ANNALES
POLONICI MATHEMATICI
XLIII (1983)

Holomorphic extension from the sphere to the ball in Hilbert space*

by J. A. Cima and J. A. Pfaltzgraff (Chapel Hill, North Carolina)

Abstract. Agranovskii and Valskii [1] and Nagel and Rudin [3] have shown that if f is a complex valued continuous function on the boundary of B_n, the unit ball in C^n (with Euclidean structure), and if f has the one-dimensional extension property, then f extends to a holomorphic function in B_n. In this note we generalize their extension theorem to Banach space-valued continuous functions with the one-dimensional extension property defined on the boundary of the unit ball in a complex Hilbert space. It is interesting to note that our proof uses only the two-dimensional case of the result of [1], [3], and hence even the n-dimensional result of [1], [3] for $n > 2$ follows from our work and the two-dimensional result.

We let $\mathscr{H}$ denote a complex Hilbert space with inner product $\langle\ ,\ \rangle$ and norm $\|\ \|$, and

$$B = \{x \in \mathscr{H}\colon \|x\| < 1\}, \qquad S = \{x \in \mathscr{H}\colon \|x\| = 1\}.$$

If Y is a complex Banach space, then $C(S) = C(S, Y)$ denotes the set of continuous functions from S into Y. A complex line L in a complex linear space is a translate of a one dimensional subspace, i.e. $L = L(x, y) = \{zx + +y\colon z \in C\}$ for fixed x, y in the linear space. We say that $f \in C(S, Y)$ has the *one dimensional extension property* if for every complex line L in $\mathscr{H}$ such that $L \cap B \neq \emptyset$ the restriction of f to the circle $L \cap S$ has a (Y-valued) holomorphic extension to the open disk $L \cap B$ that is continuous on its closure. The function $f\colon B \to Y$ is *holomorphic in B* means it has a Fréchet derivative at every point in B. Finally we let Y^* denote the dual space of continuous linear functionals on Y.

---

* AMS (MOS) subject classification (1970) 58C10, 32A10.

Key words and phrases: Holomorphic, F-differentiable, G-differentiable, Hilbert space, Banach space, continuous linear functional.

The research of the second author was supported in part by the National Science Foundation under grant MCS76-06989 A01.

THEOREM. *If $f \in C(S, Y)$ has the one-dimensional extension property, then f has a holomorphic extension to B.*

Proof. We prove this theorem by establishing the following three assertions:

(i) the various one dimensional extensions to a point $x \in B$ define a unique value $\hat{f}(x)$;

(ii) The extended function $\hat{f}(x)$ is locally bounded at some point in B and hence holomorphic in B [4];

(iii) The extended function is continuous on the closed ball, $\bar{B}$.

In the finite dimensional case ([1], [3]) (iii) is an automatic consequence of the holomorphic extension, but an additional argument is needed in the infinite dimensional case since $\bar{B}$ is not compact.

To prove (i) we fix $x \in B$, $x \neq 0$ and let $\hat{f}_1$ denote the one dimensional extension of f to $L_1 \cap B$, where $L_1 = \{\lambda x: \lambda \in C\}$. Suppose x is also on the complex line L_2 determined by two points $u_1, u_2 \in S$ ($x = \beta u_1 + (1-\beta) u_2$ for some $\beta \in C$) and let $\hat{f}_2$ denote the extension of f to $L_2 \cap B$. Note that it suffices to assume only one of the lines, L_2, is in general position missing the origin since we will show $\hat{f}_2(x) = \hat{f}_1(x)$ and hence that all possible extension values at x are uniquely defined by $\hat{f}_1(x)$. Since $L_1 \neq L_2$, u_1 and u_2 are linearly independent and $[u_1, u_2]$, the complex linear subspace spanned by u_1, u_2, is two dimensional. Clearly v_1, v_2 defined by

$$v_1 = u_1, \quad v_2 = (u_2 - \alpha u_1)/\sqrt{1-|\alpha|^2}, \quad \alpha = \langle u_2, u_1 \rangle,$$

are orthogonal unit vectors that enable us to define explicitly a linear isometric isomorphism T: $C^2 \to [u_1, u_2]$. If F denotes the restriction of f to $S \cap [u_1, u_2]$ and $\psi \in Y^*$, then $\psi \circ F \circ T$ is a complex valued continuous function on the unit sphere in C^2 which has the one dimensional extension property since T is a linear isometry. There is a unique point p in the unit ball of C^2 such that $x = Tp$ and there are lines Λ_1, Λ_2 in C^2 that correspond to L_1, L_2 via T. The one dimensional extensions $\hat{f}_1, \hat{f}_2$ induce corresponding one dimensional extensions of $\psi \circ F \circ T$ to the lines Λ_1, Λ_2 in C^2. By the two-dimensional case of the result in [1], [3], all such extensions yield the same value at p, and therefore $\psi(\hat{f}_1(x)) = \psi(\hat{f}_2(x))$. This is true for arbitrary $\psi \in Y^*$ hence $\hat{f}_1(x) = \hat{f}_2(x)$. We can take care of the value at $x = 0$ by an obvious continuity argument.

Having established our claim (i), we now have an extended function $\hat{f}$ defined throughout the closed ball $\bar{B}$ such that $\hat{f} = f$ on S and $\hat{f}$ is Gâteaux differentiable at each point in B since it is holomorphic on lines. By a theorem of Zorn [4], p. 582, $\hat{f}$ will be holomorphic in B if we can produce at least one point in B with a neighbourhood in which $\hat{f}$ is bounded, i.e. (ii) holds. It is worth mentioning that the most commonly used form of Zorn's theorem [2] (Theorem 3.17.1) states that G-differentiability plus local boundedness at every point in a domain implies F-differentiability, but we are using the stronger result which requires local boundedness only at a single point.

To prove (ii) we fix $p \in S$ and let $P = \{\lambda p: |\lambda| = 1, \lambda \in C\}$. Then P is compact and $\max_P \|f(\lambda p)\| = M < \infty$. To each $\lambda p \in P$ there corresponds $\delta = \delta(\lambda) > 0$ such that $\|f(x)\| \leqslant 2M$ if $x \in S$ and $\|x - \lambda p\| < \delta(\lambda)$ since f is continuous on S at the points λp. The sets

$$D_\lambda = \{x \in S: \|x - \lambda p\| < \delta(\lambda)\} \quad \text{for } |\lambda| = 1, \ \lambda \in C,$$

form an open cover of the compact set P. Let D_j, $j = 1, 2, \ldots, m$, be a finite subcover of P. Then there exists $\varepsilon > 0$ such that

$$A = \{x \in S: \operatorname{dist}(x, P) < \varepsilon\} \subset \bigcup_{j=1}^{m} D_j$$

and consequently $\|f(x)\| \leqslant 2M$ for all $x \in A$.

We can complete the proof of (ii) by showing that $\|\hat{f}(x)\| \leqslant 2M$ for all $x \in \Omega = \{x \in B: \|x - p\| < \varepsilon/2\}$ since this then shows any point $x \in \Omega$ has a neighbourhood contained in B in which $\hat{f}$ is bounded. Let $x \in \Omega$ and write $\hat{x} = x/\|x\|$. Then $1 - \|x\| = \|p\| - \|x\| \leqslant \|x - p\| < \varepsilon/2$ and

$$\|\hat{x} - p\| \leqslant \|\hat{x} - x\| + \|x - p\| = 1 - \|x\| + \|x - p\| < \varepsilon.$$

Clearly $\{\lambda\hat{x}: \lambda \in C, |\lambda| = 1\}$ is a subset of A since

$$\operatorname{dist}(\lambda\hat{x}, P) \leqslant \|\lambda\hat{x} - \lambda p\| = |\lambda| \, \|\hat{x} - p\| < \varepsilon \quad (|\lambda| = 1).$$

The function $g(\lambda) = \hat{f}(\lambda\hat{x})$ is Y-valued and holomorphic for complex λ in the open unit disk and by the maximum principle

$$\max_{|\lambda| < 1} \|\hat{f}(\lambda\hat{x})\| \leqslant \max_{|\lambda| = 1} \|g(\lambda)\| = \max_{|\lambda| = 1} \|f(\lambda\hat{x})\| \leqslant 2M.$$

Hence $\|\hat{f}(x)\| = \big\|\hat{f}(\|x\| \, \hat{x})\big\| \leqslant 2M$ for all $x \in \Omega$.

Finally we shall show that the extended function $\hat{f}$ is continuous on $\bar{B}$. Fix $x \in S$ and let $\{x_n\} \subset B - \{0\}$ with $x_n \to x$. Writing $x_n^* = \|x_n\| \, x$ and $\hat{x}_n = x_n/\|x_n\|$ $(n = 1, 2, \ldots)$ we have

$$\|\hat{f}(x_n) - \hat{f}(x)\| \leqslant \|\hat{f}(x_n) - \hat{f}(x_n^*)\| + \|\hat{f}(x_n^*) - \hat{f}(x)\|.$$

The second term tends to zero as $n \to \infty$ because all the x_n^* lie on the same complex line through x. For the first term we have

$$\|\hat{f}(x_n) - \hat{f}(x_n^*)\| = \big\|\hat{f}(\|x_n\| \, \hat{x}_n) - \hat{f}(\|x_n\| \, x)\big\| \leqslant \max_{|\zeta| = 1} \|\hat{f}(\zeta\hat{x}_n) - \hat{f}(\zeta x)\| < \varepsilon$$

for all $n > N = N(\varepsilon)$. The first inequality is a consequence of the maximum principle applied to the function $g(\zeta) = \hat{f}(\zeta\hat{x}_n) - \hat{f}(\zeta x)$ on the unit disk, $|\zeta| < 1$. Given $\varepsilon > 0$ there exists N depending only on ε such that the second inequality holds for $n > N$ because the continuous function $f = \hat{f}|_S$ is uniformly continuous on the compact set

$$(\bigcup_{n=1}^{\infty} \{\zeta \hat{x}_n\}) \cup \{\zeta x\} \subset S \quad (|\zeta| = 1).$$

An interesting consequence of our theorem is the following identity principle.

COROLLARY. *Suppose $f \in C(S, Y)$ has the one-dimensional extension property. If f is constant on an open subset of S, the f is constant on S.*

Proof. By hypothesis there is a point $p \in S$ and $\varepsilon > 0$, such that f is constant on the set $N(p, \varepsilon) = \{x \in S: \|x - p\| < \varepsilon\}$. We shall show that $\hat{f}$, the holomorphic extension of f to B, is constant on the open set $\Omega = \{x \in B: \|x - p\| < \varepsilon/4\}$, and hence, by the standard analytic continuation argument, that $\hat{f}$ is constant throughout B. Let $x \in \Omega$, $\hat{x} = x/\|x\|$ and $\zeta \in C$ with $|\zeta| = 1$ and $|\zeta - 1| < \varepsilon/2$. Then $\zeta \hat{x} \in N(p, \varepsilon)$ since $\|\hat{x} - p\| \leqslant 2\|x - p\|$ and

$$\|\zeta \hat{x} - p\| \leqslant |\zeta| \, \|\hat{x} - p\| + |\zeta - 1| \, \|p\| < \varepsilon/2 + \varepsilon/2.$$

Thus for any $\psi \in Y^*$ and fixed $x \in \Omega$ the complex valued function $g(\zeta) = \psi(\hat{f}(\zeta \hat{x}))$ is analytic in the open unit disk, $|\zeta| < 1$, continuous on its closure and constant on the arc $\{\zeta \in C: |\zeta| = 1, |\zeta - 1| < \varepsilon/2\}$. Hence, $g(\zeta) = M_\psi$, a constant depending on ψ, in $|\zeta| \leqslant 1$, and therefore

$$\psi(\hat{f}(x_2) - \hat{f}(x_1)) = M_\psi - M_\psi = 0$$

for all $x_1, x_2 \in \Omega$ and arbitrary $\psi \in Y^*$. This shows that $\hat{f}(x_1) = \hat{f}(x_2)$ for all $x_1, x_2 \in \Omega$ and proves the corollary.

In view of Hartogs' Theorem and the coordinate representation of point in B_n, one might conjecture that in the finite dimensional case the extension theorem should hold if one merely requires the existence of one-dimensional holomorphic extensions into B_n along lines parallel to the coordinate axes. To see that this is false consider the monomial $f(z, w) = |z|^2 w = z\bar{z}w$ defined for points (z, w) on S_2, the unit sphere in C^2. Clearly f is continuous on S_2 and corresponding to any fixed point $(z, w) \in S_2$, f has the one-dimensional holomorphic (in λ) extensions $f(\lambda z, w) = |z|^2 w$ and $f(z, \lambda w) = \lambda |z|^2 w$ $(|\lambda| < 1)$ respectively, along the two lines through (z, w) that are parallel to the coordinate axes. However, the next proposition shows that f cannot extend to a function that is holomorphic in B_2.

PROPOSITION. *If $f(z, w) = z^m \bar{z}^n w^p \bar{w}^r$ on the sphere $|z|^2 + |w|^2 = 1$ $(m, n, p, r$ positive integers) extends to a function holomorphic in B_2, then $n = 0 = r$.*

Proof. Fix $(z, w) \in S_2$. If f extends to a holomorphic function of λ on the disk $\{(\lambda z, w): |\lambda| < 1\}$, then the extension must be

$$\hat{F}(\lambda) = F(\lambda z, w) = z^m \bar{z}^n w^p \bar{w}^r \lambda^{m-n} = \lambda^{m-n} f(z, w)$$

since F must be holomorphic in $|\lambda| < 1$ and continuous in $|\lambda| \leqslant 1$ with boundary values $z^m \bar{z}^n w^p \bar{w}^r e^{i(m-n)\theta}$ on the circle $\lambda = e^{i\theta}$, $0 \leqslant \theta \leqslant 2\pi$. Furthermore $m \geqslant n$ since there are no negative Fourier coefficients. Similarly if $(u, v) \in S_2$ is fixed and if f extends holomorphically to the disk $\{(u, \zeta v): |\zeta| < 1\}$, then the extension must be $\hat{G}(\zeta) = G(u, \zeta v) = \zeta^{p-r} f(u, v)$ and $p \geqslant r$. Now choose points $(z, w) = (1/\sqrt{2}, 1/\sqrt{2})$, $(u, v) = (1/2, \sqrt{3}/2)$ and note that when $\lambda = 1/\sqrt{2}$, $\zeta = \sqrt{2/3}$ we have $(\lambda z, w) = (u, \zeta v) = (1/2, 1/\sqrt{2})$. Since the two extensions must agree at the common point $(1/2, 1/\sqrt{2})$ we must have $\lambda^{m-n} f(z, w) = \zeta^{p-r} f(u, v)$ when $\lambda = z = w = 1/\sqrt{2}$, $u = 1/2$, $v = \sqrt{3}/2$ and $\zeta = \sqrt{2/3}$. By a simple calculation this yields the equation $2^{n+r} = 3^r$ and consequently $r = 0 = n$.

Finally it is natural to ask whether the extension theorem can be generalized further by removing the assumption that the domain space $\mathscr{H}$ be a Hilbert space. We have used the inner product structure of $\mathscr{H}$ only in establishing claim (i) of our proof, i.e., the existence of a uniquely determined extension value $\hat{f}(x)$. The inner product structure enabled us to construct a linear isomorphism between the subspace $[u_1, u_2] \subset \mathscr{H}$ and C^2 which was also an isometry and therefore enabled us to use the known [1] finite dimensional result for the ball in C^2. If $\mathscr{H}$ is merely a complex Banach space there is still the obvious linear isomorphism $T(z_1, z_2) = z_1 u_1 + z_2 u_2$ $(u_1, u_2 \in \mathscr{H})$ between C^2 and $[u_1, u_2]$, but instead of the ball in C^2 one now must consider the convex domain

$$D = \{(z_1, z_2) \in C^2: \|z_1 u_1 + z_2 u_2\|_{\mathscr{H}} < 1\}.$$

If D has a $\mathscr{C}^2$ boundary, then one can apply a theorem of E. L. Stout (Duke Math. J. 44 (1977), p. 105–108) to prove our claim (i) and hence establish the extension theorem. For example, if $\mathscr{H} = C^n$ with l^p norm $(2 \leqslant p \leqslant \infty)$, then the extension theorem holds. We conjecture that in general the extension theorem is true if $\mathscr{H}$ is a complex Banach space with "sufficiently smooth" unit ball, but we do not have a precise result at present.

The authors thank the referee for suggesting the addition of a reference to the recent work of E. L. Stout.

References

[1] M. L. Agranovskii and R. E. Valskii, *Maximality of invariant algebras of functions*, Sibirskii Mat. Zhurnal 12 (1971), p. 3–12.

[2] E. Hille and R. Phillips, *Functional analysis and semigroups*, Amer. Math. Soc. Colloq. Publ. 31 (1957).

[3] A. Nagel and W. Rudin, *Moebius-invariant function spaces on balls and spheres*, Duke Math. J. 43 (1976), p. 841–865.
[4] M. Zorn, *Gâteaux differentiability and essential boundedness*, ibidem 12 (1945), p. 579–583. p. 579–583.

UNIVERSITY OF NORTH CAROLINA AT CHAPEL HILL
CHAPEL HILL, NORTH CAROLINA

Reçu par la Rédaction le 5.4.1979