Distributive partially ordered sets

by

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Abstract. In this paper we introduce a definition of "distributive" which applies to arbitrary partially ordered sets. We give several characterizations of this notion, one of which is the following: a partially ordered set P is distributive in our sense if and only if the lattice of lower ends of P closed under existent finite suprema is distributive. The definition and the characterizations extend to a notion of $\varkappa$-distributivity, where $\varkappa$ is a regular cardinal. However our notion has the property that if λ and μ are two regular cardinals, with $\lambda < \mu$, then there exists a partially ordered set which is μ-distributive but not λ-distributive. This destroys another potential characterization, and provides strong evidence that no single good notion of distributivity exists for arbitrary partially ordered sets.

Introduction. In this paper we give a notion of distributivity which applies to arbitrary partially ordered sets. Our starting point is the following notion of distributivity for (lower) semilattices: a lower semilattice S is *distributive* if for any finite number n and any $a_1, \ldots, a_n, b \in S$, if $a_1 \vee \ldots \vee a_n$ exists, then $(b \wedge a_1) \vee \ldots \vee (b \wedge a_n)$ exists and equals $b \wedge (a_1 \vee \ldots \vee a_n)$. This notion has been studied under the name "weak distributivity" by Cornish and Hickman [2]. It extends straightforwardly to a notion of $\varkappa$-distributivity for semilattices, where $\varkappa$ is any regular cardinal. We introduce a notion of $\varkappa$-distributivity for arbitrary partially ordered sets which generalizes the notion of $\varkappa$-distributive semilattice.

Cornish and Hickman characterize distributive semilattices through the lattice of ideals of a semilattice (where an ideal is a lower end closed under existent finite suprema): a semilattice is distributive if and only if the lattice of ideals is distributive. In Section 2 we extend their results to arbitrary partially ordered sets by looking at the lattice of $\varkappa$-ideals (a $\varkappa$-ideal is a lower end closed under existent suprema of sets of size less than $\varkappa$). In particular we show that a partially ordered set is $\varkappa$-distributive if and only if its lattice of $\varkappa$-ideals is a locale (i.e. complete Heyting algebra).

In Section 3 we study a different construction which associates to a partially ordered set P a locale we call $\mathscr{L}\mathscr{D}_\varkappa(P)$, where $\varkappa$ is a regular cardinal. We show that P is $\varkappa$-distributive if and only if $\mathscr{L}\mathscr{D}_\varkappa(P)$ coincides

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with the lattice of $\varkappa$-ideals of P. By letting $\varkappa$ be sufficiently large we obtain a locale we call $\mathscr{L}\mathscr{D}_{\varkappa}(P)$, which is the smallest join-completion of P which is a locale.

In Section 4 we study another result from Cornish and Hickman: they show that a semilattice S is distributive if and only if there is a semilattice morphism preserving existent finite suprema which embeds S in a distributive lattice. We define a notion of *strong embedding* of a partially ordered set to replace the notion of semilattice morphism. We show that if a partially ordered set P is $\varkappa$-distributive then there is a strong embedding preserving suprema of sets of size less than $\varkappa$ of P into a $\varkappa$-distributive lattice. The converse however fails, and the reason is that for regular cardinals λ and μ, $\lambda < \mu$, it is possible to have a partially ordered set which is μ-distributive but not λ-distributive. This indicates to us that there is no single notion which extends to arbitrary partially ordered sets the concept "$\varkappa$-distributive semilattice".

1. Preliminaries. Throughout the paper P will denote an arbitrary partially ordered set, and $\varkappa$ an arbitrary regular cardinal or ∞ (where ∞ may be thought of as exceeding every cardinal). A subset of P of cardinality less than $\varkappa$ is called *$\varkappa$-small.* (Every set is ∞-small.) A subset $A \subseteq P$ is called a *lower end* of P if $a \in A$ and $b \leqslant a$ implies $b \in A$. The empty set is considered to be a lower end. We write $\mathscr{L}(P)$ for the collection of all lower ends of P. If $B \subseteq P$ then $(B]$ denotes the *lower end generated by* B: thus

$$(B] = \{p \in P\colon (\exists b \in B)(p \leqslant b)\}.$$

If $B = \{b\}$ is a one-element set, we write $(b]$ instead of $(\{b\}]$; such a lower end is called *principal.* If $A \subseteq P$ and $b \in P$, we write $b \geqslant A$ if $b \geqslant a$ for all $a \in A$. If I is a lower end and $a \in P$, by $I|a$ we mean the lower end $\{b\colon b \in I$ and $b \leqslant a\}$.

If P and Q are partially ordered sets, by an *embedding* of P into Q we mean a function $\varphi\colon P \to Q$ such that $a \leqslant b$ if and only if $\varphi(a) \leqslant \varphi(b)$. (It follows that φ is $1:1$.)

Lattices. A complete lattice L is called *$\varkappa$-distributive* if for each $\varkappa$-small $A \subseteq L$ and any $b \in L$,

$$\left(\bigvee A\right) \wedge b = \bigvee \{a \wedge b\colon a \in A\}.$$

A complete lattice which is ∞-distributive is called a *locale.* (Locales are also called *complete Heyting algebras*, but in this context we avoid the term "complete Heyting algebra" because we do not consider implication operators.)

Join extensions. Let P be a partially ordered set and E an extension of P (that is the identity mapping is an embedding of P in E). E is called a *join extension* of P if every element of E is the supremum in E of some subset of P. It is known (see Schmidt [3]) that each join extension of P is isomorphic to a subset of $\mathscr{L}(P)$ which contains all the principal lower ends of P; we therefore assume that all join extensions of P are subsets of $\mathscr{L}(P)$. A join extension of P which is a complete lattice is called a *join completion.*

Closure operations. A *closure operation* $\mathscr{C}$ on P is an operation which assings a lower end $\mathscr{C}(A)$ to each subset A of P such that

i) $A \subseteq \mathscr{C}(A)$,

ii) $A \subseteq B$ implies $\mathscr{C}(A) \subseteq \mathscr{C}(B)$,

iii) $\mathscr{C}(\mathscr{C}(A)) = \mathscr{C}(A)$.

It should be noted that this definition is non-standard: usually $\mathscr{C}(A)$ is only required to be a subset of P, not necessarily a lower end.

A lower end of the form $\mathscr{C}(A)$ is called *$\mathscr{C}$-closed.* An arbitrary intersection of $\mathscr{C}$-closed lower ends is again $\mathscr{C}$-closed; conversely if $\mathscr{A}$ is a family of lower ends which is closed under arbitrary intersections, then $\mathscr{A}$ determines a closure operation $\mathscr{C}_{\mathscr{A}}$, where

$$\mathscr{C}_{\mathscr{A}}(A) = \bigcap \{I \in \mathscr{A}\colon I \supseteq A\}.$$

From now on we require that if $\mathscr{C}$ is a closure operation then each principal lower end is $\mathscr{C}$-closed; with this requirement there is a one-to-one correspondence between closure operations on P and join completions of P (see Schmidt [3]). We write $\mathscr{L}\mathscr{C}(P)$ for the complete lattice of $\mathscr{C}$-closed lower ends of P (so $\mathscr{L}\mathscr{C}(P)$ is the join completion of P which corresponds to $\mathscr{C}$). For any family $\{A_i\colon i \in I\}$ of elements of $\mathscr{L}\mathscr{C}(P)$,

$$\bigvee_i A_i = \mathscr{C}\left(\bigcup_i A_i\right) \quad \text{and} \quad \bigwedge_i A_i = \bigcap_i A_i.$$

There is a canonical embedding of P into $\mathscr{L}\mathscr{C}(P)$ given by the map which takes $a \in P$ to $(a]$. It is easy to see that this embedding preserves all infima which exist in P. However in general it does not preserve even finite suprema.

We note a condition which implies that $\mathscr{L}\mathscr{C}(P)$ is a locale.

PROPOSITION 1.1. *Let $\mathscr{C}$ be a closure operation on P such that if $A \subseteq P$ is arbitrary and $B \subseteq P$ is $\mathscr{C}$-closed then $\mathscr{C}(A \cap B) = \mathscr{C}(A) \cap B$. Then $\mathscr{L}\mathscr{C}(P)$ is a locale.*

Proof.

$$\bigvee_i (A_i \cap B) = \mathscr{C}\left(\bigcup_i (A_i \cap B)\right) = \mathscr{C}\left(\left(\bigcup_i A_i\right) \cap B\right) = \left(\bigvee_i A_i\right) \cap B. \ \blacksquare$$

Ideals. A lower end $I \subseteq P$ is called a *$\varkappa$-ideal* if I is closed under existent suprema of $\varkappa$-small sets, that is if $A \subseteq I$ is $\varkappa$-small and $\bigvee A$ exists in P, then $\bigvee A \in I$. An $\aleph_0$-ideal is simply called an *ideal.* For a fixed $\varkappa$ the collection of all $\varkappa$-ideals is a join completion of P; the corresponding closure operation is denoted by $\mathscr{I}_\varkappa$. We write $\mathscr{I}$ instead of $\mathscr{I}_{\aleph_0}$, and we note that $\mathscr{I}_\infty$ is the closure operation corresponding to lower ends closed under arbitrary existent suprema.

Proposition 1.2. *Let I be any lower end. Define*

$$I' = \{b\colon (\exists A \subseteq I)(|A| < \varkappa \text{ and } b \leqslant \bigvee A)\}.$$

For an arbitrary lower end J, $\mathscr{I}_\varkappa(J)$ can be described as follows. Set

$$J_0 = J,$$
$$J_{\beta+1} = (J_\beta)' \quad \text{for any ordinal } \beta,$$
$$J_\gamma = \bigcup_{\beta<\gamma} J_\beta \quad \text{for } \gamma \text{ a limit ordinal.}$$

Then $\mathscr{I}_\varkappa(J) = J_\varkappa$. ∎

Figure 1 shows a partially ordered set P. If we take the lower end J to

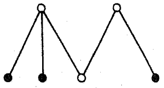

Fig. 1

consist of the three dark elements, we see that $J_1 \neq P$ but $J_2 = P$. So in this case the process of Proposition 1.2 takes two steps for completion. Examples can be considered for which λ steps are needed, for any fixed regular cardinal λ.

Proposition 1.3. *The canonical embedding of P into $\mathscr{L}\mathscr{I}_\varkappa(P)$ preserves existent suprema of $\varkappa$-small sets.*

Proof. Suppose that $A \subseteq P$ is $\varkappa$-small and that in P, $b = \bigvee A$. Evidently $(b] \supseteq \bigvee_{a\in A} (a]$. For the reverse inclusion, set $I = \bigcup_{a\in A} (a]$. Then, in the terminology of Proposition 1.2, $b \in I'$. Hence

$$\bigvee_{a\in A} (a] = \mathscr{I}_\varkappa(I) \supseteq (b]. \ ∎$$

Semilattices. For us "semilattice" will always mean lower semilattice. There are already several different concepts of distributivity for semilattices; we are concerned with the following.

Definition 1.4. A semilattice S is *$\varkappa$-distributive* if for any $\varkappa$-small $A \subseteq S$ and any $b \in S$, if $\bigvee A$ exists then $\bigvee\{b \wedge a\colon a \in A\}$ exists and equals $b \wedge (\bigvee A)$. An $\aleph_0$-distributive lattice will be called *distributive*.

We note that if L is a complete lattice then L is $\varkappa$-distributive according to Definition 1.4 if and only if L is $\varkappa$-distributive according to the definition on page 152.

The following theorem about distributive semilattices is proved in Cornish and Hickman [2].

Theorem 1.5. *For every semilattice S, the following are equivalent.*

i) *S is distributive.*

ii) *For every lower end A of S, $\mathscr{I}(A) = \{\bigvee B\colon B \subseteq A,\ B$ finite$\}$.*

iii) *For every pair of ideals I, J of S the supremum of I and J in $\mathscr{L}\mathscr{I}(S)$ is given by $I \vee J = \{\bigvee B\colon B \subseteq I \cup J,\ B$ finite$\}$.*

iv) *$\mathscr{L}\mathscr{I}(S)$ is a distributive lattice.*

v) *The map $\varphi\colon \mathscr{L}(S) \to \mathscr{L}\mathscr{I}(S)$ given by $\varphi(I) = \mathscr{I}(I)$ for I an arbitrary lower end is a lattice homomorphism.* ∎

In the next section Theorem 1.5 is generalized to arbitrary partially ordered sets.

2. A definition of "distributive" for partially ordered sets.

Definition 2.1. A lower end I is *$\varkappa$-descending* if $\bigvee I$ exists and for each $b \leqslant \bigvee I$ there exists a $\varkappa$-small set $B \subseteq I$ such that $\bigvee B = b$.

Definition 2.2. A partially ordered set P is called *$\varkappa$-distributive* if whenever $A \subseteq P$ is $\varkappa$-small and $\bigvee A$ exists, then $(A]$ is $\varkappa$-descending.

This definition extends that of $\varkappa$-distributive semilattice, as follows.

Proposition 2.3. *A semilattice is $\varkappa$-distributive in the sense of Definition 1.4 if and only if it is $\varkappa$-distributive in the sense of Definition 2.2.*

Proof. Suppose S is $\varkappa$-distributive in the sense of Definition 1.4, $A \subseteq S$ is $\varkappa$-small and $\bigvee A$ exists. Set $I = (A]$; clearly $\bigvee I = \bigvee A$. Suppose $b \leqslant \bigvee I$. Then $b \leqslant \bigvee A$, so $b = b \wedge (\bigvee A) = \bigvee\{b \wedge a\colon a \in A\}$. Since $\{b \wedge a\colon a \in A\}$ is $\varkappa$-small, $I = (A]$ is $\varkappa$-descending as required.

Conversely suppose that S is $\varkappa$-distributive in the sense of Definition 2.2, $A \subseteq S$ is $\varkappa$-small and $\bigvee A$ exists. Take $b \in S$, and set $b' = b \wedge (\bigvee A)$. By distributivity in the sense of Definition 2.2 there is a $\varkappa$-small set $B \subseteq (A]$ such that $b' = \bigvee B$. We wish to show that $b' = \bigvee\{b \wedge a\colon a \in A\}$. Now, if $c \in B$ there is some $a \in A$ such that $c \leqslant a$, and also $c \leqslant b' \leqslant b$, so $c \leqslant b \wedge a$. Thus $b' = \bigvee B \leqslant \bigvee\{b \wedge a\colon a \in A\}$. But clearly $b' \geqslant \bigvee\{b \wedge a\colon a \in A\}$, so we are finished. ∎

The concept of $\varkappa$-distributivity is not self-dual. This is well-known for lattices with $\varkappa > \aleph_0$. The two partially ordered sets in Figure 2 are $\varkappa$-

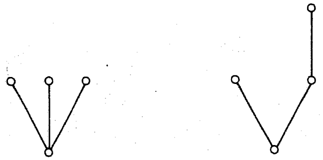

Fig. 2

distributive for all $\varkappa$, but their duals are not $\varkappa$-distributive for any $\varkappa$. We also note that an unordered set is $\varkappa$-distributive for all $\varkappa$.

If $\varkappa = \infty$ the definition of $\varkappa$-descending lower end and $\varkappa$-distributive partially ordered set simplify a little, leading to the following very easy propositions.

PROPOSITION 2.4. *A lower end I is ∞-descending if and only if $\bigvee I$ exists and for each $b \leqslant \bigvee I$, $b = \bigvee (I|b)$.*

PROPOSITION 2.5. *A partially ordered set P is ∞-distributive if and only if for every $A \subseteq P$ such that $\bigvee A$ exists, $(A]$ is ∞-descending.*

We now proceed to our generalization of Theorem 1.5.

LEMMA 2.6. *Let I be a lower end. Then $b \in \mathscr{I}_\varkappa(I \cap (b])$ if and only if $b = \bigvee B$ for some $\varkappa$-small $B \subseteq I$.*

Proof. Let J be an arbitrary lower end. Recall from Proposition 1.2 the definition of the sequence J_α, for $\alpha \leqslant \varkappa$. If $x \in \mathscr{I}_\varkappa(J) = J_\varkappa$, then $x \in J_\alpha$ for some $\alpha < \varkappa$. Let α_x be the least α such that $x \in J_\alpha$ (note that α_x is either 0 or a successor ordinal). To each $x \in J_\varkappa$ we associate a set $Q_x \subseteq J$, defined by induction on α_x, as follows.

i) If $\alpha_x = 0$, $Q_x = \{x\}$.

ii) If $\alpha_x = \beta + 1$, then there is some $\varkappa$-small $Y \subseteq J_\beta$ such that $x \leqslant \bigvee Y$. Set $Q_x = \bigcup_{y \in Y} Q_y$.

Note that in step ii) there is some arbitrariness in the choice of Y. This does not matter.

It is easy to establish by induction on α_x that $|Q_x| < \varkappa$ and also that if $c \in P$ and $c \geqslant Q_x$ then $c \geqslant x$. Suppose that $b \in \mathscr{I}_\varkappa(I \cap (b])$. Set $J = I \cap (b]$ and construct $Q_b \subseteq J$ as above. Our last observation shows that if $c \geqslant Q_b$ then $c \geqslant b$. But $b \geqslant Q_b$, so $b = \bigvee Q_b$, and thus b is the supremum of a $\varkappa$-small subset of I.

The converse is trivial. ■

THEOREM 2.7. *The following are equivalent.*

i) *P is $\varkappa$-distributive.*

ii) *For every lower end I, $\mathscr{I}_\varkappa(I) = \{\bigvee A \colon A \subseteq I, |A| < \varkappa\}$.*

iii) *Let $\{I_i \colon i \in X\}$ be a family of lower ends, $|X| < \varkappa$. Then in $\mathscr{L}\mathscr{I}_\varkappa(P)$, $\bigvee_i I_i = \{\bigvee A \colon A \subseteq \bigcup_i I_i$ and $|A| < \varkappa\}$.*

iv) *$\mathscr{L}\mathscr{I}_\varkappa(P)$ is a locale.*

v) *$\mathscr{L}\mathscr{I}_\varkappa(P)$ is a $\varkappa$-distributive lattice.*

vi) *The map $\varphi \colon \mathscr{L}(P) \to \mathscr{L}\mathscr{I}_\varkappa(P)$ given by $\varphi(I) = \mathscr{I}_\varkappa(I)$ is a lattice homomorphism.*

Proof. We argue in the circles i) ⇒ ii) ⇒ iii) ⇒ v) ⇒ i) and i) ⇒ vi) ⇒ iv) ⇒ v) ⇒ i).

i) ⇒ ii). It follows from i) that $\{\bigvee A \colon A \subseteq I, |A| < \varkappa\}$ is a lower end. Since it is clearly closed under existent suprema of $\varkappa$-small sets, it is $\mathscr{I}_\varkappa$-closed. So $\mathscr{I}_\varkappa(I) \subseteq \{\bigvee A \colon A \subseteq I, |A| < \varkappa\}$. The reverse inclusion is trivial.

ii) ⇒ iii). Let $\{I_i \colon i \in X\}$ be an arbitrary family of lower ends (without any restriction on the cardinality of X). Then

$$\bigvee_i I_i = \mathscr{I}_\varkappa\big(\bigcup_i I_i\big) = \{\bigvee A \colon A \subseteq \bigcup_i I_i \text{ and } |A| < \varkappa\}$$

by ii).

iii) ⇒ v). Let $\{A_i \colon i \in X\}$ be a family of elements of $\mathscr{L}\mathscr{I}_\varkappa(P)$, where $|X| < \varkappa$, and $B \in \mathscr{L}\mathscr{I}_\varkappa(P)$. We wish to show that $B \wedge (\bigvee_i A_i) = \bigvee_i (B \wedge A_i)$ in $\mathscr{L}\mathscr{I}_\varkappa(P)$; it is enough to show that $B \cap (\bigvee_i A_i) \subseteq \bigvee_i (B \cap A_i)$. Suppose $b \in B \cap (\bigvee_i A_i)$. Then $b \in B$ and by iii) there is some $\varkappa$-small $D \subseteq \bigcup_i A_i$ such that $b = \bigvee D$. Since $b \in B$, $D \subseteq B$ and hence $D \subseteq \bigcup_i (B \cap A_i)$, so $b \in \bigvee_i (B \cap A_i)$.

v) ⇒ i). We show that if i) is false, so is v). Suppose P is not $\varkappa$-distributive. Then there is a $\varkappa$-small set A such that $a = \bigvee A$ exists and there is some $b \leqslant a$ such that b is not the supremum of any $\varkappa$-small set $B \subseteq (A]$. By Proposition 1.3 the canonical embedding of P into $\mathscr{L}\mathscr{I}_\varkappa(P)$ preserves existent suprema of $\varkappa$-small subsets of P, so $(a] = \bigvee_{x \in A} (x]$. Thus

$$\big(\bigvee_{x \in A} (x]\big) \wedge (b] = (a] \wedge (b] = (b].$$

On the other hand,

$$\bigvee_{x \in A} ((x] \wedge (b]) = \mathscr{I}_\varkappa\big(\bigcup_{x \in A} ((x] \cap (b])\big) = \mathscr{I}_\varkappa((A] \cap (b]),$$

and $b \notin \mathscr{I}_\varkappa((A] \cap (b])$ by Lemma 2.6. Hence $\mathscr{L}\mathscr{I}_\varkappa(P)$ is not $\varkappa$-distributive.

This completes the circle i) ⇒ ii) ⇒ iii) ⇒ v) ⇒ i).

i) ⇒ vi). All closure operations preserve arbitrary suprema, so it remains to show that φ preserves finite infima. Let I and J be lower ends. It suffices to prove that

$$\mathscr{I}_\varkappa(I) \cap \mathscr{I}_\varkappa(J) \subseteq \mathscr{I}_\varkappa(I \cap J),$$

so let $b \in \mathscr{I}_\varkappa(I) \cap \mathscr{I}_\varkappa(J)$. By ii) (we have already shown i) ⇒ ii)) there are $\varkappa$-small sets $A \subseteq I$ and $B \subseteq J$ such that $b = \bigvee A = \bigvee B$. For each $a \in A$, $a \leqslant b$, so by i) there is a $\varkappa$-small set $B_a \subseteq (B]$ such that $a = \bigvee B_a$. Let $K = \bigcup \{B_a \colon a \in A\}$. Then K is $\varkappa$-small, $K \subseteq I \cap J$ and

$$\bigvee K = \bigvee \{\bigvee B_a \colon a \in A\} = \bigvee A = b.$$

Hence $b \in \mathscr{I}_\varkappa(I \cap J)$.

vi) ⇒ iv) and iv) ⇒ v) are trivial, while v) ⇒ i) was part of the first circle. Thus the second circle is complete. ■

Theorem 2.7 generalizes Theorem 1.5, from "distributive" to "$\varkappa$-distributive" and from semilattices to partially ordered sets.

3. Another characterization of distributivity. Theorem 2.7 shows that a partially ordered set P is $\varkappa$-distributive if and only if $\mathscr{L}\mathscr{I}_\varkappa(P)$ is a locale. We now introduce another closure operation $\mathscr{D}_\varkappa$ which is such that $\mathscr{L}\mathscr{D}_\varkappa(P)$ is always a locale; it turns out that P is $\varkappa$-distributive if and only if $\mathscr{L}\mathscr{I}_\varkappa(P) = \mathscr{L}\mathscr{D}_\varkappa(P)$.

Definition 3.1. A lower end A is *$\varkappa$-distinguished* if $I \subseteq A$ and I a $\varkappa$-descending lower end implies $\bigvee I \in A$.

Clearly an arbitrary intersection of $\varkappa$-distinguished lower ends is again $\varkappa$-distinguished, so we have a closure operation $\mathscr{D}_\varkappa$.

Proposition 3.2. *Let A be a lower end. Then*

$$\mathscr{D}_\varkappa(A) = \{\bigvee I\colon\ I \subseteq A,\ I \text{ a } \varkappa\text{-descending lower end}\}.$$

Proof. Set $A^+ = \{\bigvee I\colon I \subseteq A,\ I \text{ a } \varkappa\text{-descending lower end}\}$. It is easy to check that A^+ is a lower end containing A. Suppose $J \subseteq A^+$ is a $\varkappa$-descending lower end. For each $j \in J$ there is a $\varkappa$-descending lower end $I_j \subseteq A$ such that $j = \bigvee I_j$. Then $J^* = \bigcup_{j\in J} I_j$ is a lower end contained in A and $\bigvee J^* = \bigvee J$. We claim that J^* is $\varkappa$-descending. If $x \leqslant \bigvee J^*$ then since J is $\varkappa$-descending there is $B \subseteq J$, $|B| < \varkappa$ such that $\bigvee B = x$. Also each I_j is $\varkappa$-descending so there is $B_j \subseteq I_j$, $|B_j| < \varkappa$ such that $\bigvee B_j = \bigvee I_j = j$. Set $K_x = \bigcup_{j\in B} B_j$. Then $K_x \subseteq J^*$, $|K_x| < \varkappa$ and $\bigvee K_x = x$. This shows that J^* is $\varkappa$-descending, as claimed. It follows that $\bigvee J = \bigvee J^* \in A^+$, and this establishes that A^+ is $\varkappa$-distinguised. Since A^+ is obviously contained in any $\varkappa$-distinguished lower end containing A, we have that $A^+ = \mathscr{D}_\varkappa(A)$. ■

Lemma 3.3. *If A and B are lower ends and B is $\varkappa$-distinguished then $\mathscr{D}_\varkappa(A) \cap B = \mathscr{D}_\varkappa(A \cap B)$.*

Proof. It suffices to show that $\mathscr{D}_\varkappa(A) \cap B \subseteq \mathscr{D}_\varkappa(A \cap B)$. Suppose then that $x \in \mathscr{D}_\varkappa(A) \cap B$. By Proposition 3.2 there is a $\varkappa$-descending lower end $J \subseteq A$ with $\bigvee J = x$. Since $x \in B$, $J \subseteq B$. Hence $J \subseteq A \cap B$ and so $x \in \mathscr{D}_\varkappa(A \cap B)$. ■

Theorem 3.4. i) *$\mathscr{L}\mathscr{D}_\varkappa(P)$ is a locale.*

ii) *$\mathscr{L}\mathscr{I}_\varkappa(P) \subseteq \mathscr{L}\mathscr{D}_\varkappa(P)$.*

iii) *For each $A \subseteq P$ such that $\bigvee A$ exists in P, the embedding $P \to \mathscr{L}\mathscr{D}_\varkappa(P)$ preserves $\bigvee A$ if and only if $(A]$ is $\varkappa$-descending.*

Proof. i) is immediate from Lemma 3.3 and Proposition 1.1.

ii) is trivial.

iii) (a) Let $a = \bigvee A$ where $(A]$ is $\varkappa$-descending. Clearly $(a]$ is the smallest $\varkappa$-distinguished lower end containing $(A]$. So in $\mathscr{L}\mathscr{D}_\varkappa(P)$, $(a] = \bigvee_{b\in A}(b]$.

(b) Suppose $a = \bigvee A$ where $(A]$ is not $\varkappa$-descending. Then there exists $x \in P$ such that $x \leqslant a$ and yet x is not the supremum of any $\varkappa$-small subset of $(A]$. Hence (by Proposition 3.2) $x \notin \mathscr{D}_\varkappa((A]) = \bigvee_{b\in A}(b]$. Thus $(a] \neq \bigvee_{b\in A}(b]$ in $\mathscr{L}\mathscr{D}_\varkappa(P)$. ■

Theorem 3.5. *The following are equivalent.*

i) *P is $\varkappa$-distributive.*

ii) *$\mathscr{L}\mathscr{I}_\varkappa(P) = \mathscr{L}\mathscr{D}_\varkappa(P)$.*

iii) *The embedding $P \to \mathscr{L}\mathscr{D}_\varkappa(P)$ preserves all existent suprema of $\varkappa$-small sets.*

Proof. i) $\Rightarrow$ ii). If P is $\varkappa$-distributive it follows trivially from the definitions that $\mathscr{L}\mathscr{D}_\varkappa(P) \subseteq \mathscr{L}\mathscr{I}_\varkappa(P)$. The result now follows from Theorem 3.4 ii).

ii) $\Rightarrow$ iii). This follows immediately from Proposition 1.3.

iii) $\Rightarrow$ i). From ii) and Theorem 3.4 iii) it follows that if $A \subseteq P$ is $\varkappa$-small and $\bigvee A$ exists in P then $(A]$ is $\varkappa$-descending. This is just the definition of $\varkappa$-distributivity. ■

Theorem 3.5 provides a characterization of $\varkappa$-distributive partially ordered sets additional to that given in Theorem 2.7. We give two examples to illustrate these characterizations. Figure 3 shows a partially ordered set P

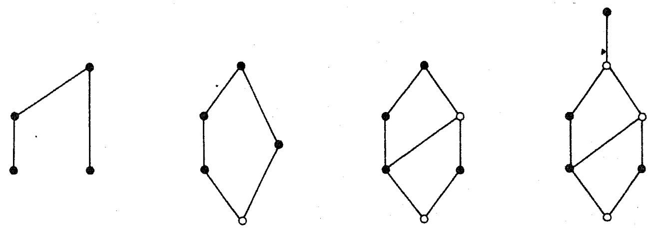

Fig. 3

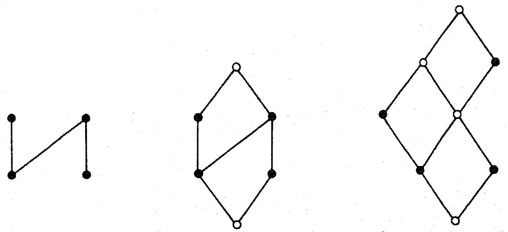

Fig. 4

together with the lattice of ideals $\mathcal{L}\mathcal{I}(P)$, the lattice of $\aleph_0$-distinguished lower ends $\mathcal{L}\mathcal{D}_{\aleph_0}(P)$ and the lattice of all lower ends $\mathcal{L}(P)$. The embedding of P into each of the three lattices is shown by black dots. P is not distributive: we see that $\mathcal{L}\mathcal{I}(P)$ is not distributive (Theorem 2.7), $\mathcal{L}\mathcal{I}(P) \neq \mathcal{L}\mathcal{D}_{\aleph_0}(P)$ and the embedding of P into $\mathcal{L}\mathcal{D}_{\aleph_0}(P)$ does not preserve all the suprema existing in P (Theorem 3.5). Figure 4 shows a distributive partially ordered set Q: here $\mathcal{L}\mathcal{I}(Q)$ is distributive, $\mathcal{L}\mathcal{I}(Q) = \mathcal{L}\mathcal{D}_{\aleph_0}(Q)$ and the embedding of Q into $\mathcal{L}\mathcal{D}_{\aleph_0}(Q)$ does preserve all the suprema existing in Q.

At any rate when $\varkappa = \infty$, the notion of $\varkappa$-distinguished lower end has another use apart from its rôle in Theorem 3.5. We will show that, for any partially ordered set P, $\mathcal{L}\mathcal{D}_{\infty}(P)$ is the smallest join completion of P which is a locale. In what follows $\mathcal{C}$ is any closure operation on P such that $\mathcal{L}\mathcal{C}(P)$ is a locale.

LEMMA 3.6. *Let $A \subseteq P$ be a lower end. If $p \in \mathcal{C}(A)$ then $\bigvee(A \mid p)$ exists and equals p.*

Proof. Certainly $A = \bigcup_{a\in A} (a]$, so in $\mathcal{L}\mathcal{C}(P)$ we have that $\mathcal{C}(A) = \bigvee_{a\in A} (a]$. If $p \in \mathcal{C}(A)$, $(p] \subseteq \bigvee_{a\in A} (a]$, so since $\mathcal{L}\mathcal{C}(P)$ is a locale we have that

$$(p] = \Big(\bigvee_{a\in A} (a]\Big) \cap (p] = \bigvee_{a\in A} \big((a] \cap (p]\big).$$

Let $x \in P$ be such that $x \geqslant A \mid p$. Then $(x] \supseteq (a] \cap (p]$ for each $a \in A$, so in $\mathcal{L}\mathcal{C}(P)$, $(x] \geqslant \bigvee_{a\in A} \big((a] \cap (p]\big)$. Thus $(x] \geqslant (p]$, whence $x \geqslant p$ and $\bigvee(A \mid p) = p$. ∎

THEOREM 3.7. $\mathcal{L}\mathcal{D}_{\infty}(P) \subseteq \mathcal{L}\mathcal{C}(P)$.

Proof. Let $A \subseteq P$ be an ∞-distinguished lower end. We will show that $\mathcal{C}(A) = A$ and hence $A \in \mathcal{L}\mathcal{C}(P)$. Take $p \in \mathcal{C}(A)$. Then, in P, $p = \bigvee(A \mid p)$ by Lemma 3.6. It remains to show that $A \mid p$ is ∞-descending. Let $q \leqslant p$. Then $q \in \mathcal{C}(A)$ so

$$q = \bigvee(A \mid q) = \bigvee\big((A \mid p) \mid q\big),$$

whence $A \mid p$ is ∞-descending. Since A is ∞-distinguished, $p \in A$. ∎

In the subject of this paper there is often difficulty with the question of whether the empty set should be admitted as a lower end. For us the empty set $\varnothing$ is a lower end, and we note the following:

i) $\varnothing$ has a supremum if and only if P has a zero.

ii) $\varnothing$ is ∞-descending if and only if P has a zero.

iii) $\varnothing$ is ∞-distinguished if and only if P does not have a zero.

Thus $\mathcal{L}\mathcal{D}_{\infty}(P)$ introduces a new zero if and only if P does not already have a zero. The same is true of $\mathcal{L}\mathcal{D}_{\varkappa}(P)$, $\mathcal{L}\mathcal{I}_{\varkappa}(P)$ and $\mathcal{L}\mathcal{I}_{\infty}(P)$.

4. A failed characterization. Section 2 of this paper was devoted to generalizing Theorem 1.5, which was taken from Cornish and Hickman [2]. In the same paper Cornish and Hickman prove the following theorem.

THEOREM 4.1. *The following conditions on a semilattice S are equivalent.*

i) *S is distributive.*

ii) *There is a semilattice monomorphism preserving existent finite suprema which embeds S into a distributive lattice.*

iii) *There is a semilattice monomorphism preserving existent finite suprema which embeds S into a ring of sets.* ∎

In this section we investigate some possible extensions of Theorem 4.1 to $\varkappa$-distributive partially ordered sets.

In order to extend Theorem 4.1 we need a suitable replacement for the concept of "semilattice monomorphism". A first thought is to replace it by "embedding of partially ordered sets preserving existent finite infima", but the following example shows that this is unsatisfactory.

EXAMPLE 4.2. Figure 5 exhibits a partially ordered set which is not distributive embedded into a Boolean algebra by an embedding preserving existent infima and suprema.

Fig. 5

In order to block this sort of example we make the following definition.

DEFINITION 4.3. Let P be a partially ordered set, L a complete lattice. An embedding $\varphi\colon P \to L$ is called *strong* if for every finite n and every $a_1, \ldots, a_n \in P$

$$\varphi(a_1) \wedge \ldots \wedge \varphi(a_n) = \bigvee\{\varphi(z)\colon z \in P \text{ and } z \leqslant a_1, \ldots, z \leqslant a_n\}.$$

(Note that if $\{z \in P\colon z \leqslant a_1, \ldots, z \leqslant a_n\}$ is empty, then

$$\bigvee\{\varphi(z)\colon z \in P \text{ and } z \leqslant a_1, \ldots, z \leqslant a_n\}$$

is taken to be the zero of L.)

The following three facts are easily seen.

(i) A strong embedding preserves existent finite infima.

(ii) If S is a semilattice and L a complete lattice, an embedding of partially ordered sets $\varphi\colon S \to L$ is strong if and only if it is a semilattice monomorphism.

(iii) The embedding in Example 4.2 is not strong.

We now apply the theory of prime ideals, developed by Balbes for semilattices (see [1]) to partially ordered sets. Recall that by "ideal" we mean $\aleph_0$-ideal, that is lower end closed under existent finite suprema.

DEFINITION 4.4. An ideal J of a partially ordered set P is called *prime* if for every finite n and every $a_1, \ldots, a_n \in P$, $(a_1] \cap \ldots \cap (a_n] \subseteq J$ implies that $a_i \in J$ for some i.

DEFINITION 4.5. A partially ordered set P is said to have *enough prime ideals* if for each $a, b \in P$ with $a \not\leqslant b$ there is a prime ideal J of P with $a \in J$ and $b \notin J$.

THEOREM 4.6. *If P has enough prime ideals then there is a strong embedding preserving existent finite suprema of P into a ring of sets.*

Proof. For $a \in P$, set

$$\mathscr{X}_a = \{J\colon J \text{ a prime ideal and } a \notin J\}.$$

Our embedding φ is given by $\varphi(a) = \mathscr{X}_a$ for $a \in P$. We note that $a \leqslant b$ if and only if $\mathscr{X}_a \subseteq \mathscr{X}_b$ (since P has enough prime ideals), and thus φ is an embedding of partially ordered sets. It remains to show that φ is strong and preserves existent finite suprema.

To show that φ is strong. Let $a_1, \ldots, a_n \in P$. I claim

$$\mathscr{X}_{a_1} \cap \ldots \cap \mathscr{X}_{a_n} = \bigcup \{\mathscr{X}_z\colon z \in P \text{ and } z \leqslant a_1, \ldots, z \leqslant a_n\}.$$

Clearly the left-hand side contains the right. If equality does not hold, there is a prime ideal J containing every $z \leqslant a_1, \ldots, a_n$ but not containing any of $a_1, \ldots, a_n$. This contradicts the definition of primality.

To show that φ preserves existent finite suprema. Suppose $b = a_1 \vee \ldots \vee a_n$ in P. Clearly $\mathscr{X}_b \supseteq \mathscr{X}_{a_1} \cup \ldots \cup \mathscr{X}_{a_n}$. If equality does not hold, there is a prime ideal J such that $a_i \in J$ for each i but $b \notin J$. Since J is an ideal (closed under existent finite suprema) this is impossible. ∎

THEOREM 4.7. *A distributive partially ordered set has enough prime ideals.*

Proof. Let P be a distributive partially ordered set and suppose that $a, b \in P$ and $a \not\leqslant b$. Set

$$\mathscr{Z} = \{I\colon I \text{ is an ideal, } a \in I \text{ and } b \notin I\}.$$

$\mathscr{Z}$ is non-empty (as $(a] \in \mathscr{Z}$) so by Zorn's lemma $\mathscr{Z}$ has maximal elements. Let J be maximal; we claim J is prime. Indeed suppose $(a_1] \cap \ldots \cap (a_n] \subseteq J$ and yet $a_i \notin J$ for each i. Then for each i the ideal $(a_i] \vee J$ (in the lattice $\mathscr{L}\mathscr{I}(P)$) contains b, since J is maximal in $\mathscr{Z}$. Thus $b \in \bigcap_i ((a_i] \vee J)$. But $\mathscr{L}\mathscr{I}(P)$ is distributive (by Theorem 2.7(v)), so

$$\bigcap_i ((a_i] \vee J) = ((a_1] \cap \ldots \cap (a_n]) \vee J = J.$$

Thus $b \in J$, a contradiction. ∎

THEOREM 4.8. *If P is a distributive partially ordered set then there is a strong embedding preserving existent finite suprema of P into a ring of sets.*

Proof. Immediate from Theorems 4.6 and 4.7. ∎

This is the only result we have extending the equivalence of Theorem 4.1 parts (i) and (iii) to arbitrary partially ordered sets. We will show later that the converse of Theorem 4.8 is false, but we now turn our attention to the question of embedding distributive partially ordered sets in distributive lattices.

THEOREM 4.9 *Let P be a $\varkappa$-distributive partially ordered set. There is a strong embedding, preserving existent suprema of $\varkappa$-small sets, of P into a $\varkappa$-distributive complete lattice.*

Proof. The embedding is the canonical embedding $\varphi\colon P \to \mathscr{L}\mathscr{D}_\varkappa(P)$ (given by $\varphi(a) = (a]$). Since P is $\varkappa$-distributive, φ preserves existent suprema of $\varkappa$-small sets, by Theorem 3.5. It remains only to show that φ is strong.

Suppose $a_1, \ldots, a_n \in P$. Since $(a_1], \ldots, (a_n]$ are $\varkappa$-distinguished lower ends, it follows from Lemma 3.3 that

$$(1)\qquad \mathscr{D}_\varkappa((a_1] \cap \ldots \cap (a_n]) = \mathscr{D}_\varkappa((a_1]) \cap \ldots \cap \mathscr{D}_\varkappa((a_n]).$$

Now

$$\begin{aligned}\mathscr{D}_\varkappa((a_1] \cap \ldots \cap (a_n]) &= \mathscr{D}_\varkappa(\{z\colon z \leqslant a_1, \ldots, z \leqslant a_n\})\\ &= \mathscr{D}_\varkappa(\bigcup \{(z]\colon z \leqslant a_1, \ldots, z \leqslant a_n\})\\ &= \bigvee \{(z]\colon z \leqslant a_1, \ldots, z \leqslant a_n\} \text{ in } \mathscr{L}\mathscr{D}_\varkappa(P).\end{aligned}$$

So

$$(2)\qquad \mathscr{D}_\varkappa((a_1] \cap \ldots \cap (a_n]) = \bigvee \{\varphi(z)\colon z \leqslant a_1, \ldots, z \leqslant a_n\}.$$

Also

$$\mathscr{D}_\varkappa((a_1]) \cap \ldots \cap \mathscr{D}_\varkappa((a_n]) = (a_1] \cap \ldots \cap (a_n] = \varphi(a_1) \wedge \ldots \wedge \varphi(a_n).$$

This last result together with (1) and (2) shows that φ is strong. ∎

After Theorem 4.9 it seems reasonable to make the following conjecture.

CONJECTURE. Let P be a partially ordered set, $\varkappa$ either a regular cardinal or ∞. P is $\varkappa$-distributive if and only if there is a strong embedding, preserving existent suprema of $\varkappa$-small sets, of P into a $\varkappa$-distributive complete lattice.

This conjecture is false, because of the following example.

EXAMPLE 4.10. Let λ be a regular cardinal, X a set of cardinality λ, $\mathscr{F}$ a non-principal ultrafilter on X. Let q denote an object not in X. We describe a partially ordered set P. P has underlying set $\mathscr{F} \cup \{\{x\}\colon x \in X\} \cup \{q\}$; the ordering on P is given by the following conditions.

i) $\mathscr{F} \cup \{\{x\}\colon x \in X\}$ is ordered by set inclusion.

ii) $q < X$.
iii) $q > \{x\}$ for each $x \in X$.
iv) q is unrelated to every other element of P.

It is fairly easy to check that P is μ-distributive for every $\mu > \lambda$. However P is not λ-distributive. For let $A \in \mathscr{F}$ be such that $|A| = \lambda$ and $|X - A| = \lambda$, take $b \in X - A$ and set $B = X - \{b\}$. Consider the supremum $A \vee q = X$. If I is the lower end $(\{A, q\}]$ then $\bigvee I = X$ and

$$I = \{F \in \mathscr{F}\colon F \subseteq A\} \cup \{\{x\}\colon x \in X\} \cup \{q\}.$$

Now $B \leqslant X$ but X is not the supremum in P of any λ-small subset of I. ∎

THEOREM 4.11. *The conjecture is true for at most one value of* $\varkappa$.

Proof. Suppose the Conjecture is true for $\varkappa = \lambda$ and for $\varkappa = \mu$, with $\lambda < \mu$. Consider the set P of Example 4.10. P is μ-distributive, so there is a strong embedding, preserving existent suprema of μ-small sets, of P into a μ-distributive lattice. A fortiori there is a strong embedding, preserving existent suprema of λ-small sets, of P into a λ-distributive lattice. Since P is not λ-distributive, the conjecture fails for $\varkappa = \lambda$. ∎

As well as destroying the Conjecture, Example 4.10 can be used to show that the converse of Theorem 4.8 is false, as follows.

THEOREM 4.12. *There is a partially ordered set P which is not distributive, but such that there is a strong embedding preserving existent finite suprema of P into a ring of sets.*

Proof. We take P to be the set of Example 4.10, with $\lambda = \aleph_0$, and we use the terminology of that example. P is not distributive. However it is readily verified that $(q]$ is a prime ideal and also, if we set $Y_x = X - \{x\}$ for $x \in X$, that $(Y_x]$ is a prime ideal. It now follows that P has enough prime ideals (in the sense of Definition 4.5), and so by Theorem 4.6 there is a strong embedding preserving existent finite suprema of P into a ring of sets. ∎

We now return to the Conjecture. We know that it holds for at most one value of $\varkappa$, and in fact it is true for $\varkappa = \infty$.

THEOREM 4.13. *A partially ordered set P is ∞-distributive if and only if there is a strong embedding preserving all existent suprema of P into a locale.*

Proof. After Theorem 4.9 it suffices to prove that if there is a strong embedding φ of P into a locale L such that φ preserves existent suprema, then P is ∞-distributive. Suppose then that $A \subseteq P$, that $\bigvee A$ exists in P and equals x, and that $b \leqslant x$. We have to show that in P $\bigvee(A \mid b) = b$, that is if $d \in P$ and $d \geqslant c$ for every $c \in A \mid b$, then $d \geqslant b$. Now in L

$$\varphi(b) = \varphi(b) \wedge \varphi(x) = \varphi(b) \wedge \big(\bigvee_{a \in A} \varphi(a)\big),$$

the last equality holding because φ preserves existent suprema. Since L is a locale we then have

$$\varphi(b) = \bigvee_{a \in A} (\varphi(b) \wedge \varphi(a)). \tag{1}$$

Now since φ is strong, $\varphi(b) \wedge \varphi(a) = \bigvee\{\varphi(c)\colon c \leqslant a, b\}$. Thus

$$\bigvee_{a \in A} (\varphi(b) \wedge \varphi(a)) = \bigvee_{a \in A} \bigvee_{c \leqslant a, b} \varphi(c) = \bigvee_{c \in A \mid b} \varphi(c).$$

This together with (1) gives us

$$\varphi(b) = \bigvee_{c \in A \mid b} \varphi(c). \tag{2}$$

Since $d \geqslant c$ for every $c \in A \mid b$, $\varphi(d) \geqslant \bigvee_{c \in A \mid b} \varphi(c)$. From (2) $\varphi(d) \geqslant \varphi(b)$, so since φ is an embedding $d \geqslant b$ as required. ∎

It is easily checked that the Conjecture is true (for all values of $\varkappa$) for semilattices, so when we move from semilattices to partially ordered sets the situation with respect to the Conjecture changes completely.

We close this section by remarking that for *finite* partially ordered sets the concepts "distributive" and "∞-distributive" coincide, so we obtain the following result.

THEOREM 4.14. *The following conditions on a finite partially ordered set P are equivalent.*

i) *P is distributive.*

ii) *There is a strong embedding preserving existent finite suprema of P into a distributive lattice.*

iii) *There is a strong embedding preserving existent finite suprema of P into a ring of sets.*

Proof. We show i)$\Rightarrow$iii)$\Rightarrow$ii)$\Rightarrow$i). Theorem 4.8 shows that i)$\Rightarrow$iii) holds, while iii)$\Rightarrow$ii) is trivial. The proof of Theorem 4.13 shows that ii)$\Rightarrow$i); since P is finite the various suprema appearing in the proof are all over finite sets, so the locale L in the proof can be replaced by a mere distributive lattice. ∎

Thus we recover Theorem 4.1 for finite partially ordered sets.

5. Discussion. The purpose of this paper has been to introduce a new notion, that of $\varkappa$-distributive partially ordered set. We think that the results of Sections 2 and 3 show that this notion is worth investigation. However in Section 4 we saw that there are properties of $\varkappa$-distributive semilattices which do not extend to arbitrary $\varkappa$-distributive partially ordered sets. It seems to us that the important new occurrence in the move from $\varkappa$-distributive semilattices to $\varkappa$-distributive partially ordered sets is the possibility of partially ordered sets like Example 4.10 (where μ-distributivity does not imply λ-distributivity even though $\mu > \lambda$). It may be that the notion "strong embedding" introduced in Section 4 is not the correct replacement for "semilattice monomorphism", but the reasoning of Theorem 4.11 is quite general, and should go through with other notions of strong embedding. We conclude that Example 4.10 is a fundamental obstacle to finding a single concept of $\varkappa$-distributivity which will extend both Theorem 1.5 and Theorem 4.1 to arbitrary partially ordered sets. As a final remark, we note that if we restrict ourselves to finite partially ordered sets, all the difficulties just mentioned disappear.

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Axiomatization of the forcing relation with an application to Peano Arithmetic

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Abstract. In the paper we describe those formal properties of forcing which set theoretical forcing and the method of indicators in Peano Arithmetic have in common.

Introduction. We describe a formal similarity between forcing in set theory and the indicator method in Peano Arithmetic.

Let $\boldsymbol{P}$ be a set of forcing conditions, $\langle \mathscr{Y}, \mathcal{O}\rangle$ a topological space in $V^{\boldsymbol{P}}$ which is V-codable, i.e. such that there is a set $\langle 0, \leqslant\rangle$ and an isomorphism φ in $V^{\boldsymbol{P}}$ of $\langle 0, \leqslant\rangle$ and $\langle \mathcal{O}, \subseteq\rangle$ such that the relation "$y \in \varphi(\check{q})$" for a $q \in 0$ is an absolute relation of y and q w.r.t. V and $V^{\boldsymbol{P}}$ (see [1]). We shall always assume that $q_1 q_2 \in \mathcal{O} \,\&\, q_1 \cap q_2 \neq \emptyset \Rightarrow q_1 \cap q_2 \in \mathcal{O}$.

Let $y \in V^{\boldsymbol{P}}$ be an element of $\mathscr{Y}$. Let us identify in $V^{\boldsymbol{P}}$ $\langle 0, \leqslant\rangle$ with $\langle \mathcal{O}, \subseteq\rangle$.

We formulate two systems of axioms characterizing, respectively, the following relations:

$$R(p, q) \text{ defined as } p \Vdash (\underline{y} \in \check{q})$$

and

$$R'(p, q) \text{ defined as } p \wedge \|\underline{y} \in \check{q}\| \neq \emptyset.$$

Then R satisfies the first system of axioms iff there is a $\underline{y} \in V^{\boldsymbol{P}}$ such that R is the relation $p \Vdash (\underline{y} \in \check{q})$ and R' satisfies the second system iff there is a $\underline{y} \in V^{\boldsymbol{P}}$ such that R' is the relation $p \wedge \|\underline{y} \in \check{q}\| \neq \emptyset$. We call relations of the first type *forcing relations* and those of the second type *consistency relations*.

We show that the Kirby–Paris indicator for models of PA defined by means of a game where questions are Gödel numbers of formulas naturally determines a relation satisfying the second group of axioms, i.e., a consistency relation.

We also show that there is a strict correspondence between consistency and forcing relations.

A consistency relation canonically determines a forcing relation and conversely. Thus the Kirby–Paris indicator determines a forcing relation. This explains certain analogies between the forcing and the indicator method.

The forcing relation determined by the indicator is not definable within