, this ideal does not consist locally of joint topological divisors of zero. The algebra A has no topological divisors of zero except the zero element (cf. [4], [14]).

The following proposition gives yet another type of a non-removable ideal:

2.17. Proposition. *Let K be any class of topological algebras and let $A \in K$. Let I_1 and I_2 be any two ideals in A, with $I_2 \subset I_s(A)$. Then the ideal $I = I_1+I_2$ is K-non-removable if and only if I_1 is such an ideal.*

Proof. It is sufficient to show that I_1 is a K-removable ideal in A if and only if I is such an ideal. Suppose then that the ideal I_1 can be removed by an extension $B \supset A$, $B \in K$. Since I contains I_1, the same extension also removes I. Suppose now that the ideal I can be removed by an extension B of A with $B \in K$. Thus there are elements $x_1, \ldots, x_n \in I$ and elements $b_1, \ldots, b_n \in B$ such that

$$\sum_{i=1}^{n} x_i b_i = e.$$

Since each x_i can be written in the form

$$x_i = u_i + v_i, \quad i = 1, 2, \ldots, n,$$

with $u_i \in I_1$, $v_i \in I_2$, we have

$$\sum_{i=1}^{n} u_i b_i + \sum_{i=1}^{n} v_i b_i = e. \tag{19}$$

But the element $\sum_1^n v_i b_i$ has small powers, and so, by a result in [14], it is contained in the radical of B. Consequently the element $b = e - \sum_1^n v_i b_i$ is an invertible element in B. This implies, in view of (19),

$$\sum_{i=1}^{n} u_i b_i b^{-1} = e,$$

and since $u_i \in I_1$, the extension B removes the ideal I_1. The conclusion follows.

As a corollary we obtain the following:

2.18. Proposition. *Let A be a topological algebra and let I be an ideal in A contained in an ideal of the form*

$$J = I_1 + I_s(A), \tag{20}$$

where I_1 consists locally of joint topological divisors of zero. Then I is a T-non-removable ideal in A.

This is the most general type of a non-removable ideal, known to the author.

§ 3. A characterization of LC-non-removable ideals. The result given here is a generalization of the main result given in [15], where we characterized permanently singular elements in commutative locally convex algebras. Also the proof is similar; its origin is in paper [13]. Before proving the characterization theorem we introduce some notation useful in the sequel. We denote by Z the set of all integers and by N the subset of all non-negative numbers in Z. We fix a natural number n. Any element μ of N^n will be called a *multiindex*. Thus

$$\mu = (i_1, \ldots, i_n), \tag{21}$$

where $i_j \geqslant 0$. On multiindices we can perform coordinatewise addition and subtraction; the latter, however, may lead from elements in N^n to elements in Z^n. We write also $|\mu| = i_1 + \ldots + i_n$ for μ of the form (21). We shall use some special multiindices, setting

$$\delta_j = (\delta_j^1, \ldots, \delta_j^n), \tag{22}$$

where δ_j^i is the Kronecker symbol. So δ_j has the jth coordinate equal to 1 and other coordinates equal to zero. A multisequence with values in a set S is any function from N^n to S. As S we shall take a locally convex algebra. A multisequence will be denoted by (x_μ); sometimes we shall treat x_μ as a function defined on Z^n and supported by a subset in N^n, i.e., $x_\mu = 0$ for $\mu \in Z^n \setminus N^n$.

Let $t = (t_1, \ldots, t_n)$ be an n-tuple of variables. For a multiindex μ of the form (21) we write

$$t^\mu = t_1^{i_1}, \ldots, t_n^{i_n}.$$

Similar notation will be used for an n-tuple $b = (b_1, \ldots, b_n)$ in an algebra B.

For an algebra A denote by $A(t)$ the algebra of all polynomials in variables $t = (t_1, \ldots, t_n)$ with coefficients in A. Any polynomial in $A(t)$ can be written as

$$p = p(t) = \sum_\mu a_\mu t^\mu,$$

where a_μ is a multisequence supported by a finite subset in N^n, i.e., $a_\mu = 0$ for $|\mu|$ sufficiently large. We shall write also

$$p = p(t) = \sum_{j\geqslant 0} p_j(t), \tag{23}$$

where $p_j(t)$ is a homogeneous polynomial of degree j, i.e.,

$$p_j(t) = \sum_{|\mu|=j} a_\mu t^\mu.$$

The product of two polynomials $p = \sum a_\mu t^\mu$, $q = \sum b_\mu t^\mu$ is given by

$$pq = \sum c_\mu t^\mu,$$

where

$$c_\mu = \sum_\nu a_{\mu-\nu} b_\nu,$$

i.e., the multisequence (c_μ) is a convolution of multisequences (a_μ) and (b_μ).

A part of our construction is contained in the following lemma.

3.1. Lemma. *Let A be a B_0-algebra with topology given by means of a sequence $(|x|_i)$ of seminorms satisfying conditions* (3) *and* (4). *Let $(a_j^{(i)})$, $i = 1, 2, \ldots$, $j = 0, 1, 2, \ldots$ be a matrix of positive real numbers satisfying the following conditions:*

$$a_0^{(i)} = 1 \tag{24}$$

for $i = 1, 2, \ldots$,

$$a_j^{(i)} \leqslant a_j^{(i+1)} \tag{25}$$

for $j = 0, 1, \ldots, i = 1, 2, \ldots$, and

$$a_{k+l}^{(i)} \leqslant a_k^{(i+1)} a_l^{(i+1)} \tag{26}$$

for $k, l \geqslant 0$, $i = 1, 2, \ldots$

Introduce a sequence of seminorms on $A(t)$, given by

$$\|p\|_i = \sum_{j\geqslant 0} a_j^{(i)} |p_j|_i, \tag{27}$$

where $p, p_j \in A(t)$, p_j are related to p by formula (23), *and*

$$|p_j|_i = \sum_{|\mu|=j} |x_\mu|_i, \tag{28}$$

where x_μ are the coefficients in p.

The algebra $A(t)$ is then an (incomplete) algebra of type B_0 with the seminorms (27). *Moreover, these seminorms satisfy conditions* (3) *and* (4), *and $A(t)$ is a B_0-extension of A.*

Proof. Relation (4) for seminorms (27) follows immediately from the same relation for the seminorms $(|x|_i)$ and from inequalities (25). To prove relation (3) take $p = \sum x_\mu t^\mu$, $q = \sum y_\mu t^\mu$. By (26) and (3) for the seminorms $(|x|_i)$ we have

$$\begin{aligned}\|pq\|_i &= \sum_j a_j^{(i)} \sum_{|\mu|=j} \Big|\sum_\nu x_{\mu-\nu} y_\nu\Big|_i \\ &\leqslant \sum_j \sum_{|\mu|=j} \sum_\nu a_{|\mu-\nu|}^{(i+1)} a_{|\nu|}^{(i+1)} |x_{\mu-\nu}|_{i+1} |y_\nu|_{i+1} \\ &= \sum_{\mu,\nu} a_{|\mu-\nu|}^{(i+1)} |x_{\mu-\nu}|_{i+1} a_{|\nu|}^{(i+1)} |y_\nu|_{i+1} = \|p\|_{i+1} \|q\|_{i+1}.\end{aligned}$$

By (24) we see that each seminorm $\|p\|_i$ restricted to the algebra A treated as the subalgebra of constant polynomials in $A(t)$ coincides with $|x|_i$ for $p = x \in A$. The conclusion follows.

For the sake of completeness we reproduce the proof of the following lemma (cf [13], Lemma 3).

3.2. Lemma. *Let $(a_n)_0^\infty$ be a sequence of positive real numbers with $a_0 = 1$. There exists a sequence $(b_i)_0^\infty$ of positive real numbers such that $b_0 = 1$ and*

$$a_{i+j} \leqslant b_i b_j \tag{29}$$

for all non-negative integers i and j. Assuming, in particular, $j = 0$ we have

$$a_i \leqslant b_i \tag{30}$$

for $i = 0, 1, 2, \ldots$

Proof. Put $b_0 = 1$ and proceed by induction. If we already have $b_0, b_1, \ldots, b_{n-1}$, we put

$$b_n = \max\{a_n, a_{n+1}/b_1, a_{n+2}/b_2, \ldots, a_{2n-1}/b_{n-1}, a_{2n}^{1/2}\}.$$

We can now formulate our main result. According to Proposition 1.3 we can formulate it in terms of LC-regular tuples.

3.3. THEOREM. *Let A be a commutative locally convex algebra (resp. an algebra of type B_0) with unit element e. Let $(|x|_a)$, $a \in \mathfrak{A}$ (resp. $(|x|_i)$, $i = 1, 2, \ldots$) be a system of seminorms giving the topology of A and satisfying conditions* (1) *and* (2) *(resp.* (3) *and* (4)*).*

An n-tuple of elements of A, $(x_1, \ldots, x_n)$, is LC*-regular (resp. B_0-regular) if and only if for each index $a \in \mathfrak{A}$ there is an index $\beta \in \mathfrak{A}$ and a sequence (C_i) of positive numbers such that for each finite multisequence (a_μ), $\mu \in N^n$, of elements of A (i.e., $a_\mu = 0$ for $|\mu|$ sufficiently large) the following inequality holds true:*

$$|a_{(0,\ldots,0)}|_a \leqslant \sum_{j=1}^{\infty} C_j \sum_{|\mu|=j} \Big| \sum_{s=1}^{n} a_{\mu-\delta_s} x - a_\mu \Big|_\beta, \tag{31}$$

where δ_s is given by formula (22).

Proof. Suppose first that $(x_1, \ldots, x_n)$ is an LC-regular (resp. B_0-regular) n-tuple in A. Thus there is an extension B of the algebra A, $B \in \mathrm{LC}$ (resp. $B \in B_0$) such that

$$\sum_{i=1}^{n} x_i b_i = e$$

for a certain n-tuple $(b_1, \ldots, b_n) \subset B$. Let $(|||x|||_\beta)$, $\beta \in \mathfrak{B}$, be a family of seminorms on B defining its topology and satisfying relations (1) and (2) (resp. (3) and (4)). For an a in $\mathfrak{A}$ there is a $\gamma \in \mathfrak{B}$ and a positive constant C such that

$$|x|_a \leqslant C\,|||x|||_\gamma \tag{32}$$

for all x in A. Such a γ exists, because the system $(|||x|||_\beta)$, restricted to A, gives its topology. By (1) (resp. (3)) we can find a $\delta \in \mathfrak{B}$ with

$$|||xy|||_\gamma \leqslant |||x|||_\delta |||y|||_\delta \tag{33}$$

for all $x, y \in A$. Let (a_μ), $\mu \in N^n$, be an arbitrary finite multisequence of elements of A. We have

$$\begin{aligned} 0 &= \Big|\Big|\Big| \Big(e - \sum_{1}^{n} x_i b_i\Big) \sum_{|\mu|\geqslant 0} a_\mu b^\mu \Big|\Big|\Big|_\gamma \\ &= \Big|\Big|\Big| a_{(0,\ldots,0)} - \sum_{j\geqslant 1} \sum_{|\mu|=j} \Big(\sum_{s=1}^{n} (a_{\mu-\delta_s} x_s - a_\mu)\Big) b^\mu \Big|\Big|\Big|_\gamma \\ &\geqslant |||a_{(0,\ldots,0)}|||_\gamma - \Big|\Big|\Big| \sum_{j\geqslant 1} \sum_{|\mu|=j} \Big(\sum_{s=1}^{n} a_{\mu-\delta_s} x_s - a_\mu\Big) b^\mu \Big|\Big|\Big|_\gamma, \end{aligned} \tag{34}$$

where $b = (b_1, \ldots, b_n)$. In the same way as we obtained (32) we find an index $\beta \in \mathfrak{A}$ and a positive constant D so that

$$|||x|||_\delta \leqslant D\,|x|_\beta \tag{35}$$

for all x in A. Formulas (1) (resp (3)), (32), (33), (34) and (35) imply now

$$\begin{aligned} |a_{(0,\ldots,0)}|_a &\leqslant C\,|||x_{(0,\ldots,0)}|||_\gamma \leqslant C \Big|\Big|\Big| \sum_{|\mu|\geqslant 1} \Big(\sum_{s=1}^{n} a_{\mu-\delta_s} x_s - a_\mu\Big) b^\mu \Big|\Big|\Big|_\gamma \\ &\leqslant C \sum_{|\mu|\geqslant 1} \Big|\Big|\Big| \Big(\sum_{s=1}^{n} a_{\mu-\delta_s} x_s - a_\mu\Big) b^\mu \Big|\Big|\Big|_\gamma \leqslant C \sum_{|\mu|\geqslant 1} |||b^\mu|||_\delta \Big|\Big|\Big| \sum_{s=1}^{n} a_{\mu-\delta_s} x_s - a_\mu \Big|\Big|\Big|_\delta \\ &\leqslant CD \sum_{|\mu|\geqslant 1} |||b^\mu|||_\delta \Big| \sum_{s=1}^{n} a_{\mu-\delta_s} x_s - a_\mu \Big|_\beta \leqslant \sum_{j\geqslant 1} C_j \sum_{|\mu|=j} \Big| \sum_{s=1}^{n} a_{\mu-\delta_s} x_s - a_\mu \Big|_\beta, \end{aligned}$$

where $C_j = CD \max\limits_{|\mu|=j} |||b^\mu|||_\delta$. Thus relation (31) holds true.

Conversely, suppose that the condition given by formula (31) is satisfied. We have to construct a locally convex (resp. of type B_0) extension B of A such that

$$\sum x_i b_i = e$$

for a certain n-tuple $(b_1, \ldots, b_n) \subset B$. We shall construct the algebra B in the form of the quotient algebra of the algebra $A(t)$ of polynomials, topologized similarly as in Lemma 3.1. Consider first the case where A is a B_0-algebra. Without loss of generality we can assume that the sequence of seminorms $(|x|_i)$, giving the topology of A and satisfying conditions (3) and (4) satisfies moreover the following condition: For $a = i$, the suitable β satisfying (31) equals $i+1$ (otherwise we could pass to a subsequence of the sequence $(|x|_i)$). Relation (31) implies now that there exists a matrix $(C_j^{(i)})$, $i, j \geqslant 1$, of positive entries such that

$$|a_{(0,\ldots,0)}|_i \leqslant \sum_{j=1}^{\infty} C_j^{(i)} \sum_{|\mu|=1} \Big| \sum_{s=1}^{n} a_{\mu-\delta_s} x_s - a_\mu \Big|_{i+1}, \tag{36}$$

$i = 1, 2, \ldots$, for each finite multisequence (a_μ), $\mu \in N^n$, of elements of A. By means of the matrix $C_j^{(i)}$ we shall construct another matrix $(a_j^{(i)})$, $i = 1, 2, \ldots$, $j = 0, 1, \ldots$, satisfying conditions (24), (25) and (26). This is done in the following way:

Assume first

$$a_0^{(1)} = 1, \quad a_j^{(1)} = C_j^{(1)}, \quad j = 1, 2, \ldots$$

Setting $a_j = a_j^{(1)}$ in Lemma 3.2, we obtain a certain sequence b_j, and we put

$$a_0^{(2)} = 1, \quad a_j^{(2)} = \max(b_j, C_j^{(2)}), \quad j = 1, 2, \ldots$$

We proceed by an induction. If we have already defined the numbers $a_j^{(i)}$, $i < n$, $j = 0, 1, 2, \ldots$, we put in Lemma 3.2 $a_j = a_j^{(n-1)}$, $j = 0, 1, \ldots$; obtain a suitable b_j, $j = 0, 1, \ldots$, and put

$$a_0^{(n)} = 1 \quad \text{and} \quad a_j^{(n)} = \max(b_j, C_j^{(n)}), \quad j = 1, 2, \ldots$$

The matrix $(a_j^{(i)})$ obtained in this way evidently safisfies condition (24), while conditions (25) and (26) follow from conditions (29) and (30), respectively. The construction gives also the following relation:

$$a_j^{(i)} \geqslant C_j^{(i)} \tag{37}$$

for $i, j = 1, 2, 3, \ldots$ Using the matrix $(a_j^{(i)})$ we topologize the algebra $A(t)$ by means of the seminorms given by formula (27). We denote these seminorms by $(\|p\|_i^{(1)})$. We have $\|x\|_i^{(1)} = |x|_i$ for all i and all x in A. We now put

$$\|p\|_i^{(2)} = \sum_{j=0}^{\infty} a_j^{(i)} |p_j|_{i+1}, \quad i = 1, 2, \ldots$$

for any $p \in A(t)$ of form (23). Relations (4) and (25) imply

$$\|p\|_1^{(1)} \leqslant \|p\|_1^{(2)} \leqslant \|p\|_2^{(1)} \leqslant \|p\|_2^{(2)} \leqslant \ldots,$$

and so the new sequence $(\|p\|_i^{(2)})$ is equivalent to the sequence $(\|p\|_i^{(1)})$. Finally we define on $A(t)$ yet another sequence of seminorms, equivalent to the previous ones. Namely, for any polynomial $p \in A(t)$ written in form (23) we put

$$\|p\|_i = |p_0|_i + \sum_{j \geqslant 1} a_j^{(i)} |p_j|_{i+1}, \tag{38}$$

$i = 1, 2, \ldots$, where $|p_j|_i$ is given by formula (28). We have

$$\|p\|_i^{(1)} \leqslant \|p\|_i \leqslant \|p\|_i^{(2)}, \quad i = 1, 2, \ldots,$$

and so the seminorms (38) give the same topology as the seminorms (27).

Consider now the case of a non-metrizable locally convex algebra A. For a given $\alpha \in \mathfrak{A}$, where $\mathfrak{A}$ is the index set for a system of seminorms giving the topology of A, we form a sequence $(|x|_i^{(\alpha)})$ of seminorms of A in the following way. We put $|x|_1^{(\alpha)} = |x|_\alpha$ and find indexes $\beta_1, \beta_2 \in \mathfrak{A}$ so that α together with β_1 satisfies relation (1) and α together with β_2 satisfies relation (31). Relation (2) now implies that there is an index $\beta \in \mathfrak{A}$ such that together with α relations (1) and (31) are both satisfied and, moreover, $|x|_\alpha \leqslant |x|_\beta$ for all x in A. We then put $|x|_2^{(\alpha)} = |x|_\beta$ and proceed by induction. If we have already chosen the seminorms $|x|_i^{(\alpha)}$, $1 \leqslant i \leqslant n-1$, satisfying

$$|x|_1^{(\alpha)} \leqslant |x|_2^{(\alpha)} \leqslant \ldots \leqslant |x|_{n-1}^{(\alpha)}$$

for all $x \in A$, we construct $|x|_n^{(\alpha)}$ by means of $|x|_{n-1}^{(\alpha)}$ exactly in the same way as we constructed $|x|_2^{(\alpha)}$ by means of $|x|_1^{(\alpha)}$. The sequence $(|x|_i^{(\alpha)})$ is contained in the family $(|x|_\alpha)$, $\alpha \in \mathfrak{A}$, and satisfies the following relations:

$$|x|_1^{(\alpha)} = |x|_\alpha, \tag{39}$$

for all $x \in A$,

$$|xy|_i^{(\alpha)} \leqslant |x|_{i+1}^{(\alpha)} |y|_{i+1}^{(\alpha)}, \quad i = 1, 2, \ldots, \tag{40}$$

for all $x, y \in A$, and

$$|a_{(0,\ldots,0)}|_i^{(\alpha)} \leqslant \sum_{j \geqslant 1} C_j^{(i)}(\alpha) \sum_{|\mu|=1} \Big| \sum_{s=1}^{n} a_{\mu-\delta_s} x_s - a_\mu \Big|_{i+1}^{\alpha} \tag{41}$$

for $i = 1, 2, \ldots$ and all finite multisequences $(a_\mu) \subset A$. Here $C_j^{(i)}(\alpha)$ is the matrix obtained from condition (31) exactly in the same way as in proving relation (36). By means of the sequence $(|x|_i^{(\alpha)})$ we now form a sequence $(\|p\|_i^{(\alpha)})$ of seminorms on $A(t)$ given by formula (38). The system of seminorms $(\|p\|_i^{(\alpha)})$, $i = 1, 2, \ldots$, $\alpha \in \mathfrak{A}$, gives the desired topology on $A(t)$. Each seminorm in this family is in fact of the form

$$\|p\|_{(\alpha,\beta)} = |p_0|_\alpha + \sum_{j \geqslant 1} C_j(\alpha, \beta) |p_j|_\beta, \tag{42}$$

where $p = \sum p_j$ as in formula (23) and the indexes α, β together with the sequence $\big(C_j(\alpha, \beta)\big)$ satisfy relation (31). The same holds true if A is a B_0-algebra.

We now construct the desired extension B of A in the form of a quotient of the algebra $A(t)$. We put

$$I = \overline{\Big(e - \sum_{i=1}^{n} x_i t_i\Big) A(t)},$$

which is a closed ideal in A, and define

$$B = A(t)/I.$$

There is a natural imbedding of A into B given by $x \to [x]$, where $[x]$ is the coset of the constant polynomial x. This map is clearly an algebra homomorphism, and we shall be done if we show that it is also a homeomorphism, since we have in B

$$e_B = \sum_{i=1}^{n} [x_i][t_i].$$

The topology in B is given by the quotient seminorms obtained from the seminorms (42). They are given by

$$\|[p]\|_{(\alpha,\beta)} = \inf\Big\{\Big|\Big|p+q\Big(e-\sum_{1}^{n} x_i t_i\Big)\Big|\Big|_{(\alpha,\beta)}\colon\ q\in A(t)\Big\}. \tag{43}$$

We obviously have

$$\|[p]\|_{(\alpha,\beta)} \leqslant \|p\|_{(\alpha,\beta)},$$

and, in particular, for each x in A we have

$$\|[x]\|_{(\alpha,\beta)} \leqslant \|x\|_{(\alpha,\beta)} = |x|_\alpha. \tag{44}$$

On the other hand, for any x in A and an arbitrary q in $A(t)$, $q = \sum_\mu y_\mu t^\mu$, relations (31) and (42) imply

$$\Big|\Big|x+\Big(e-\sum_{1}^{n} x_i t_i\Big)\sum_\mu y_\mu t^\mu\Big|\Big|_{(\alpha,\beta)} = |x-y_{(0,\ldots,0)}|_\alpha +$$

$$+\sum_{j\geqslant 1} C_j(\alpha,\beta)\sum_{|\mu|=1}\Big|\sum_{s=1}^{n} y_{\mu-\delta_s}x_s - y_\mu\Big|_\beta$$

$$\geqslant |x-y_{(0,\ldots,0)}|_\alpha + |y_{(0,\ldots,0)}|_\alpha \geqslant |x|_\alpha.$$

Thus

$$\|[x]\|_{(\alpha,\beta)} \geqslant |x|_\alpha,$$

which, together with (44), gives

$$\|[x]\|_{(\alpha,\beta)} = |x|_\alpha$$

for all x in A. Since α is an arbitary index in $\mathfrak{A}$, the map $x\to[x]$ of A into B is a homeomorphic imbedding. The conclusion follows.

§ 4. Final remarks and open problems. The characterization of LC-non-removable ideals given in the previous section is similar to the characterization of B-non-removable ideals given in [2]. It seems to be rather formal and not easy for applications, but it is the only characterization we have in hand. On the other hand, we have clearer sufficient conditions formulated in Section 3. We now formulate the remark closing Section 2 in the form of the following problem, in which and in the sequel by a non-removable ideal we mean a T-non-removable ideal.

PROBLEM 1. Let A be a topological algebra and let I be a non-removable ideal in A. Is it true that I is contained in an ideal of the form (20)?

It will be useful in the sequel to have the following concept.

4.1. DEFINITION. Let P be any property of an ideal of a topological algebra A. We say that an ideal $I\subset A$ is *maximal with respect to the property* P, or is a *maximal P-ideal*, if for any ideal $J\subset A$ with $I\subset J$ either we have $I=J$, or the property P fails for the ideal J.

The next two propositions show that there exist maximal ideals containing a given ideal having the property P in the case where P means "K-non-removable", or "consists locally of topological divisors of zero".

4.2. PROPOSITION. *Let K be any class of topological algebras. Let $A\in K$ and let I be a K-non-removable ideal in A. There exists a maximal K-non-removable ideal J in A with $I\subset J$.*

Proof. Let F be a family of ideals in A such that each ideal in F contains I and is a K-non-removable ideal. The family F is a partially ordered set with inclusion as the order relation. Let (I_α) be a chain of elements in F and put $J = \bigcup_\alpha I_\alpha$. We claim that J is a K-non-removable ideal in A. It is clearly an ideal, and if some K-extension B removes J, then

$$\sum_{i}^{n} x_i b_i = e$$

for some n-tuple $(x_1,\ldots,x_n)\subset J$ and $(b_1,\ldots,b_n)\subset B$. But since (I_α) is a chain, there is an index α_0 with $(x_1,\ldots,x_n)\subset I_{\alpha_0}$. This is a contradiction since I_{α_0} is K-non-removable. Thus any chain in F has an upper bound there and the conclusion follows from the Kuratowski–Zorn lemma.

In a similar way one can prove the following result, whose details are left to the reader.

4.3. PROPOSITION. *Let A be a topological algebra, and let I be an ideal in A consisting locally of joint topological divisors of zero* (cf. Definition 2.10). *Then I is contained in an ideal J maximal with respect to the property that it consists locally of topological divisors of zero.*

We can now rewrite Problem 1 in the following way:

PROBLEM 1a. Let A be a topological algebra and let I be a maximal non-removable ideal in A. Is it true that

$$I = I_l + I_s(A), \tag{45}$$

where I_l is maximal among ideals consisting locally of joint topological divisors of zero and $I_s(A)$ is the ideal consisting of all elements in A having small powers?

PROBLEM 2. Is it true that every LC-non-removable ideal of a locally convex algebra A contained in an ideal is of the form (45)?

A positive answer to Problem 2 would give a positive answer to the following one.

PROBLEM 3. Is it true that the concept of LC-removability has an absolute character? (cf. Definition 1.4).

The next problems are connected with the question of removability of families of ideals. The definition below and some subsequent problems are analogous to the ones given in [1].

4.4. DEFINITION. Let K be a class of topological algebras and let $A \in K$. Let (I_a) be a family of K-removable ideals in A. The family (I_a) is said to be *K-removable* if there is a K-extension B of A which removes all ideals in this family.

It is known ([5]) that any finite number of removable ideals of a Banach algebra A is a B-removable family; the same holds true for countable families of ideals (V. Müller—oral communication).*

The problem of removability of finite families of removable ideals can be formulated in several ways, as can be seen from the following result proved in [11] in a more general, purely algebraic situation.

4.5. PROPOSITION. *Let K be any class of topological algebras and let $A \in K$. The following conditions are equivalent:*

(i) *Every finite family of K-removable ideals in A is K-removable.*

(ii) *Every family consisting of two K-removable ideals in A is K-removable.*

(iii) *Every maximal K-non-removable ideal in A is a prime ideal.*

By [5] and [6] the above conditions are satisfied for $K = B$.

PROBLEM 4. Are conditions (i)–(iii) satisfied in the classess M_0, M, B_0, LC, T?

The positive answer to the above question would follow from the positive answer to the following

PROBLEM 5. Let K be one of the classess M_0, M, B_0, LC, T, and let $A \in K$. Is it true that every maximal K-non-removable ideal in A is a maximal ideal?

Unfortunately, the answer to the above question is in negative even in the case of the smallest class M_0. Indeed, from Example 2.14 it follows that the only non-removable ideal in the algebra E of all entire functions is the zero ideal.

* (*Added in proof*). This result will be published in [16].

PROBLEM 6. Let K be as in Problem 5 and let $A \in K$. Let $(I_n)_1^\infty$ be a sequence of K-removable ideals in A. Is (I_n) a K-removable family?

To formulate yet another version of Problem 4 we need the following

4.6. DEFINITION. Let A be an algebra. The *product* of an n-tuple $(x_1, \ldots, x_n) \subset A$ and an k-tuple $(y_1, \ldots, y_k) \subset A$ we define as the nk-tuple $(x_1y_1, x_1y_2, \ldots, x_1y_k, x_2y_1, \ldots, x_ny_k)$.

The positive answer to Problem 4 would follow from the positive answer to the following

PROBLEM 7. Let K be as in Problem 5 and let $A \in K$. Is it true that the product of two K-regular tuples is K-regular?

As remarked in [10], the Bollobás construction in [3] shows that there is a Banach algebra A such that the family of all removable ideals in A is not a B-removable family. It easily follows that the same family is not M-removable either, but it is still unclear whether the job can be done by an extension of a larger class. So we pose the following

PROBLEM 8. Let $K = \text{LC}$ or T, and let $A \in K$. Is the family of all K-removable ideals in A a K-removable family?

The answer is not known even if A is a Banach algebra.

4.7. Remark. The construction in Bollobás [3] is rather complicated The fact that the answer to Problem 8 is negative for $K = M$ or $K = M^0$ follows immediately from Example 2.14. For, if some m-convex extension removed the family of all ideals in the algebra E of entire functions, then, in particular, all non-zero elements in E would be invertible in B. Thus B would contain the field of all meromorphic functions. But, as follows from the Gelfand–Mazur type theorem for m-convex algebras (cf. e.g. [7]), this is impossible.

The following problem is open even in the class of Banach algebras (I learned it from V. Müller—oral communication).

PROBLEM 9. Let K be one of the classess B, M_0, M, B_0, LC, T, and let $A \in K$. Is the union of two K-removable families of ideals a K-removable family?

References

[1] R. Arens, *Extensions of Banach algebras*, Pacific J. Math. 10 (1960), 1–16.
[2] — *Ideals in Banach algebra extensions*, Studia Math. 31 (1968), 29–34.
[3] B. Bollobás, *Adjoining inverses to commutative Banach algebras*, Trans. Amer. Math. Soc. 181 (1973), 165–174.
[4] M. E. Kuczma, *On the problem of E. Michael concerning topological divisors of zero*, Colloq. Math. 19 (1968), 295–299.
[5] V. Müller, *Non-removable ideals in commutative Banach algebras*, Studia Math. 74 (1982), 97–104.
[6] Z. Słodkowski, *On ideals consisting of joint topological divisors of zero*, ibid. 48 (1973), 83–88.

[7] W. Żelazko, *Selected topics in topological algebras*, Aarhus University Lecture Notes N° 31 (1971).
[8] — *Banach algebras*, Warszawa 1973.
[9] — *On permanently singular elements in commutative m-convex algebras*, Studia Math. 37 (1971), 177–186.
[10] — *On a certain class of non-removable ideals in Banach algebras*, ibid. 44 (1972), 87–92.
[11] — *Concerning a problem of Arens on removable ideals in Banach algebras*, Colloq. Math. 30 (1974), 127–131.
[12] — *Concerning non-removable ideals in commutative m-convex algebras*, Demonstratio Math. 11 (1978), 239–245.
[13] — *A characterization of* LC-*non-removable ideals in commutative Banach algebras*, Pacific J. Math. 87 (1980), 241–248.
[14] — *On permanent radicals in commutative locally convex algebras*, Studia Math. 75 (1983), 265–272.
[15] — *Concerning a characterization of permanently singular elements in commutative locally convex algebras*, in: Bulgarian Academy of Sciences, Papers dedicated to Professor L. Iliev's 70th Anniversary (in print).
[16] V. Müller, *Removability of ideals in commutative Banach algebras*, Studia Math. 79 (in print).

INSTYTUT MATEMATYCZNY PAN
MATHEMATICAL INSTITUTE, POLISH ACADEMY OF SCIENCES

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Entropy numbers of r-nuclear operators between L_p spaces

by

BERND CARL (Jena)

Abstract. We show that the sequence of entropy numbers of r-nuclear operators acting from L_p into L_q, $0 < r < 1$, $1 < p, q < \infty$, belongs to the Lorentz sequence space $l_{s,r}$, where

$$1/s = 1/r + \min(1/2; 1/p) - \max(1/2; 1/q).$$

Introduction. Since the fundamental work of Grothendieck the r-nuclear operators ("operateurs á puissance r-ième" [6]) were intensively investigated. A representation of the theory of these operators can be found in the book *Operator ideals* of Pietsch [14]. A remarkable fact about the distribution of eigenvalues of r-nuclear operators was proved by H. König [8].

The aim of this paper is to determine the "degree of compactness" of r-nuclear operators in terms of entropy numbers. As an application we also get once more König's result about the behaviour of eigenvalues of r-nuclear operators acting in L_p spaces.

Let $0 < r < 1$. An operator $S \in \mathscr{L}(E, F)$ from a Banach space E into a Banach space F is called *r-nuclear* if it admits a representation

$$S = \sum_{n=1}^{\infty} a_n \otimes y_n, \qquad a_n \in E', \; y_n \in F$$

with $\sum_{n=1}^{\infty} \|a_n\|^r \|y_n\|^r < \infty$. Let

$$N_r(S) := \inf \Big(\sum_n \|a_n\|^r \|y_n\|^r \Big)^{1/r},$$

where the infimum is taken over all possible representations of S. The class of these nuclear operators is denoted by $\mathfrak{N}_r(E, F)$. $[\mathfrak{N}_r, N_r]$ forms an r-