tels que :

$$\sum_{j\in J}\varphi(s_j) < +\infty.$$

En fait, $\zeta_{\mathscr{P}}$ a alors aussi une infinité de pôles simples en les $s_j-\varphi(s_j)$. Il serait intéressant de savoir si ce résultat est encore vrai si on impose de plus à $\zeta_{\mathscr{P}}$ de ne pas avoir de pôles sur le demi-plan $\operatorname{Re}(s) > 0$ sauf en $s = 1$, ou même d'être méromorphe sur C avec un seul pôle, en $s = 1$.

Enfin, le lemme 7 permet d'utiliser des exposants non entiers, donc de limiter la possibilité de prolongement de la fonction $\exp(H_{E',\Phi})$. On obtient par exemple :

THÉORÈME 3. *Soit $U \subset \mathscr{B}$ tel que $\overline{U} = U$ et que $U \cup \mathscr{D}$ soit ouvert simplement connexe. Soit E une partie fonctionnelle finie de $\mathscr{E}(U)$, telle que $\overline{E} = E$. Alors il existe $\mathscr{P} \in$ T.N.P. tel que :*
- *$U \cup \mathscr{D}$ est $\zeta_{\mathscr{P}}$-maximal ;*
- *$E(\zeta_{\mathscr{P}}, U) = E$.*

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An improved upper bound for $G(k)$ in Waring's problem for relatively small k

by

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1. Introduction. In [3] K. Thanigasalam established the following

THEOREM 1. *$G(k) \leqslant 2[A_2+1]+[A_1+1]$ if $k \geqslant 2$ where*

$$A_1 = -\log(3k)\log^{-1}\theta, \quad A_2 = -\log(6k)\log^{-1}\theta \quad \text{and} \quad \theta = \left(1-\frac{1}{k}\right).$$

In this paper we improve this result by establishing the following

THEOREM 2. *$G(k) \leqslant [2A_2+A_1-4]$ if $k \geqslant 2$.*

For $k \leqslant 20$ Theorem 2 is established by comparison with known results. In particular, for $2 \leqslant k < 9$ confer [6]; for $9 \leqslant k \leqslant 20$ confer [4].

It is easy to see from the fact that $2A_2+A_1$ is transcendental that to establish Theorem 2 for $k > 20$ it is sufficient to establish

THEOREM 3. *$G(k) \leqslant 2(A_2+3)+(A_1+3)-(2\lambda_2+\lambda_1)$ if $k > 20$ where λ_1 and λ_2 are chosen such that $A_1-\lambda_1$ and $A_2-\lambda_2$ are integers, $2\lambda_2+\lambda_1 \leqslant 14$, $2\lambda_2+\lambda_1$ is maximal, and $[\lambda_2] = 4$.*

Theorem 3 is established by combining the admissible exponents introduced in [5] with the circle method construction introduced in [4] together with a careful estimation of the coefficients and the error terms in the Taylor polynomial expansions of two relevant functions.

2. A proof of Theorem 3. Let

$$f(s) = \frac{k^3-3k^2+k+2}{k^3+k^2-k^2\theta^{s-3}}\,\theta^{s-3} \quad \text{(cf. 2.20 in [5])} \tag{1}$$

and let

$$s_1 = A_1+3-\lambda_1 \quad \text{and} \quad s_2 = A_2+3-\lambda_2. \tag{2}$$

By the construction presented in [4] to establish Theorem 3 it is sufficient to show

$$f(s_2)+\frac{1}{2}\left(\frac{k-1}{2k-1}\right)f(s_1) < \frac{1}{2}\left(\frac{1}{2k-1}\right) \quad \text{if} \quad k > 20. \tag{3}$$

By direct calculation it follows that it is sufficient to show if $k > 20$

(4) $$A_1(k)\,\theta^{s_1-3}+A_2(k)\,\theta^{s_2-3}+A_3(k)\,\theta^{s_1-3}\,\theta^{s_2-3} \leqslant k^3+2k^2+k$$

where

(5) $$A_1(k) = k^4-3k^3+k^2+6k,$$

(6) $$A_2(k) = 4k^4-10k^3-3k^2+17k+2,$$

(7) $$A_3(k) = -5k^3+18k^2.$$

Clearly,

(8) $$A_3(k)\,\theta^{s_1-3}\,\theta^{s_2-3} \leqslant A_3(k)\left(\frac{1}{3k}\right)\left(\frac{1}{6k}\right)\theta^{-\lambda_1}\theta^{-\lambda_2} \leqslant \left(-\tfrac{5}{18}k+1\right).$$

Let

(9) $$P(k,\lambda) = \left(1+\frac{\lambda}{k}+\frac{\lambda(\lambda+1)}{2k^2}+\frac{\lambda(\lambda+1)(\lambda+2)}{6k^3}\right),$$

(10) $$R(k,\lambda) = \theta^{-\lambda}-P(k,\lambda),$$

(11) $$S(k,\lambda) = k^3R(k,\lambda).$$

It is easy to see (since $\lambda_1 \leqslant 6$ and $\lambda_2 \leqslant 5$)

(12) $$R(k,\lambda) = \frac{S(k,\lambda)}{k^3} \leqslant \frac{S(20,6)}{k^3} \leqslant \frac{7}{k^3} \quad \text{if} \quad k > 20.$$

Hence

(13) $$A_1(k)\left(\frac{1}{3k}\right)R(k,\lambda_1)+A_2(k)\left(\frac{1}{6k}\right)R(k,\lambda_2) \leqslant 7 \quad \text{if} \quad k > 20.$$

By direct calculation it follows that

(14) $$A_1(k)\left(\frac{1}{3k}\right)P(k,\lambda_1)+A_2(k)\left(\frac{1}{6k}\right)P(k,\lambda_2)$$
$$= A_4k^3+A_5k^2+A_6k+A_7+\frac{A_8}{k}+\frac{A_9}{k^2}+\frac{A_{10}}{k^3}+\frac{A_{11}}{k^4},$$

where the A_i for $i \geqslant 4$ are functions of λ_1 and λ_2.

The hypotheses imply that $3 \leqslant \lambda_1 \leqslant 6$ and $4 \leqslant \lambda_2 \leqslant 5$. These inequalities and the fact that $2\lambda_2+\lambda_1 \leqslant 14$ imply

(15) $A_4 \leqslant 1$, (16) $A_5 \leqslant 2$, (17) $A_6 \leqslant 5/6$,

(18) $A_7 \leqslant 1/3$, (19) $A_8 \leqslant -28$, (20) $A_9 \leqslant 88$,

(21) $A_{10} \leqslant 217$, (22) $A_{11} \leqslant 12$.

Theorem 3 now follows from (4), (8), (13), (14), (15), (16), (17), (18), (19), (20), (21), (22).

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