[3] J. Dudek, *Medial groupoids and Mersenne numbers*, Fund. Math. 114 (1981), pp. 235–256.
[4] — *On binary polynomials in idempotent commutative groupoids*, Fund. Math. 120 (1983), pp. 165–169.
[5] —, *Polynomials in idempotent commutative groupoids*, Dissertationes Math. (to appear).
[6] G. Grätzer, *Universal Algebra*, Springer-Verlag, 1979.
[7] — and J. Płonka, *On the number of polynomials of an idempotent algebra* I, Pacific J. Math. 32 (1970), pp. 697–709.
[8] E. Marczewski, *Independence and homomorphism in abstract algebras*, Fund. Math. 50 (1961), pp. 45–61.
[9] R. Padmanabhan, *On axioms for semilattices*, Canad. Math. Bull. 9 (1966), pp. 357–358.
[10] J. Płonka, *On equational classes of abstract algebras defined by regular equations*, Fund. Math. 64 (1969), pp. 241–247.

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On elementary cuts in recursively saturated models of Peano Arithmetic

by

Henryk Kotlarski (Warszawa)

Abstract. If M is a model of Peano Arithmetic, let $Y = \{N \subsetneq_e M\colon N \prec M\}$; we study this family under the assumption that M is countable and recursively saturated.

§ 1. Introduction and notation. Let PA denote Peano Arithmetic in any of its usual formalizations. For $M \models \mathrm{PA}$ we set $Y^M = \{N \subsetneq_e M\colon N \prec M\}$; when no confusion arises we omit the superscript M. Clearly the properties of this family depend on M; we shall study this family under the assumption that M is countable and recursively saturated. In § 2 we show that M has many cuts which have some combinatorial properties introduced by Kirby and Paris (see [1]), in § 3 we show many non-isomorphic elements of Y and in § 4 we study the connection between elementary cuts of M and automorphisms of M.

We use standard terminology and notation. We assume that the reader knows the notion of recursive saturation (see Schlipf [9] and Smorynski [10] for a survey of recursively saturated models of PA) and knows the notion of a satisfaction class studied in some depth by Krajewski [6] and in several more recent papers; also some knowledge of initial segments (= cuts) in models of PA (see e.g. Kirby [3]) is required (however, we shall define the combinatorial properties of cuts in the body of the paper). The present paper has grown out from our earlier paper [4], where Theorem 1.1 below was proved. The results of [4] and the present paper were annouced in abstract [5].

Before we state the main result of [4], we need some more notation:

$$Y_1 = \{N \in Y\colon N \text{ is not recursively saturated}\}.$$

For $a \in M$ we denote $M(a) = \{x \in M\colon$ for some parameter-free term $t(v)\ M \models x < t(a)\}$.

The following notion is taken from [1]. Two families A, B of cuts of $M \models \mathrm{PA}$ are symbiotic iff, for all $a, b \in M$

$$(\exists N \in A\ a < N < b) \equiv (\exists N \in B\ a < N < b).$$

THEOREM 1.1. *Let $M \models$ PA be countable and recursively saturated. Then*

(i) *if $A \subseteq Y$ has no greatest element, then $\bigcup A \in Y \setminus Y_1$,*

(ii) *for $N \in Y$ we have $N \in Y_1$ iff there exists an $a \in M$ such that $N = M(a)$,*

(iii) *Y and Y_1 are symbiotic,*

(iv) *Y_1 is of the order type of $1+$rationals,*

(v) *Y is of the order type of Cantor set 2^ω with its usual ordering: for $b^1, b^2 \in 2^\omega$ we put*

$$b^1 < b^2 \equiv \exists m (\forall n < m\, b_n^1 = b_n^2) \;\&\; b_m^1 = 0 \;\&\; b_m^2 = 1.$$

Proof. See [4]. ∎

§ 2. Combinational properties of elementary cuts. In this section we show that if M is countable and recursively saturated then M has many elementary cuts having some combinatorial properties introduced by Kirby and Paris [1], [2]. Let us recall these properties.

DEFINITION. Let $M \models$ PA and let $N \subsetneq_e M$.

(i) *ω codes N* iff there exists a function $f \in M$ such that all standard $n \in M$ are in dom (f) and $\forall x \in M\, x \in N \equiv \exists n \in \omega\, M \models x < f(n)$.

(ii) *$\omega \downarrow$ codes N* iff there exists a function $f \in M$ such that all standard $n \in M$ are in dom (f) and $\forall x \in M\, x \in N \equiv \forall n \in \omega\, M \models x < f(n)$.

(iii) *N is strong in M* iff for every function $f \in M$ such that every $x \in N$ is in dom (f) there exists a $b \in M \setminus N$ such that

$$\forall y \in N f(y) > N \to f(y) > b.$$

This last notion has become of fundamental importance in recent results about sentences independent of PA, cf. [8].

Our first result shows that M has many cuts coded by ω.

THEOREM 2.1. *Let M be countable and recursively saturated. Then*

(i) *for every $N \in Y_1$, ω codes N,*

(ii) *$\{N \in Y\colon N \notin Y_1$ and ω codes $N\}$ is symbiotic with Y.*

This result may be shown by constructing some recursive types; it will be more convenient to prove this by using a non-standard satisfaction class to get some uniformity of further arguments. The following fact has been known to several investigators.

LEMMA 2.2. *Let $M \models$ PA be countable and non-standard. Then M is recursively saturated iff there exists a satisfaction class S on M such that $(M, S) \models$ induction and S decides all Σ_x formulas for some non-standard $x \in M$.*

Proof. $\to$ Consider the theory:

"S is a satisfaction class" + induction in $L \cup \{S\}$ + "S decides all Σ_x formulas" + $\{x > n\colon n \in \omega\}$.

This is a theory in $L \cup \{S\} \cup \{x\}$ which is recursive and consistent with the complete diagram of M (because any of its finite subsets can be interpreted by Tr_n — the truth definition for Σ_n — formulae, provided n is sufficiently large), so the result follows from the resplendency of M (cf. [9]).

$\leftarrow$ Let p be a recursive and consistent type in a variable v and parameter $b \in M$. Let Γ represent p in PA. Now we have

$$(M, S) \models \exists v \forall \varphi < n\, \Gamma(\varphi) \to S(\varphi, v^\frown b)$$

for all $n \in \omega$, and so by overspill

$$(M, S) \models \exists v \forall \varphi < n_0\, \Gamma(\varphi) \to S(\varphi, v^\frown b)$$

for some $n_0 > \omega$.

Such v realizes p because its elements are standard formulas, and so smaller than n_0. ∎

From now on we shall write "a sat.cl." instead of "a satisfaction class with the properties stated in Lemma 2.2".

Proof of Theorem 2.1. (i) Let $N \in Y_1$. By Theorem 1.1 (ii), $N = M(a)$ for some $a \in M$. Let S be a sat. class on M. Let $\{t_i(v)\colon i \in \omega\}$ be a recursive enumeration of parameter-free terms of L. We define a function g by induction in (M, S):

$$g(0) = \text{the value of } t_0 \text{ on } a,$$
$$g(i+1) = \max\big(g(i)+1, \text{ the value of } t_{i+1} \text{ on } a\big).$$

The satisfaction class S is used to define the values of terms in (M, S). It is obvious that g has the required properties.

(ii) Let $a, b \in M$ be given. Assume that $\exists N \in Y$, $a < N < b$. In particular, $a < M(a) < b$. Once again let S be a sat.cl. on M. We define a function $g(v_1, v_2, v_3)$ by induction in (M, S). Let $t_i(v)$ be a recursive enumeration of terms.

$$g(u, w, 0) = \text{the value of } t_0 \text{ on } u.$$

$$g(u, w, i+1) = \text{the smallest } x \text{ such that for all } z \leqslant w, \text{ if } S \text{ decides the formula } t_z(g(u, w, i)) < x \text{ then } S\big(t_z(g(u, w, i)) < x\big).$$

Observe that such an x always exists because x is required to be greater than only finitely many elements of M (finiteness in the sense of M).

Moreover, for all i,

$g(u, w, i) < g(u, w, i+1)$, because S decides all Σ_0 terms.

Now we observe that for all $n \in \omega$

$(M, S) \models g(a, n, n) < b$, because $g(a, n, n) \in M(a)$.

By overspill there exists an $n_0 > \omega$ such that $(M, S) \models g(a, n_0, n_0) < b$. Let $N^1 = \{x \in M\colon$ for some $k \in \omega$, $x \leqslant g(a, n_0, k)\}$. We show that N^1 satisfies our demand. Clearly $a < N^1 < b$. Also it is clear that

$$N^1 = \bigcup_{k\in\omega} M(g(a, n_0, k)),$$

so $N^1 \in Y\setminus Y_1$ by Theorem 1.1 (i). Also the function $f(k) = g(a, n_0, k)$ shows that ω codes N^1. The converse implication is evident. ∎

Now we shall produce many cuts $N\in Y$ such that $\omega\downarrow$ codes N.

Definition. For $a\in M$ let

$$M[a] = \{x\in M\colon \text{for each parameter-free term } t(v),\ M\models t(x) < a\}.$$

For convenience we shall use this symbol only when $M[a]$ is non-empty (i.e., no definable element of M is greater than a); otherwise the symbol $M[a]$ will be treated as undefined.

It is easy to show that the family $\{M[a]\colon a\in M\}$ is symbiotic with Y and each $M[a]$ is recursively saturated (under the assumption that M is recursively saturated).

Theorem 2.3. *Let $M\models$ PA be countable and recursively saturated. Then*

(i) *for every $a\in M$ $\omega\downarrow$ codes $M[a]$,*

(ii) *Y is symbiotic with $\{N\in Y\colon \omega\downarrow$ codes N and N is not of the form $M[a]\}$.*

Proof. This argument is very similar to that of Theorem 2.1, and so we give only a very rough sketch.

(i) Use the function g defined as follows:

$$\begin{aligned} g(0) &= \max x\colon \text{the value of } t_0 \text{ on } x \text{ is } < a,\\ g(i+1) &= \max x\colon \text{for each } j\leqslant i+1, \text{ the values of } t_j \text{ on } x \text{ are smaller than } a. \end{aligned}$$

(ii) Use the following function $g(v_1, v_2, v_3)$:

$$\begin{aligned} g(u, w, 0) &= \text{the greatest } x \text{ such that the value of } t_0 \text{ on } x \text{ is smaller than } u,\\ g(u, w, i+1) &= \max x\colon \text{for all } z\leqslant w \text{ if the formula } t_z(g(u, w, i)) < x \text{ is decided by } S \text{ then } S(t_z(g(u, w, i)) < x). \ ∎ \end{aligned}$$

The following fact is known from Kirby [1].

Proposition 2.4. *We do not have (ω codes N) and ($\omega\downarrow$ codes N) for any $N \subset_e M$.*

It follows that the families considered in Theorems 2.1 and 2.3 (ii) are disjoint.

Our next goal is to show that in a sense the family $\{N\in Y\colon \omega$ codes $N\}$ is much bigger than Y_1; namely, we shall introduce the notion of a stationary family of cuts of M and show that the first of the above families is stationary but the second is not.

From now on we fix a countable recursively saturated model of PA and fix a sat.cl. S on M.

Definition. (i) A function $F\colon M\to M$ *is normal* iff F is definable in (M, S) and is strictly increasing.

(ii) A set $A\subseteq Y$ *is normal* iff, for some normal F,

$$A = \{N\in Y\colon \forall x\in N\, F(x)\in N\}.$$

(iii) A set $B\subseteq Y$ *is stationary* iff, for all normal sets, $A\subseteq Y$, $A\cap B\neq\emptyset$.

The above notions depend not only on M but also on S, and so should be S-normal and S-stationary, but we shall omit S because it will be fixed.

The above terminology is taken from set theory; cuts of M play the role of limit ordinals.

Proposition 2.5. (i) *If A and B are normal in Y then $A\cap B$ contains a normal set.*

(ii) *If A is stationary then A contains arbitrarily large cuts:*

$$\forall a\in M\,\exists N\in A\; a < N < M.$$

Proof. (i) Let

$$A = \{N\in Y\colon \forall x\in N\, F(x)\in N\}$$

and

$$B = \{N\in Y\colon \forall x\in N\, G(x)\in N\}$$

where F and G are normal functions. We define the function H:

$$\begin{aligned} H(0) &= \max(F(0), G(0)),\\ H(i+1) &= \max(1+H(i), F(i), G(i)). \end{aligned}$$

Clearly H is normal and if $N\in Y$ is closed under H then N is closed under both F and G.

(ii) Let $A\subseteq Y$ be stationary. Given $a\in M$, the function $F(x) = a+x$ is normal, and so there is an $N\in A$, $N<M$ such that $\forall x\in N\, a+x\in N$. In particular, $a\in N$. ∎

Theorem 2.6. (i) *Y_1 is not stationary.*

(ii) *$\{N\in Y\colon \omega$ codes $N\}$ is stationary.*

Proof. (i) First we construct a function $F(a, i)$ by induction in (M, S):

$$\begin{aligned} F(a, 0) &= \text{the value of } t_0 \text{ on } a,\\ F(a, i+1) &= \text{the smallest } x \text{ such that for all } j\leqslant i+1 \text{ if } S \text{ decides the formula } t_j(F(a, i)) < x \text{ then } S(t_j(F(a, i)) < x). \end{aligned}$$

Now let

$$G(x) = \begin{cases} F(\langle a, i\rangle) & \text{if } x = \langle a, i\rangle, \\ 0 & \text{if } x \text{ is not a pair} \end{cases}$$

and let H be defined by

$$H(0) = G(0) \quad \text{and} \quad H(i+1) = \max(1+H(i), G(i+1)).$$

Thus H is a normal function. But no $N \in Y_1$ is closed under H because if $i \in N$, $i > \omega$ and $N = M(a)$, then $F(a, i) \notin N$ so $H(\langle a, i\rangle) \notin N$.

(ii) One can construct a Σ_n and Π_n hierarchy for the language $L \cup \{S\}$ as usual and one can also write down the truth definitions Tr_n for Σ_n formulas of the extended language; these formulas have the properties of truth definitions because we work over a model $(M, S) \models$ induction.

Now let a normal function H be given; H is Σ_n in $L \cup \{S\}$ for some n, and the statement $\forall x \exists y\, H(x) = y$ is Π_{n+1} and so it is Σ_{n+2}. Thus it suffices to prove that (M, S) has arbitrarily large Σ_{n+2} – elementary cuts coded by ω. Let us state this as a lemma.

LEMMA 2.7. *Let $a \in M$, $k \in \omega$, $k \geqslant 1$. Then (M, S) has Σ_k – elementary cuts $(N, S \cap N)$ such that $a \in N$, and ω codes N. Moreover, $N \prec M$.*

Proof (cf. Lessan [7]). Let $N_1 = \{x \in M$: there exists a formula $\varphi(v_1, v_2) \in \Sigma_{k+1}$ such that $(M, S) \models \varphi(a, x)\ \&\ \forall y < x \neg \varphi(a, y)\}$. Clearly $(N_1, S \cap N_1)$ is a Σ_{k+1} – elementary submodel of (M, S). Let $N = \{y \in M$: there exists an $x \in N_1$ such that $y \leqslant x\}$. Clearly N is a cut of M.

We claim that $(N, S \cap N)$ is a Σ_k – elementary cut of (M, S) and ω codes N. It is easy to verify that N is closed under the pairing function. Now let a formula $\varphi(v_1, v_2) \in \Sigma_k$ be given, let $z \in N$ and assume that $(M, S) \models \exists v_2 \varphi(z, v_2)$.

Let $c \in N_1$, $z \leqslant c$. Consider the formula $\xi(v_1, v_3)$:

$$\mathrm{seq}(v_1)\ \&\ \mathrm{lh}(v_1)$$
$$\geqslant v_3\ \&\ \forall i < v_3 \big[\big(\exists v_4\, \varphi(i, v_4) \to \varphi(i, (v_1)_i)\big)\ \&\ \big(\neg \exists v_4\, \varphi(i, v_4) \to (v_1)_i = 0\big)\big].$$

(v_1 is the Skolem function for φ with domain v_3).

Now ξ is Σ_{k+1}, and so by Σ_{k+1}-elementarity

$$(N_1, S \cap N_1) \models \forall v_3 \exists v_1\, \xi(v_1, v_3)$$

because

$$(M, S) \models \forall v_3 \exists v_1\, \xi(v_1, v_3).$$

In particular, $(N_1, S \cap N_1) \models \exists v_1\, \xi(v_1, c)$. Let $b \in N_1$ be such that $(N_1, S \cap \cap N_1) \models \xi(b, c)$. But now $(b)_z \in N$ because $(b)_z < b$ and $(M, S) \models \varphi(z, (b)_z)$. Now we show that ω codes N. The idea is the same as in the proof of 2.1 (i),

but we use Tr_{k+1} instead of S. To be more specific, let $\{t_i\}$ be an enumeration of Σ_{k+1} terms of $L \cup \{S\}$ and we define g by $g(0) =$ the value of t_0 on a,

$$g(n+1) = \max(1+g(n), \text{the value of } t_{n+1} \text{ on } a).$$

Once again, Tr_{k+1} is used to define the value of the Σ_{k+1} term t on a. Clearly g is coded in M; in fact, it is defined in (M, S). Also $\mathrm{rng}(g)$ is cofinal in N_1, and hence N as well.

Thus it remains to show that $N \prec M$. In fact, the statement "S is a satisfaction class" is Π_2 in $L \cup \{S\}$, and so it is preserved since by assumption $k \geqslant 1$. Thus $S \cap N$ is a satisfaction class on N. But all satisfaction classes coincide on standard formulas, and so for $a_1, \ldots, a_l \in N$ and standard $\varphi \in L$ we have

$$\begin{aligned} M \models \varphi[a_1, \ldots, a_l] &\equiv \langle \ulcorner \varphi \urcorner, \langle a_1, \ldots, a_l\rangle\rangle \in S \\ &\equiv \langle \ulcorner \varphi \urcorner, \langle a_1, \ldots, a_l\rangle\rangle \in S \cap N \\ &\equiv N \models \varphi[a_1, \ldots, a_l]. \ \blacksquare \end{aligned}$$

Now we shall prove that there are many strong cuts in Y.

THEOREM 2.8. *Let $Z = \{N \in Y$: N is strong in $M\}$. Then*

(i) *Z is stationary,*

(ii) *Z is symbiotic with Y,*

(iii) *Z is of cardinality $2^{\aleph_0}$.*

Proof. We give only a sketch because the proof is long and uses ideas which are well known from Kirby [1] and Paris [8]. One first uses the construction of Kirby and Paris [2] to obtain a model $(M_1, S_1) \succ (M, S)$ such that $M \subsetneq_e M_1$ and M is strong in M_1. (Their argument works for (M, S) because this model satisfies induction). The next step is to define indicators Z_k for families $\{N \subsetneq_e M$: $(N, S \cap N) \prec_{\Sigma_k} (M, S)$ and N is strong in $M\}$.

This is done by defining games in which the 1st player asks two sorts of questions:

(a) questions which ensure that the cut produced in the usual manner is strong,

(b) questions about the truth of Σ_k formulas of the language $L \cup \{S\}$. The definition of "2nd player wins" ensures that he produces a satisfaction class for Σ_k formulas of $L \cup \{S\}$ and that he gets no contradiction by means of the combinatorial part of his answers. Given indicators Z_k, one shows (ii) exactly as in other constructions using indicators, (i) is proved exactly as in the proof of 2.7; (iii) is obtained by means of the usual trick of splitting strategies. $\blacksquare$

§ 3. **Isomorphisms of elementary cuts.** The following observation is commonly known.

THEOREM 3.1. *Let $M \models \mathrm{PA}$, be countable and recursively saturated, and let $N_1, N_2 \in Y \backslash Y_1$. Then N_1 is isomorphic to N_2.*

Proof. See e.g. Smorynski [10]. ∎

The question if all cuts $N \in Y_1$ are isomorphic has been posed by Roman Kossak; the aim of this section is to answer this question negatively. For $a \in M$ we denote $H(a) =$ the Skolem closure of $\{a\}$.

We say that $a \in M$ is *minimal* iff $H(a)$ has only two elementary submodels (i.e., $H(0)$ and $H(a)$). The main result of this section is

THEOREM 3.2. *There exist two countable infinite families p_k, q_k, $k \in \omega$, of parameter-free recursive types of* PA *such that*

(i) *if Γ is a finite subset of some p_k then*

$$\mathrm{PA} \vdash \forall b \exists a > b \bigwedge\!\!\bigwedge \Gamma(a);$$

(ii) *if Γ is a finite subset of some q_k then*

$$\mathrm{PA} \vdash \forall b \exists a > b \bigwedge\!\!\bigwedge \Gamma(a);$$

(iii) *for every $M \models \mathrm{PA}$, if a realizes p_k and b realizes p_j in M and $k \neq j$ then no $u \in M(a) \backslash M[a]$ realizes the type of b;*

(iv) *for every $M \models \mathrm{PA}$, if a realizes q_k and b realizes q_j in M and $k \neq j$ then no $u \in M(a) \backslash M[a]$ realizes the type of b;*

(v) *for every $M \models \mathrm{PA}$ if a realizes some q_k in M then a is minimal;*

(vi) *for every $M \models \mathrm{PA}$, if a realizes some p_k in M then no $u \in M(a) \backslash M[a]$ is minimal.*

COROLLARY 3.3. (This result was also obtained by Smorynski [11] by other methods.) *If $M \models \mathrm{PA}$ is countable and recursively saturated then there exists an infinite family $A \subseteq Y_1$ such that if $N_1, N_2 \in A$ then N_1 is not isomorphic with N_2.*

Proof of the corollary. Let $M \models \mathrm{PA}$ be countable and recursively saturated; we choose a_k to be any element realizing p_k in M and choose b_k to be any element realizing q_k in M. Let

$$A = \{M(a_k):\ k \in \omega\} \cup \{M(b_k):\ k \in \omega\}.$$

To show that no distinct $N_1, N_2 \in A$ are isomorphic, if suffices to observe that if g is an isomorphism of $M(c)$ onto $M(d)$ then $g(c) \in M(d) \backslash M[d]$. ∎

Before proving Theorem 3.2 we need some auxiliary facts. For $n \in \omega$, Tr_n denotes the natural truth definition for Σ_n — formulae. We define the following functions F_n in PA:

$$F_n(0) = \text{The Gödel number of the formula } v_2 = v_1 + 1.$$
$$F_n(x+1) = \min y:\ \forall \varphi \leqslant F_n(x)\, \forall u \leqslant F_n(x)\, \varphi \in \Sigma_n \rightarrow \big(\exists w\, \mathrm{Tr}_n(\varphi, u^\frown w) \rightarrow \exists w \leqslant y\, \mathrm{Tr}_n(\varphi, u^\frown w)\big).$$

Thus $F_n(x+1)$ is the maximum of all examples for all Σ_n formulas $\varphi \leqslant F_n(x)$ with all parameters $u \leqslant F_n(x)$.

The simplest properties of the functions F_n are

LEMMA 3.4. (i) $\mathrm{PA} \vdash \forall a\, F_n(a) < F_n(a+1)$,

(ii) *the formula $y = F_n(x)$ is Σ_{n+1},*

(iii) *if t is a Σ_n term then for some a $\mathrm{PA} \vdash \forall b > a\, t(b) < F_n(b)$.*

Proof. Obvious. ∎

Let C_n be Range (F_n); formally $C_n(x)$ is the formula $\exists y\, x = F_n(y)$ of PA.

Let $l_n(x) = \max(C_n \cap < x)$ and $p_n(x) = \min(C_n \cap > x)$. The following lemma is obvious.

LEMMA. 3.5. *The following sentences are provable in* PA:

$$\forall x \exists y \big[C_n(x) \rightarrow \big(l_n(x) = F_n(x)\ \&\ p_n(x) = F_n(x+2)\big)\ \&$$
$$\&\ \neg C_n(x) \rightarrow \big(l_n(x) = F_n(x)\ \&\ p_n(x) = F_n(x+1)\big)\big].\ ∎$$

The main lemma about the functions F_n is

LEMMA 3.6. *Let $M \models \mathrm{PA}$ and let $a \in M$ be greater than any definable element of M and such that, for all $n \in \omega$,*

$$M \models F_{n-1}\big(l_n(a)\big) < a\ \&\ F_{n-1}(a) < p_n(a).$$

Then

$$M(a) \backslash M[a] = \bigcup_n \big(l_n(a),\, p_n(a)\big).$$

Proof. $\supseteq$. Pick $u \in \big(l_n(a), p_n(a)\big)$, i.e., $l_n(a) < u < p_n(a)$. Now $p_n(a)$ is definable from u ($p_n(a)$) is either $\min(C_n \cap > u)$ or $\min(C_n \cap > u+1)$), and so $u \in M(a)$.

The same observation shows that $u \notin M[a]$.

$\subseteq$. Let $u \in M(a) \backslash M[a]$.

Case 1. $u < a$. By the assumption there exists a term $t(v)$ such that $t(u) \geqslant a$ (otherwise $u \in M[a]$); this t is Σ_{n-1} for some n. We claim that $l_n(a) < u$. Indeed, otherwise $u \leqslant l_n(a)$, and so $F_{n-1}(u) \leqslant F_{n-1}\big(l_n(a)\big) < a$. But we have $F_{n-1}(u) > t(u)$ by 3.4 (iii), and so we have a contradiction: $a \leqslant t(u) < F_{n-1}(u) \leqslant F_{n-1}\big(l_n(a)\big) < a$. Thus $l_n(a) < u < a < p_n(a)$ so $u \in \big(l_n(a), p_n(a)\big)$.

Case 2. $a \leqslant u$. There exists a term $t(v)$ such that $u < t(a)$ (otherwise $u \notin M(a)$); this t is Σ_{n-1} for some n. We claim that $u < p_n(a)$. Indeed, otherwise, $p_n(a) \leqslant u$ and we obtain a contradiction: $t(a) < F_{n-1}(a) < p_n(a)$. Thus $l_n(a) < a \leqslant u < p_n(a)$, so $u \in \big(l_n(a), p_n(a)\big)$. ∎

The following lemma shows that the functions F_n increase very fast.

LEMMA 3.7. *For $j, n \in \omega$, $j, n > 1$ we have*

$$\mathrm{PA} \vdash \exists b\, \forall y > b\, F_{n-1}\big(j^{F_{n-1}(F_n(y))}\big) < F_n(y+1).$$

Proof. Let b be the Gödel number of the formula $\varrho(x, w)$: $x = F_{n-1}(j^{F_{n-1}(w)})$. By 3.4 (ii) ϱ is Σ_n, and so the result follows from the definition of F_n. ∎

Proof of Theorem 3.2. We shall first construct the types p_k. Let α_k be the kth prime. Let $\xi_k(x)$ be the formula "x is a power of α_k". We put

$$p_k = \{x > F_n(n)\colon n\in\omega\} \cup \tag{1}$$

$$\cup \{F_{n-1}(l_n(x)) < x \;\&\; F_{n-1}(x) < p_n(x)\colon n\in\omega,\ n>0\} \cup \tag{2}$$

$$\cup \{\exists z\, \xi_k(z) \;\&\; l_n(x) = F_n(z)\colon n\in\omega,\ n>0\} \cup \tag{3}$$

$$\cup \{t(l_n(x)) < l_{n-1}(x) \vee t(l_n(x)) > p_{n-1}(x)\colon \tag{4}$$

$$n\in\omega,\ t \text{ is a parameter-free term}\}.$$

We shall verify that these types satisfy (i), (iii) and (vi) of Theorem 3.2.

We first verify (i). So let a finite $\Gamma \subseteq p_k$ be given. Let $M \models \mathrm{PA}$. As Γ is finite, there exists and $a\in M$ such that every $b > a$ satisfies all formulas of the form $x > F_n(n)$ which occur in Γ. Let $A = \{b\in M\colon a < b\}$. Now let n be the greatest natural number such that some formula of the form (2), (3) or (4) (with some term t) occurs in Γ. Pick any by $b\in A$ and let $z = F_n(\alpha_k^b)$. If we take b to be large enough, we can ensure that in M we have

$$\forall y \leqslant b\, F_{n-1}(\alpha_k^{F_{n-1}(F_n(y))}) < F_n(y+1)$$

by Lemma 3.7.

Consider the elements

$$d_1 = F_{n-1}(\alpha_k^{F_{n-1}(F_n(\alpha_k^b))})$$

$$d_2 = F_{n-1}(\alpha_k^{F_{n-1}(F_n(\alpha_k^{b-1}))})$$

$$\dots\dots\dots\dots\dots$$

$$d_r = F_{n-1}(\alpha_k^{F_{n-1}(F_n(\alpha_k^{b-r}))}).$$

By taking b large enough we may ensure that $d_r > F_{n-1}(F_n(\alpha_k^b))$ for some r which is greater than the number of terms t such that some formula of the form (4) with indices t, n occurs in Γ. Thus, for some $i \leqslant r$, every x such that

$$F_{n-1}(d_i) < x < F_{n-1}(d_i+1)$$

satisfies all formulas of form (2), (3), (4) with index n which occur in Γ. Moreover, $\exists z\, d_i = F_{n-1}(\alpha_k^z)$, and so one can repeat the construction starting with $n-1$ and so on. This shows (i) of 3.2.

Now we prove (iii) of 3.2. Let $M \models \mathrm{PA}$ be given, let a realize p_k in M and let b realize p_j in M, where $k \neq j$. Let $u\in M(a)\backslash M[a]$. Now a satisfies all formulas of the form (2), and so by Lemma 3.6 there exists an n such that $l_n(a) < u < p_n(a)$.

By Lemma 3.5, $l_n(u) = l_n(a)$. Thus, in M

$$\exists s\, l_n(u) = F_n(\alpha_k^s) \quad\text{and}\quad \exists z\, l_n(b) = F_n(\alpha_j^z).$$

But F_n is one-to-one by Lemma 3.4 (i), and so u does not realize p_j.

Now we verify (vi) of Theorem 3.2. Let b realize p_k in M. Pick $u\in M(b)\backslash M[b]$. Once again, for some n $l_n(b) = l_n(u) < u$, and so, for every term $t(v)$, either $t(l_n(u)) < l_{n-1}(u)$ or $t(l_n(u)) > p_{n-1}(u)$; thus $H(l_n(b))$ is a proper submodel of $H(u)$.

Our next aim is to define the types q_k. Let $A_n^k(x)$ be the formula

$$\exists z\, lh(z) = n+1 \;\&\; x = k+F_0(z_0) \;\&\; z_0 = k+F_1(z_1) \;\&\; \dots \;\&\; z_{n-1} = k+F_n(z_n).$$

Let

$$\Gamma_k = \{x > F_n(n)\colon n\in\omega\} \cup \{A_n^k\colon n\in\omega\}.$$

The types Γ_k have properties (ii) and (iv) of 3.2; in order to ensure the minimality condition (i.e. (vi)) we use a standard trick of constructing types which give minimal extensions.

Let $\{t_n\colon n\in\omega\}$ be a recursive enumeration of parameter-free terms in one free variable. For a formula $\psi(v)$ and $n\in\omega$, by $\alpha^{\psi,n}(v)$ we denote the formula

$$\{\exists b[Qy(\psi(y) \;\&\; t_n(y) = b) \;\&\; \forall e < b \neg Qy(\psi(y) \;\&\; t_n(y) = e) \;\&\; t_n(v) = b]\} \vee$$

$$\vee \{\forall b \neg Qy(\psi(y) \;\&\; t_n(y) = b) \;\&\; [\exists w(\mathrm{Seq}(w) \;\&\; (w)_0 = \min e\colon \psi(e) \;\&$$

$$\& \forall i < lh(w)(w)_{i+1} = \min e\colon [\psi(e) \;\&\; t_n(e) > (w)_i] \;\&\; \exists i\, v = (w)_i)]\}.$$

Here $Qy\,\delta(y)$ is an abbreviation of $\forall x \exists y > x\, \delta(y)$. Thus $\alpha^{\psi,n}(v)$ expresses that either there exists a b such that $Qx(\psi(x) \;\&\; t_n(x) = b)$ and b is the smallest number with this property and $t_n(v) = b$ or $Qx(\psi(x) \;\&\; t_n(x) = b)$ for no b and v is an element of the natural subset of (the set defined by) ψ on which t_n is strictly increasing.

Now we define

$\beta_0(v)$ is $\alpha^{v=v,0}$,

$\beta_{n+1}(v)$ is $\alpha^{\beta_n,n+1}$.

Let $q_k = \Gamma_k \cup \{\beta_n(x)\colon n\in\omega\}$; we shall verify that these types have the desired properties. We prove first (ii) of Theorem 3.2. Let Δ be a finite subset of some q_k, and let $M \models \mathrm{PA}$.

Let us observe that

$$\mathrm{PA} \vdash Qx\, A_n^k(x) \quad \text{for all } n$$

and

$$\mathrm{PA} \vdash \forall x \big(A_n^k(x) \to A_{n-1}^k(x)\big).$$

Pick the greatest n such that A_n^k is in Δ. Thus unboundedly many x satisfy A_n^k, and so also all formulas of the form A_m^k which occur in Δ.

We may assume that all such x satisfy also all formulas of the form $x > F_n(n)$ which occur in Δ.

We may assume that $\beta_0, \ldots, \beta_n$ are all the other formulas which occur in Δ. By the standard trick used to construct minimal types, $\mathrm{PA} \vdash Qx\psi(x) \to Qx\alpha^{\psi,n}(x)$ for all n.

Let B_r, $r \leqslant m$ be the following sets:

$$B_{-1} = \{x\colon A_n^k(x)\}, \quad B_0 = \alpha^{B_{-1},0}, \quad B_{r+1} = \alpha^{B_r,t_r}.$$

All of them are defined without parameters and unbounded. But every $x \in B_m$ realizes Δ, and so 3.2 (ii) is proved.

Verification of 3.2 (iv) is similar to the case of the types p_k: we give a rough sketch. If b realizes some q_k then let z be a sequence such that

$$b = k + F_0(z_0) \;\&\; z_0 = k + F_1(z_1) \;\&\; \ldots$$

Now one verifies that

$$l_n(b) = z_n = F_{n+1}(k + z_{n+1}) \quad \text{and} \quad p_n(b) = F_{n+1}(k+1+z_{n+1}).$$

Moreover, the inequalities $F_{n-1}\big(l_n(b)\big) < b \;\&\; F_{n-1}(b) < p_n(b)$ hold. Now let b realize q_k and a realize q_j in M, where $k \neq j$. By the above inequalities and Lemma 3.6, for $u \in M(a) \backslash M[a]$ there exists an n such that $l_n(a) < u < p_n(a)$, and so in M

$$\exists w\, l_n(u) = F_n\big(j + F_{n+1}(w)\big).$$

But b does not satisfy this formula as $k \neq j$. Thus it remains to show 3.2 (v). Let $b \in M$ realize some q_k. Let $e = t(b)$ be any element of $H(b)$.

By construction, either e is definable without parameters or e is the value of a term which is strictly increasing (and so one-to-one) on (the set defined by) some formula $\psi \in q_k$; thus there is a term $s(v)$ such that

$$M \models \psi(x) \to \big(\forall y\, s(y) = x \equiv t(x) = y\big).$$

In the first case $e \in H(0)$, in the second $b \in H(e)$, and so $H(b)$ has only trivial elementary submodels. Thus Theorem 3.2 is proved. ∎

§ 4. Automorphisms and elementary cuts. For any model M we denote by Aut (M) the group of all automorphisms of M. The following notion is taken from the Galois Theory. $X \subseteq M$ is closed iff for each $b \in M \backslash X$ there exists a $g \in \mathrm{Aut}(M)$ such that $g(b) \neq b$ and, for all $x \in X$, $g(x) = x$.

Clearly if $M \models \mathrm{PA}$ and $X \subseteq M$ is closed then X is (the universe of) an elementary submodel of M.

The aim of this section is to give some information about closed elementary cuts. The results which we have in mind are the following.

THEOREM 4.1. *If M is countable and recursively saturated and $N \in Y$ is not closed then there exists a $b \in M$ such that $N = M[b]$. It follows that all $N \in Y$ except countably many are closed. The natural question if models of the form $M[b]$ are closed is settled in the following way.*

THEOREM 4.2. *There exists a recursive consistent type q such that, for every $M \models \mathrm{PA}$ and every $b \in M$ which realizes q, $M[b]$ is not closed.*

THEOREM 4.3. *There exists a recursive consistent type p such that, for every countable and recursively saturated $M \models \mathrm{PA}$ and every b realizing p in M, $M[b]$ is closed.*

We first prove Theorem 4.2. Let q be the type q_0 defined in the proof of Theorem 3.2, (in fact, the type Γ_0 suffices here). Consistency of q was proved in § 3; so let $M \models \mathrm{PA}$ and let b realize q in M, we show that $M[b]$, is not closed.

CLAIM. *For each $c \in M(b) \backslash M[b]$, if c and b realize the same parameter-free type, then $c = b$.*

Indeed, if $c \in M(b) \backslash M[b]$ then, for some n, $l_n(b) < c < p_n(b)$ and $l_n(b) = F_{n+1}(z_{n+1}) = l_n(c)$.

Now if $b \neq c$ then c does not satisfy the formula $x = F_0 \circ F_1 \circ \ldots \circ F_n\big(l_n(x)\big)$ but b satisfies this formula, and so the claim is proved.

Theorem 4.2 follows from the above claim, because if $g \in \mathrm{Aut}(M)$ is such that $\forall x \in M[b]\, g(x) = x$ then $g(b) \in M(b) \backslash M[b]$, and so $g(b) = b$ by the claim. ∎

We define the term $t(v) = \max\{y\colon \forall z \subseteq [0, y]\; z < v\}$ (we freely use the Σ_1 formula $\in$ in PA; this gives the notion of inclusion; thus $t(v)$ is the greatest y such that (codes of) all subsets of $\{x\colon x \leqslant y\}$ are smaller then v). We also put $t^0(v) = v$ and $t^{n+1}(v) = t\big(t^n(v)\big)$. Clearly, for each n, t^n is Σ_1.

LEMMA 4.4. (A similar result for models of Alternative Set Theory has also been obtained by Alena Vencovska in Prague). *Let $M \models \mathrm{PA}$ be countable and recursively saturated. Let $a, b, c, d \in M$ be such that*

(i) *$M \models t^n(b) > a$ for all n,*

(ii) *$M \models \forall x < b\;\; \varphi(x, c) \equiv \varphi(x, d)$ for all formulas φ.*

Then there exists an automorphism g of M such that $g(c) = d$ and, for all $x < a$, $g(x) = x$.

Proof. Let S be a sat.cl. on M (cf. Lemma 2.2). We claim that, for every $n \in \omega$ and every two finite sequences $\bar{k}$, $\bar{l}$ of elements of M, if

$$\max\{r\colon \forall x < t^n(b)\, \forall\varphi < r\, S(\varphi, x^\frown c^\frown \bar{k}) \equiv S(\varphi, x^\frown d^\frown \bar{l})\}$$

is non-standard, then for each $e \in M$ there exists an $f \in M$ such that

$$\max\{r\colon \forall x < t^{n+1}(b)\, \forall\varphi < r\, S(\varphi, x^\frown c^\frown \bar{k}^\frown e) \equiv S(\varphi, x^\frown d^\frown \bar{l}^\frown f)\}$$

is non-standard. Indeed, let $\bar{k}, \bar{l}$ satisfy the assumption and let $e \in M$. Assume that, for every $f \in M$,

$$\max\{r\colon \forall x < t^{n+1}(b)\, \forall \varphi < r\, S(\varphi, x^\frown c^\frown \bar{k}^\frown e) \equiv S(\varphi, x^\frown d^\frown \bar{l}^\frown f)\}$$

is standard. It follows that

$$\max\{r\colon \exists f\, \forall x < t^{n+1}(b)\, \forall \varphi < r\, S(\varphi, x^\frown c^\frown \bar{k}^\frown e) \equiv S(\varphi, x^\frown c^\frown \bar{l}^\frown f)\} = r_0$$

is standard. Let $\varphi_0, \ldots, \varphi_m$ be all formulas $\leqslant r_0$. Thus we have

$$M \models \forall f\, \exists x < t^{n+1}(b) \bigvee_{i=1}^{m} \neg [\varphi_i(x, c, \bar{k}, e) \equiv \varphi_i(x, d, \bar{l}, f)].$$

For $i = 1, \ldots, m$ we define in M

$$w_i = \{x < t^{n+1}(b)\colon \varphi_i(x, c, \bar{k}, e)\}.$$

Obviously $M \models w_i < t^n(b)$. But now we have

$$M \models \exists z\, \forall x < t^{n+1}(b) \bigwedge_{i=1}^{m} [\varphi_i(x, c, \bar{k}, z) \equiv x \in w_i]$$

namely $z = e$ has this property, but

$$M \models \forall f\, \exists x < t^{n+1}(b) \bigvee_{i=1}^{m} \neg [\varphi_i(x, d, \bar{l}, f) \equiv x \in w_i].$$

Thus we have distinguished the sequence $c, \bar{k}$ from the sequence $d, \bar{l}$ by means of a standard formula with parameters w_i and $t^{n+1}(b) < t^n(b)$ which contradicts the assumption of the claim, and so the claim is proved.

Clearly the claim allows us to construct an automorphism g of M as desired by the standard back and forth argument. ∎

LEMMA 4.5. *If M is countable and recursively saturated and $N \in Y$ is not closed then $\omega\!\downarrow$ codes N.*

We first derive Theorem 4.1 from Lemma 4.5. Let $b \in M \setminus N$ be such that, for every $g \in \mathrm{Aut}(M)$, if $\forall x \in N\, g(x) = x$ then $g(b) = b$; we show that $N = M[b]$. The inclusion $\subseteq$ is obvious, and so assume that $N \subsetneq M[b]$. By Theorem 1.1 there exists an $N_1 \in Y$ such that $N \subsetneq N_1 \subsetneq M[b]$ (cuts of the form $M[b]$ are unions of smaller elementary cuts). Pick $c \in N_1 \setminus N$ and consider the cut $M(c)$. By Theorem 2.1 ω codes $M(c)$, and so it is not true that $\omega\!\downarrow$ codes $M(c)$; by Lemma 4.5 $M(c)$ is closed, i.e., there exists a $g \in \mathrm{Aut}(M)$ such that $g(b) \neq b$ and, for all $a \in M(c)$, $g(a) = a$. But $N \subseteq M(c)$, and so this g is an identity on N. We have got a contradiction and Theorem 4.1 follows. ∎

Proof of Lemma 4.5. Let $N \in Y$ be such that it is not true that $\omega\!\downarrow$ codes N; let $b \in M \setminus N$ be given. We shall find a $g \in \mathrm{Aut}(M)$ such that $g(b) \neq b$ and, for all $a \in N$, $g(a) = a$.

We claim first that for each $n \in \omega$ and $u \in N$

$$(M, S) \models \exists c \neq b \quad \forall x < u \quad \forall \varphi < n \quad S(\varphi, x^\frown b) \equiv S(\varphi, x^\frown c),$$

where S is a sat.cl. on M.

Assume that the claim does not hold, let $n \in \omega$ and $u \in N$ be such that

$$(M, S) \models \forall c \neq b\, \exists x < u\, \exists \varphi < n\, \neg [S(\varphi, x^\frown b) \equiv S(\varphi, x^\frown c)].$$

Exactly as in the proof of Lemma 4.4, let $\varphi_1, \ldots, \varphi_m$ be all formulas $\leqslant n$; thus

$$M \models \forall c \neq b\, \exists x < u \bigvee_{i=1}^{m} \neg [\varphi_i(x, b) \equiv \varphi_i(x, c)].$$

Let $w_i = \{x < u\colon \varphi_i(x, b)\}$. Clearly $w_i \in N$ because N is an elementary cut of M. But now b is defined in M with parameters from N; namely, the formula $\bigwedge_{i=1}^{m} \forall x < u\, [\varphi_i(x, z) \equiv x \in w_i]$ defines b and has parameters $u, w_1, \ldots, w_m \in N$. We have a contradiction: $b \in N$ because $N \prec M$, and so the claim follows.

For $n \in \omega$ we define

$$u_n = \max\{u\colon \exists c \neq b\, \forall x < u\, \forall \varphi < n\, S(\varphi, x^\frown b) \equiv S(\varphi, x^\frown c)\}.$$

By the claim, for each $n \in \omega$, $u_n > N$, and so there exists a $w \in M$ such that $w \notin N$ and, for all $n \in \omega$, $w < u_n$ (otherwise $\omega\!\downarrow$ codes N). We define the sequence w_n as follows: $w_0 = w$ was chosen above and $w_{n+1} = t(w_n)$. Obviously, for all $n \in \omega$, $w_n \notin N$, and so once again by assumption, there exists a $y \in M \setminus N$ such that, for all $n \in \omega$, $y < w_n$.

For these y and w we have

(i) $t^n(w) > y$ and

(ii) there exists a $c \in M$, $c \neq b$ such that $M \models \forall x < w\, \varphi(x, b) \equiv \varphi(x, c)$ for each formula φ because $w < u_n$ for all n.

By Lemma 4.4 there exists a $g \in \mathrm{Aut}(M)$ such that $g(b) = c$ (so $g(b) \neq b$) and $\forall x < w\, g(x) = x$, and so in particular $\forall x \in N\, g(x) = x$ because N is a cut and $w > N$. ∎

Now we prove Theorem 4.3. The idea is similar to that of § 3 but now we need a type with a somewhat stronger property than the types p_k from Theorem 3.2, because we must allow parameters in the construction.

The idea is the following. The type $p(v)$ will ensure that if b realizes p in M then

(i) every $c \in M(b) \setminus M[b]$ satisfies some inequality of the form $l_n(b) < c < p_n(b)$,

(ii) there exists a z such that $l_{n+1}(b) < z < p_{n+1}(b)$ and z realizes the same type as $l_n(b)$; this will allow us to find an automorphism g of M which moves $l_n(b)$, but $\forall x < l_{n+1}(b)\ g(x) = x$.

In this situation it is obvious that for $c \in M$ such that $l_n(b) < c < p_n(b)$

we have $g(c) \neq c$ and $g \upharpoonright M[b]$ is the identity. Now we define the type p. For a natural number n, let $A_n(w)$ be the formula

$$F_n(l_{n+1}(w)) < w \ \& \ F_n(w) < p_{n+1}(w).$$

For a natural number n and a sequence $\varphi_1, \ldots, \varphi_r$ of formulas in two variables, by $B^n_{\varphi_1,\ldots,\varphi_r}$ we denote the formula

$$\exists z \neq l_n(w) \, \forall x < F_n(l_{n+1}(w)) \bigwedge_{j=1}^{r} [\varphi_j(x, z) \equiv \varphi_j(x, l_n(w))].$$

We define

$$p = \{A_n(w)\colon n > 1\} \cup \{B^n_{\varphi_1,\ldots,\varphi_r}\colon n > 1,\ \varphi_1, \ldots, \varphi_r \text{ is a finite sequence of parameter-free formulas in two free variables}\}.$$

Clearly p is a recursive set of formulas. Now we show that for any finite $\Gamma \subseteq p$ we have $\mathrm{PA} \vdash \forall x \exists w > x \bigwedge\!\!\bigwedge \Gamma(w)$.

We need a lemma.

Lemma 4.6. *Let r be a natural number. Then for $n > 2$ there exists a natural number a such that* $\mathrm{PA} \vdash \forall b > a$ "Card$\big[C_{n-1} \cap (F_{n-1}(l_n(b)), \max\{e\colon F_{n-1}(e) < p_n(b)\})\big]$ *is greater than* $2^{r \cdot F_{n-1}(l_n(b))}$".

Intuitively Lemma 4.6 states that between $l_n(b)$ and $p_n(b)$ (in fact, between $F_{n-1}(l_n(b))$ and $\max\{e\colon F_{n-1}(e) < p_n(b)\}$ there are very many values of the function F_{n-1}).

Proof of Lemma 4.6. Consider the formula $\varrho(u, w)$:

$$\begin{aligned} \exists h \{\mathrm{Seq}(h) \ \& \ \mathrm{lh}(h) > 2^{r \cdot F_{n-1}(u)} \ \& \ h(0) > F_{n-1}(u) \ \& \\ \& \big(\forall i < \mathrm{lh}(h) - 1 \ h(i) < h(i+1)\big) \ \& \\ \& \ \big(\forall i < \mathrm{lh}(h) \, C_{n-1}(h(i))\big) \ \& \ w = F_{n-1}\big(h(\mathrm{lh}(h) - 1)\big)\}. \end{aligned}$$

It is easy to verify that ϱ is Σ_n(F_{n-1} is Σ_n by Lemma 3.4, and so C_{n-1} is a Σ_n formula, whence also exponentiation $2^{r \cdot F_{n-1}(u)}$ is Σ_n). Let a be the value of F_n on the Gödel number of ϱ; we show that this a satisfies our demand. Let $b > a$ be given. We apply the definition of F_n to the parameter $u = l_n(b)$ and the Σ_n formula ϱ and find that there exists a $w < p_n(b)$ such that $\varrho(u, w)$ (in fact, if $l_n(b) = F_n(z)$ then $p_n(b) = F_n(z+1)$). For this w there exists an h as above, but then all values of h are in C_{n-1}, all of them are in the interval $\big(F_{n-1}(l_n(b)), \max\{e\colon F_{n-1}(e) < p_n(b)\}\big)$ and there are more than $2^{r \cdot F_{n-1}(l_n(b))}$ such values. ∎

Now we observe that if $\varphi_1, \ldots, \varphi_r$ is a subsequence of $\psi_1, \ldots, \psi_s$, then, for each n, $\mathrm{PA} \vdash B^n \psi_1, \ldots, \psi_s \to B^n_{\varphi_1,\ldots,\varphi_r}$.

Let Γ be any finite subset of p. Pick the greatest n such that some formula of the form $B^n_{\varphi_1,\ldots,\varphi_r}$ is in Γ. By the remark above we may assume that $B^n_{\varphi_1,\ldots,\varphi_r}$ is the only formula of this form with index n which occurs in Γ.

Let any x be given. By Lemma 4.6 if x is sufficiently big there are more than $2^{r \cdot F_n(F_{n+1}(x))}$ elements of $C_n \cap \big(F_n(F_{n+1}(x)), \max\{e\colon F_n(e) < F_{n+1}(x+1)\}\big)$. Thus at least two of them cannot be distinguished by means of r formulas $\varphi_1, \ldots, \varphi_r$ with parameters smaller than $F_n(F_{n+1}(x))$ because there are only $2^{r \cdot F_n(F_{n+1}(x))}$ sets of pairs of the form $\langle$formula, parameter$\rangle$. Let two such elements be x_n and z_n. Thus we see that any b such that $F_n(x_n) < b < F_n(x_n+1)$ satisfies $B^n_{\varphi_1,\ldots,\varphi_r}$. Clearly such a b satisfies, also A_n.

Now we iterate this procedure, i.e., apply it to $n-1$, $n-2$ and so on. This shows $\exists w > x \bigwedge\!\!\bigwedge \Gamma(w)$; in fact, we have shown a non-empty interval of such elements w. Hence Γ is consistent(1). Thus it remains to show that p has the required property, i.e., for any countable and recursively saturated $M \models \mathrm{PA}$ and b realizing p in M, $M[b]$ is closed. So let M, b satisfy the above assumptions. Let $c \notin M[b]$.

We consider two cases: $c \notin M(b)$ and $c \in M(b) \setminus M[b]$.

If $c > M(b)$, then, as $M(b)$ is not of the form $M[x]$, by Theorem 4.1 there exists a $g \in \mathrm{Aut}(M)$ such that $g(c) \neq c$ and for all $x \in M(b)$ $g(x) = x$.

But $M[b] \subseteq M(b)$, and so this g has the desired properties.

If $c \in M(b) \setminus M[b]$ then there exists an n such that $l_n(b) < c < p_n(b)$ because $M \models A_n(b)$.

Now the type $\Delta(z) = \{\forall x < F_n(l_{n+1}(b)) \, \varphi(x, z) \equiv \varphi(x, l_n(b))\colon \varphi$ is a parameter-free formula in two variables$\}$ with one parameter $l_n(b)$ is consistent because b satisfies all formulas of the form $B^n_{\varphi_1,\ldots,\varphi_r}$. By recursive saturation of M, Δ is realized by an element which we denote by z as well as the variable; we claim that there exists a $g \in \mathrm{Aut}(M)$ such that $g(l_n(b)) = z$ and, for all $x < l_{n+1}(b)$ $g(x) = x$. By Lemma 4.4 it suffices to observe that $F_n(l_{n+1}(b)) > t^k(b)$ for all k; this follows from the fact that there exist Σ_1 terms s^k such that $\mathrm{PA} \vdash \exists a \forall b > a \, s^k(t^k(b)) > b$. Hence there exists a $g \in \mathrm{Aut}(M)$ with the properties stated in the claim. This g must move c; indeed, $z < c < p_n(z)$ and the intervals $(l_n(b), p_n(b))$ and $(z, p_n(z))$ are disjoint.

Moreover, $l_{n+1}(b) \notin M[b]$ because $p_{n+1}(b) > b$ and $p_{n+1}(b)$ is definable from $l_{n+1}(b)$ as the next element of C_{n+1}. It follows that $\forall x \in M[b] \, g(x) = x$ and the proof of Theorem 4.3 is finished. ∎

§ 5. Problems and remarks.

5.1. Let $M \models \mathrm{PA}$ be countable and recursively saturated. What is the structure of the lattice of all elementary submodels of M and of the group $\mathrm{Aut}(M)$?

In particular, do they depend on M?

5.2. Conjecture. If $M \models \mathrm{PA}$ is countable and recursively saturated

(1) We may work so freely in PA because, as Γ is finite, $\Gamma \subseteq \Sigma_k$ for some k, and so we may use the truth definition $\mathrm{Tr}_k(\cdot,\cdot)$ to formalize this argument.

then $\{N \in Y\colon \forall a > \omega \exists I < a\, I$ codes $N\}$ is the complement of a set of first category in the Cantor set Y.

5.3. What is the exact distribution of the values of the functions F_n? In particular, is the type $\{x > F_n(n)\colon n \in \omega\} \cup \{C_n(x)\colon n \in \omega\}$ consistent?

5.4. For $X \subseteq M \models \mathrm{PA}$ we define the closure of X in the usual way: $b \in \mathrm{cl}(X)$ iff, for each $g \in \mathrm{Aut}(M)$, if $\forall x \in X\, g(x) = x$ then $g(b) = b$.

Conjecture. There exist two consistent extensions Δ_1, Δ_2 of the type Γ_0 (cf. the proof of Theorem 3.2) such that, for each countable and recursively saturated $M \models \mathrm{PA}$, if b_1 realizes Δ_1 in M then $\mathrm{cl}(M[b]) =$ the Skolem closure of $(M[b] \cup \{b\})$ and if b_2 realizes Δ_2 in M then $\mathrm{cl}(M[b]) \supsetneq$ the Skolem closure of $(M[b] \cup \{b\})$.

5.5. Conjecture (Smorynski [11]). Let $M \models \mathrm{PA}$ be countable and recursively saturated, and let $b_1, b_2 \in M$. If $(M, M[b_1])$ is elementary equivalent to $(M, M[b_2])$ then $M(b_1)$ is isomorphic with $M(b_2)$.

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References

[1] L. Kirby, *Initial segment of models of Arithmetic*, Doctoral thesis, Manchester 1977.
[2] — and J. Paris, *Initial segments of models of Peano axioms*, in: Set theory and hierarchy theory V, Springer Lecture Notes 619 (1977), pp. 211–226.
[3] — K. Mc Aloon and R. Murawski, *Indicators, recursive saturation and expandability*, Fund. Math. 114 (1981), pp. 127–139.
[4] H. Kotlarski, *On elementary cuts in models of arithmetic*, Fund Math. 115 (1983), pp. 27–31.
[5] — *Elementary cuts in models of arithmetic*, abstract, to appear in Bull. Acad. Polon. Sci.
[6] S. Krajewski, *Non-standard satisfaction classes*, in: Set theory and hierarchy theory, Springer Lecture Notes 537 (1976), pp. 125–144.
[7] H. Lessan, *Models of arithmetic*, Doctoral thesis, Manchester 1978.
[8] J. Paris, *Some independence results for Peano Arithmetic*, J. Symb. Logic. 43 (4) (1978), pp. 725–732.
[9] J. Schlipf, *Toward model theory through recursive saturation*, J. Symb. Logic 43 (2) (1978), pp. 183–203.
[10] C. Smorynski, *Recursively saturated non-standard models of arithmetic*, to appear in J. Symb. Logic. 46 (1981), pp. 259–286.
[11] — *Elementary extensions of recursively saturated models of arithmetic*, Notre Dame J. Formal Logic 22, pp. 193–203.

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On locally contractive fixed-point mappings

by

Ljubomir Ćirić (Beograd)

Abstract. Let (M, d) be a metric space and T a selfmapping on M. Suppose that for each $u \in M$ there exists a sphere $S(u, r(u))$ such that $x, y \in S(u, r(u))$ with $x \neq y$ implies $d(Tx, Ty) < d(x, y)$ and $Tx, Ty \in S(v, r(v))$ for some $v \in M$. Furthermore, suppose that $\{T^n x\}$ contains a convergent subsequence for some $x \in M$. Under these assumptions our main result states that the set of fixed or periodic points of T is non-void. This generalizes one result of M. Edelstein for ε-contractive mappings. A fixed point theorem for corresponding mappings on Hausdorff uniform spaces is stated also.

Introduction. Let (M, d) be a metric space and T a selfmapping on M. A mapping T is said to be *locally contractive on M* if for each $u \in M$ there exists a sphere $S(u, r(u)) = \{x\colon d(u, x) < r(u)\}$, $r(u) > 0$, such that $d(Tx, Ty) < d(x, y)$ holds for all $x, y \in S(u, r(u))$ with $x \neq y$. If there exists $\varepsilon > 0$ such that $r(u) \geq \varepsilon$ for all $u \in M$, then T is called *ε-contractive*. M. Edelstein in [3] proved that if $\lim_{i\to\infty} T^{n_i}x = u \in M$ for some $x \in M$, then an ε-contractive mapping has fixed or periodic points. On compact spaces locally contractive mappings are ε-contractive, and therefore have fixed or periodic points. However, M. Edelstein in [3] and S. Naimpally in [4] have constructed examples which show that if M is not compact, then locally contractive mappings may be without fixed or periodic points, even though $\lim_{i\to\infty} T^{n_i}x = u \in M$ for some $x \in M$.

Our aim is to present a subclass of locally contractive mappings which need not be ε-contractive, but still have fixed or periodic points in the case that $\{T^n x\}$ contrains a convergent subsequence for some $x \in M$.

Definition. A mapping T of a metric space M into itself is said to be *well locally contractive* if for each $u \in M$ there exists $S(u, r(u))$ such that $x, y \in S(u, r(u))$ with $x \neq y$ implies

$$d(Tx, Ty) < d(x, y) \quad \text{and} \quad Tx, Ty \in S(v, r(v))$$

for some $v \in M$.

1. Now we shall prove the following result.

Theorem 1. *Let T be a well locally contractive selfmapping on a metric*