Note also that a sequence $\{x_n\}$ in X converges to u iff it converges to u in p-topology for all $p \in \mathscr{P}$, or equivalently, the numbers $p(x_n, u)$ converges to zero for all $p \in \mathscr{P}$. For $x, y \in X$, $x \neq y$ iff there exists some $p \in \mathscr{P}$ such that $p(x, y) > 0$.

Now we shall present an extended form of Theorem 3.

THEOREM 4. *Let X be a Hausdorff topological space and $\mathscr{P}$ a family of pseudo-metrics which generate the topology on X. Let $T\colon X \to X$ be a mapping such that for each $u \in M$ and $p \in \mathscr{P}$ there exists an open sphere $S_p(u, r_p(u))$ such that $x, y \in S_p(u, r_p(u))$ with $p(x, y) > 0$ implies*

$$p(Tx, Ty) < p(x, y) \quad \text{and} \quad Tx, Ty \in S_p(v, r_p(v))$$

for some $v \in X$. If $\lim_{i\to\infty} T^{n_i}x \in M$ for some $x \in M$ and

$$\inf_{n>0} p(T^n x, T^{n+1}x) = 0 \tag{7}$$

holds for every $p \in \mathscr{P}$, then the set of fixed point of T is non-void.

Proof. Let p be any member of $\mathscr{P}$. If in the proof of Theorem 1 we replace $d(x, y)$ by $p(x, y)$ and $r(u)$ by $r_p(u)$, then by (7) we may choose a positive integer m such that $p(T^m x, T^{m+1}x) < \frac{1}{3} r_p(u)$ and (as $\lim_{i\to\infty} T^{n_i}x = u$) $T^m x, T^{m+1}x \in S_p(u, r_p(u))$.

Following arguments given in the proof of Theorem 1 we obtain that $p(Tu, u) = 0$. Since $p \in \mathscr{P}$ was arbitrary, it follows that $p(Tu, u) = 0$ for all $p \in \mathscr{P}$. Therefore, $Tu = u$ and the proof is complete.

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Weak-chainability of tree-like continua and the combinatorial properties of mappings

by

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Abstract. In 1951, R. H. Bing mentioned the question of the existence of atriodic tree-like continua which are not chainable. In 1972, W. T. Ingram constructed an example of an atriodic tree-like continuum with positive span which is not chainable. A. Lelek introduced the notion of weak chainability and characterized it by the property of being a continuous image of a chainable continuum. A. Lelek introduced the concept of span and proved chainable continua have span zero. The question of Ingram's example of 1972 mentioned above being weakly chainable was mentioned by W. T. Ingram in 1976.

We present a theorem in this paper that gives sufficient conditions for a continuum expressed in terms of inverse expansions in finite trees not to be weakly chainable. Since Ingram's example given in 1972 was obtained as an inverse limit on simple triods, our theorem is applied to show that this example is not weakly chainable. The argument given is not span dependent but does, however, depend upon the combinatorial properties of the bonding maps of the inverse system in question.

1. Introduction. In 1972, W. T. Ingram [1] constructed an example of an atriodic tree-like continuum with positive span. This example in [2] answered the question mentioned by R. H. Bing [1] of the existence of atriodic tree-like continua which are not chainable. A. Lelek [4] introduced the notion of weak chainability and characterized it by the property of being a continuous image of a chainable continuum. A. Lelek [5] introduced the concept of span and proved chainable continua have span zero (p. 210). The question of the continuum given in [2] being weakly chainable was mentioned by W. T. Ingram in [3]. In this paper we give a theorem that gives sufficient conditions for a continuum expressed in terms of inverse expansions in finite trees not to be weakly chainable. Since the continuum given in [2] was obtained as an inverse limit on simple triods, our theorem is applied to show that the example given in [2] is not weakly chainable. The argument given is not span dependent. The argument does, however, depend upon the combinatorial properties of the bonding maps of the inverse system in question. The bonding maps between consecutive factor spaces do not necessarily have to be identical for the main theorem in this paper to apply.

This paper makes use of the results given in 1963 by J. Mioduszewski [6] for a compact metric space to be a continuous image of another one expressed in terms of inverse expansions in polyhedra. This paper also makes use of the results given in 1966 by Whittaker [7](1). The theorem used from [7] states that if f is a continuous real valued function that does not change sign, $f(0) = f(1) = 0$, and f consists of a finite number of strictly monotone pieces, then there exist continuous mappings $\varnothing(t)$ and $\psi(t)$ from $[0, 1]$ onto $[0, 1]$ such that

1) $\varnothing(0) = 0$, $\varnothing(1) = 1$, $\psi(0) = 1$, $\psi(1) = 0$, and
2) $f\{\varnothing(t)\} = f\{\psi(t)\}$, for each t in $[0, 1]$.

Throughout this paper if f and g are functions $f \circ g(x)$ and $f[g(x)]$ will be used interchangeably.

Throughout this paper the term mapping refers to continuous function and d denotes the ordinary distance function in the plane. A continuum is said to be weakly chainable if it is the continuous image of a chainable continuum. For inverse limits, the conventions of [2] are used.

The example given in [2] and how it satisfies the hypotheses of Lemmas I and II of section 2 and the main theorem of this paper, Theorem 1 of section 3, is given in sections 4 and 5 at the end of this paper.

2. Results concerning property $W-U$. In this section, certain lemmas are developed which are used to prove Theorem 1 in section 3.

Throughout this section Y_i and Y_j denote finite trees, $\{G_i\}_{i=1}^{p}$, $p \in I^+$, $p > 2$, is a sequence of points lying in Y_i, G_1, G_2 lie in an arc V lying in Y_i such that no point of $\{G_i\}_{i=3}^{p}$ lies in V, and ε is a positive number which is less than $\frac{1}{2}\min\{d(G_i, G_j)\colon i \neq j\}$. Also throughout this section X_i and X_j denote $[0, 1]$, f denotes a mapping of X_j onto X_i, g denotes a mapping of Y_j onto Y_i, $\varnothing_i$ denotes a mapping of X_i onto Y_i, and $\varnothing_j$ denotes a mapping of X_j onto Y_j such that the diagram D defined by

$$\begin{tikzcd} X_i \arrow[d, "\varnothing_i"'] & X_j \arrow[l, "f"'] \arrow[d, "\varnothing_j"] \\ Y_i & Y_j \arrow[l, "g"] \end{tikzcd}$$

is $\frac{1}{2}\varepsilon$-commutative. Also throughout this section $I_1, I_2, \ldots, I_p$ denote mutually exclusive closed intervals lying in Y_i containing $G_1, G_2, \ldots, G_p$, respectively, such that if m and n are in $\{1, 2, \ldots, p\}$, then $d[I_m, I_n] \geqslant \varepsilon$.

Definition 1. Suppose that v is a positive integer greater than two and $\{P_i\}_{i=1}^{v}$ is a finite sequence such that for each i, $1 \leqslant i \leqslant v-1$, $P_i \neq P_{i+1}$. The statement that $\{P_i\}_{i=1}^{v}$ has *property* $W-U$ (property $W-U$ extended) means there is a positive integer k, $1 < k < v$, and two sequences of positive integers $\{a(i)\}_{i=1}^{t}$ and $\{b(i)\}_{i=1}^{t}$ ($\{a(i)\}_{i=0}^{t}$ and $\{b(i)\}_{i=0}^{t}$) where t is a positive integer greater than one, such that each of the following is true:

1. $a(1) = k-1$, $b(1) = k+1$,
(1. $a(1) = k-1$, $b(1) = k+1$, $a(0) = k$, $b(0) = k$,)
2. $|a(i+1)-a(i)| = |b(i+1)-b(i)| = 1$ for $1 \leqslant i \leqslant t-1$,
(2. $|a(i+1)-a(i)| = |b(i+1)-b(i)| = 1$ for $0 \leqslant i \leqslant t-1$,)
3. $P_{a(i)} = P_{b(i)}$ for $1 \leqslant i \leqslant t$, and
(3. $P_{a(i)} = P_{b(i)}$ for $0 \leqslant i \leqslant t$,)
4. $a(i) < b(i)$ for $1 \leqslant i \leqslant t$,
(4. $a(i) < b(i)$ if $a(i) \neq k \neq b(i)$, $0 \leqslant i \leqslant t$, and,)
(5. $a(i) = b(i)$ if and only if $a(i) = b(i) = k$, $0 \leqslant i \leqslant t$.)

If the sequence $\{P_i\}_{i=1}^{v}$ has property $W-U$, (property $W-U$ extended) then P_k is called the *starting point*, $P_{a(t)}$ is called the *left-finishing point*, $P_{b(t)}$ is called the *right-finishing point*, $\{a(i)\}_{i=1}^{t}$ ($\{a(i)\}_{i=0}^{t}$) is called the *left-winding sequence*, and $\{b(i)\}_{i=1}^{t}$ ($\{b(i)\}_{i=0}^{t}$) is called the *right-winding sequence*.

Lemma I. *Suppose I'_1 and I'_2 are mutually exclusive closed intervals lying in Y_j such that Q_2 is in I'_2, $g[I'_1]$ is a subset of I_k and $g[I'_2]$ is a subset of I_h. Then if P_1 and P_2 are in X_j, $f(P_1) = f(P_2)$, $\varnothing_j(P_1)$ is in I'_1, and $\varnothing_j(P_2)$ is in I'_2 then $h = k$, $1 \leqslant h, k \leqslant p$.*

Proof. Suppose $h \neq k$. By hypothesis, $d[\varnothing_i(f(P_1)), g(\varnothing_j(P_1))] < \frac{1}{2}\varepsilon$ and $d[g(\varnothing_j(P_2)), \varnothing_i(f(P_2))] < \frac{1}{2}\varepsilon$. Since $f(P_1) = f(P_2)$, $d[g(\varnothing_j(P_2)), g(\varnothing_j(P_1))] < \varepsilon$. However, $g(\varnothing_j(P_1))$ is in I_k and $g(\varnothing_j(P_2))$ is in I_h which contradicts $d[I_h, I_k] \geqslant \varepsilon$. This completes the proof of Lemma I.

Remark. The statement of Lemma I remains true if X_i and X_j denote any interval of real numbers. Lemma I requires that we assume only $\frac{1}{2}\varepsilon$-commutativity on the set $\{P_1, P_2\}$ with respect to the diagram D.

Definition 2. Suppose $\{P_i\}_{i=1}^{V}$ is a finite sequence such that for each i, $1 \leqslant i \leqslant V-1$, $P_i \neq P_{i+1}$. The statement that $\{P_i\}_{i=1}^{V}$ has *generalized property $W-U$ with respect to two sequences of positive integers* $\{a(i)\}_{i=1}^{t}$ and $\{b(i)\}_{i=1}^{t}$ means that

1) $a(1) = k$, $b(1) = k$, $1 \leqslant k \leqslant V$,
2) $|a(i+1)-a(i)| = |b(i+1)-b(i)| = 1$ for $1 \leqslant i \leqslant t-1$,
3) $P_{a(i)} = P_{b(i)}$ for $1 \leqslant i \leqslant t$,
4) $a(i) \leqslant b(i)$ for $1 \leqslant i \leqslant t$.

If the sequence $\{P_i\}_{i=1}^{V}$ has generalized property $W-U$ with respect to $\{a(i)\}_{i=1}^{t}$ and $\{b(i)\}_{i=1}^{t}$ then P_k is called the *starting point*, $P_{a(t)}$ is called the *L-finishing point*, $P_{b(t)}$ is called the *R-finishing point*, $\{a(i)\}_{i=1}^{t}$ is called the *L-winding sequence*, and $\{b(i)\}_{i=1}^{t}$ is called the *R-winding sequence*.

Lemma II. *Suppose*

1) *$\{Q_i\}_{i=1}^{n}$ is a finite set of points lying in an arc $\overline{DH}$ which is a subset of Y_j, $Q_1 < Q_2 < \ldots < Q_n$, in the order from D to H, $g(Q_i)$ is in $\{G_i\}_{i=1}^{p}$, $1 < i < n$, and $g(Q_i) \neq g(Q_{i+1})$ for $1 < i < n-1$,*

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(1) This theorem was known earlier: T. Homma, Kodei Math. Seminar 1 (1952), pp. 13–16; see also R. Sikorski and K. Zarankiewicz, Fund. Math. 41 (1954), pp. 339–344.

2) $I'_1, I'_2, \ldots, I'_{n-1}$ *and* I'_n *are mutually exclusive closed intervals lying in* $\overline{DH}$ *and containing* $Q_1, Q_2, \ldots, Q_{n-1}$ *and* Q_n, *respectively. Also, if* e *is in* $\{1, 2, \ldots, n\}$ *and* $g(Q_e) = G_u$ *for* $u\{1, 2, \ldots, p\}$ *then* $g[I'_e]$ *is a subset of* I_u, *and if* x *is in* $\overline{DH}$ *and* $g(x)$ *is in* I_u, u *in* $\{1, 2, \ldots, p\}$, *then there exists* v *in* $\{1, 2, \ldots, n\}$ *such that* x *is in* I'_v.

3) There is a finite subset $\{P_i\}_{i=1}^m$ *of* X_j *such that* $m \geqslant 3$, $P_1 < P_2 < \ldots < P_m$, $P_1 = 0$, $P_m = 1$, $0 = f(P_1) = f(P_m)$, *for* $1 \leqslant i \leqslant m-1$, $f(P_i) \neq f(P_{i+1})$, *and* $f^{-1} \circ f[\{P_i\}_{i=1}^m] = \{P_i\}_{i=1}^m$.

4) $\varnothing_j(P_1)$ *is in* I'_1 *and for each* i, $1 \leqslant i \leqslant m$, $\varnothing_j(P_i)$ *is in* $\bigcup_{i=1}^{n} I'_i$.

5) If $\varnothing_j(P_i)$ *is in* I'_v, $1 \leqslant v \leqslant n$, $1 \leqslant i \leqslant m$, *then* $\varnothing_j(P_{i+1})$ *is not in* I'_v.

6) If $\varnothing_j(P_i)$ *is in* I'_v, $1 \leqslant v \leqslant n$, $1 \leqslant i \leqslant m$, x *is in* X_j *such that* $P_i < x < P_{i+1}$ *for* $1 \leqslant i \leqslant m-1$, *and* $\varnothing_j(x)$ *is in* $\bigcup_{t=1}^{n} I'_t$, *then* $\varnothing_j(x)$ *is in* $I'_v \cup I'_w$ *where* $\varnothing_j(P_{i+1})$ *is in* I'_w, $1 \leqslant v, w \leqslant n$, *(note* $w = v+1$ *or* $w = v-1$*).*

Then A) there are two mappings $\varnothing$ *and* ψ *from* $[0, 1]$ *onto* $[0, 1]$ *with* $\varnothing(0) = 0$, $\varnothing(1) = 1$, $\psi(0) = 1$, $\psi(1) = 0$, *an increasing sequence* $\{t_i\}_{i=1}^k$ *of positive numbers with* $t_1 = 0$, $t_k < 1$, *such that the sequence* $\{f(p_i)\}_{i=1}^m$ *has property* $W-U$ *with left-finishing point* $f(P_1)$, *right-finishing point* $f(P_m)$, *starting point* $f(P_s)$, *where* $\varnothing(t_k) = \psi(t_k) = P_s$, *and respectively. left-and right-winding sequences* $\{a(i)\}_{i=1}^k$ *and* $\{b(i)\}_{i=1}^k$ *where* $\varnothing(t_{k-i+1}) = P_{a(i)}$, $\psi(t_{k-i+1}) = P_{b(i)}$ *for each* i, $1 \leqslant i \leqslant k-1$, *and* $1 \leqslant a(i), b(i) \leqslant m$, *and*

B) if $1 \leqslant h \leqslant k-1$ *and* S *is the sequence* $f \circ \varnothing(t_1), \ldots, f \circ \varnothing(t_{k-h+1})$, $f \circ \varnothing(t_{k-h}), \ldots, f \circ \varnothing(t_{k-1})$, $f \circ \varnothing(t_k) = f \circ \psi(t_k)$, $f \circ \psi(t_{k-1}), \ldots, f \circ \psi(t_{k-(h+1)})$, $f \circ \psi(t_{k-h}), \ldots, f \circ \psi(t_2), f \circ \psi(t_1)$, *then the sequence* $\{g(Q_i)\}_{i=1}^r$ *has generalized property* $W-U$ *with L-finishing point* $g(Q_{v(h)})$, *R-finishing point* $g(Q_{u(h)})$, *starting point* $g(Q_{s'})$, *L-winding sequence* $\{a'(i)\}_{i=1}^{h+1}$, *R-winding sequence* $\{b'(i)\}_{i=1}^{h+1}$, *with* $a'(i+1) = v(i)$ *where* $\varnothing_j(P_{a(i)})$ *is in* $I'_{v(i)}$ *and* $b'_{(i+1)} = u(i)$ *where* $\varnothing_j(p_{b(i)})$ *is in* $I'_{u(i)}$, $1 \leqslant i \leqslant h$, *and* $a'(i) = b'(i) = s'$ *where* $\varnothing_j(P_s) = \varnothing_j[\varnothing(t_k)] = \varnothing_j[\psi(t_k)]$ *is in* $I'_{s'}$.

Remark. Condition A) of the conclusion of Lemma II requires that we assume only condition 3) of its hypothesis and that f is a mapping of X_i onto X_j. Furthermore, since Lemma I requires that we assume only $\frac{1}{2}\varepsilon$-commutativity on the set $\{P_1, P_2\}$ with respect to the diagram D, Lemma II requires that we assume only $\frac{1}{2}\varepsilon$-commutativity on the set $\{P_i\}_{i=1}^m$ with respect to the diagram D.

Proof. First, a proof of A) is given. Let F be a piecewise linear map of $[0, 1]$ onto $[0, 1]$ such that for each i, $1 \leqslant i \leqslant m$, $F(P_i) = f(P_i)$ and each local extreme value of F is in $\{f(P_i)\}_{i=1}^m$, and such that $F^{-1} \circ F\{P_i\}_{i=1}^m = \{P_i\}_{i=1}^m$. (That there is such a piecewise linear map F having the property that $F^{-1} \circ F\{P_i\}_{i=1}^m = \{P_i\}_{i=1}^m$ follows from the hypothesis that $f^{-1} \circ f\{P_i\}_{i=1}^m = \{P_i\}_{i=1}^m$.)

By a theorem of J. V. Whittaker [2, Th. 1, p. 1], there exist two mappings $\varnothing$ and ψ of $[0, 1]$ onto $[0, 1]$ such that $\varnothing(0) = 0$, $\varnothing(1) = 1$, $\psi(0) = 1$ and $\psi(1) = 0$, and for each t in $[0, 1]$, $F[\varnothing(t)] = F[\psi(t)]$.

Let $t_E < 1$ be the first number t in $[0, 1]$ such that $\varnothing(t) = \psi(t)$. There exist a positive integer $k \geqslant 2$ and a finite set $T = \{t_1, t_2, \ldots, t_k\}$ of positive numbers such that

1) $t_1 = 0$,

2) t_{n+1} is the first number t in $[0, 1]$ after t_n such that $t \leqslant t_E$, $\varnothing(t)$ is in $\{P_i\}_{i=1}^m$ and $\varnothing(t) \neq \varnothing(t_n)$, $n \leqslant k-1$.

It follows from the properties of $\varnothing$ and ψ that

1) If t is in $[0, 1]$ and $t < t_E$ then $\varnothing(t) < \psi(t)$. Since $F^{-1} \circ F(\{P_i\}_{i=1}^m) = \{P_i\}_{i=1}^m$ and $F(\varnothing(t)) = F(\psi(t))$ for each t in $[0, 1]$, it follows that if $1 \leqslant n \leqslant k$, then $\psi(t_n)$ is in $\{P_i\}_{i=1}^m$.

Since 1) $F(\varnothing(t)) = F(\psi(t))$ for each t in $[0, 1]$, 2) $F(P_i) \neq F(P_{i+1})$ for i in $\{1, 2, \ldots, m\}$, 3) F is a piecewise linear map, and 4) if t is in $[0, 1]$ and $t < t_E$ then $\varnothing(t) < \psi(t)$, we now prove that if $\varnothing(t_E) = \psi(t_E) = x$, then $F(x)$ is a local extrema value of F. Suppose that $F(x)$ is not a local extreme value of F. It follows from the fact that $F(x)$ is not a local extreme value of F and from 2) and 3) above that there is an interval $I \subset [0, 1]$ containing x such that if y_1 and y_2 are two numbers in I then either $F(y_1) < F(y_2)$ or $F(y_2) < F(y_1)$. Since $\varnothing$ and ψ are continuous functions, there is a number t_1 in $(0, 1)$, $t_1 < t_E$, such that both $\varnothing(t_1)$ and $\psi(t_1)$ are in I. Since $t_1 < t_E$, $\varnothing(t_1) < \psi(t)$. Thus, $F[\varnothing(t_1)] < F[\psi(t_1)]$ or $F[\varnothing(t_1)] > F[\psi(t_1)]$. However by 1) above $F[\varnothing(t_1)] = F[\psi(t_1)]$. This contradiction implies that $F(x)$ is a local extreme value of F.

Since by construction the set of all local extreme values of F is a subset of $F\{P_i\}_{i=1}^m$, it then follows from the fact that $F(x)$ is a local extreme value of F that there is an integer s in $\{1, 2, \ldots, m\}$ such that $\varnothing(t_E) = \psi(t_E) = P_s$.

We now prove that $t_k = t_E$. Suppose that $t_k < t_E$. Now $\varnothing(t_k) = \varnothing(t_E)$ because otherwise t_k is not the last number in T. Thus, $\varnothing(t_k) = \varnothing(t_E) = \psi(t_E) = P_s$ where s is in $\{1, 2, \ldots, m\}$. Let g be the positive integer, $1 \leqslant g \leqslant m$, such that $\psi(t_k) = P_g$. Since $\varnothing(t_k) < \psi(t_k)$ and $\{P_i\}_{i=1}^m$ is an increasing sequence, $g > s$. Let t_2 be the last number t in $(0, 1)$ such that $t_k < t < t_E$ and $\psi(t) = P_{s+1}$. Since $F^{-1} \circ F\{P_i\}_{i=1}^m = \{P_i\}_{i=1}^m$, $\varnothing(t_2) = P_q$ for q in $\{1, 2, \ldots, m\}$. Now $q = s$ for otherwise t_k is not the last number in T. Thus, since $F[\psi(t_2)] = F[\varnothing(t_2)]$, it follows that $F(P_{s+1}) = F(P_s)$ and this contradicts the fact that $F(P_i) \neq F(P_{i+1})$ for i in $\{1, 2, \ldots, m\}$. Thus $t_k = t_E$.

From the properties of $\{t_i\}_{i=1}^k$ and the continuity of $\varnothing$ and ψ it follows that

II) If $1 \leqslant n \leqslant k$, $\varnothing(t_n) = P_L$, $\psi(t_n) = P_I$, $\varnothing(t_{n+1}) = P_M$ and $\psi(t_{n+1}) = P_T$ then $|L-M| = 1$ and $|I-T| = 1$, $1 \leqslant L, I, M, T \leqslant m$. We now complete the proof of A) by showing:

III) If $1 \leqslant h \leqslant k-1$, if $\varnothing(t_{k-h}) = P_g$, if $\psi(t_{k-h}) = P_q$ and if $\varnothing(t_k) = \psi(t_k)$

$=P_s$, $1 \leqslant g, q, s \leqslant m$, then the sequence $S' = F(P_{q+1}), \ldots, F(P_g)$, $F(P_{g+1}), \ldots$ $\ldots, F(P_s)$, $F(P_{s+1}), \ldots, F(P_q)$, $F(P_{q+1}), \ldots, F(P_m)$ has property $W-U$ with left-finishing point $F(P_g)$, and right-finishing point $F(P_q)$, left-winding sequence $\{a(i)\}_{i=1}^{n}$, right-winding sequence $\{b(i)\}_{i=1}^{n}$, with each $a(i)$ defined as the positive integer v such that $\varnothing(t_{k-i}) = P_v$ and with each $b(i)$ defined as the positive integer u such that $\psi(t_{k-i}) = P_u$, $1 \leqslant i \leqslant n$.

That III) is true is shown by induction on h. Let $h = 1$. Let the left-winding sequence be $a(1) = s-1$ and the right-winding sequence to be $b(1) = s+1$. It follows by use of I) and II) that $g = a(1)$, $q = b(1)$, and the sequence $f(P_g), f(P_{g+1}), f(P_{g+2})$ has the property of statement III) being satisfied for $h = 1$. Now suppose that $h+1$ is the least positive integer less than or equal n for which III) is not true. Consider the sequence S'. Let $\varnothing(t_{k-(h+1)}) = P_g$, $\varnothing(t_{k-h}) = P_{g'}$, $\varnothing(t_k) = (t_k) = P_s$, $\psi(t_{k-(h+1)}) = P_q$ and $\psi(t_{k-h}) = P_{q'}$. Since III) is true for h, the sequence $F(P_1), \ldots$ $\ldots, F(P_{g'}), \ldots, F(P_s), \ldots, F(P_{q'}), \ldots, F(P_m)$ has property $W-U$ with left-finishing point $F(P_{g'})$, right-finishing point $F(P_{q'})$, starting point $F(P_s)$, left-winding sequence $\{a(i)\}_{i=1}^{h}$, right-winding sequence $\{b(i)\}_{i=1}^{h}$ with each $a(i) = v(i)$ where $\varnothing(t_{k-i}) = P_{v(i)}$, and with each $b(i) = u(i)$ where $\psi(t_{k-i}) = P_{u(i)}$, $1 \leqslant i \leqslant h$. Using II) it follows that $|g-g'| = 1$ and $|q-q'| = 1$. Define $\{a''(i)\}_{i=1}^{h+1}$ such that $g = a''(h+1)$, and for each $1 \leqslant i \leqslant h$, $a''(i) = a(i)$. Define $\{b''(i)\}_{i=1}^{h+1}$ such that $q = b''(h+1)$, and for each i, $1 \leqslant i \leqslant h$, $b''(i) = b(i)$. Since from I), $P_g = \varnothing(t_{k-(h+1)}) < \psi(t_{k-(h+1)}) = P_q$, and since $P_1 < P_2 < \ldots$ $< P_m$, it follows that $g = a''(h+1) < b''(h+1) = q$. Thus, since $F[P_{a(h+1)}] = F[P_{b(h+1)}]$, it follows that $\{a''(i)\}_{i=1}^{h+1}$ and $\{b''(i)\}_{i=1}^{h+1}$ are, respectively, left- and right-winding sequences for the sequence $F\{(P_i)\}_{i=1}^{m}$ to have property $W-U$ with left-finishing point $F(P_g)$, right-finishing point $F(P_q)$ and starting point $F(P_s)$. This involves a contradiction with the definition of $h+1$, and III) follows. Part A) of the conclusion of Lemma II then follows from III) by letting $h = k$ and the fact $\varnothing(t_1) = P_1$, $\psi(t_1) = P_m$, and $F(P_i) = f(P_i)$ for each i in $\{1, 2, \ldots, m\}$.

The Proof of B) is given by induction on h. First let $h = 1$. Case 1 (for $h = 1$). Suppose $\varnothing_j(P_{a(i)})$ is in I'_e and $\varnothing_j(P_{b(i)})$ is in I'_e, $1 \leqslant e \leqslant n$. Then define $a'(1) = b'(1) = s'$, $a'(2) = e = b'(2)$. Thus, the conclusion of B) follows for this case since: $s' = e \pm 1$, $\varnothing(t_{k-1}) = P_{a(1)}$, $\psi(t_{k-1}) = P_{b(1)}$, $\varnothing_j(P_{a(1)})$ is in $I'_{v(1)}$, $\varnothing_j(P_{b(1)})$ is in $I'_{u(1)}$, $Q_{v(1)} = Q_{u(1)}$, and $I'_{v(1)} = I'_{u(1)}$. Case 2 (for $h = 1$). Suppose $\varnothing_j(P_{a(1)})$ is in I'_a, $\varnothing_j(P_{b(1)})$ is in I'_b, $1 \leqslant a, b \leqslant n$, and $a \neq b$. It follows by the use of Lemma I that $g(Q_{v(1)}) = g(Q_{u(1)})$ with $v(1) = a$ and $u(1) = b$. Thus, define $a'(1) = b'(1) = s'$, $a'(2) = a$, and $b'(2) = b$ and the conclusion of part B) follows since for this case $s' = a \pm 1$. $\varnothing(t_{k-1}) = P_{a(1)}$ and $\psi(t_{k-1}) = P_{b(1)}$.

Now suppose that $h+1$ is the least positive integer less than or equal n for which B) is not true. Since B) is true for the positive integer h, $\{g(Q_i)\}_{i=1}^{n}$ has generalized property $W-U$ with L- and R-winding sequences $\{a'(i)\}_{i=1}^{h}$ and $\{b'(i)\}_{i=1}^{h}$ such that the other conditions of the inductive hypothesis follow as in the conclusion of B). Also, $\varnothing_j(P_{a(h+1)})$ is in $I'_{v(h+1)}$ and $\varnothing_j(P_{b(h+1)})$ is in $I'_{u(h+1)}$.

Case 1. Suppose $u(h+1) = v(h+1) = M$, $1 \leqslant M \leqslant n$. Then define $a'(h+1) = M = b'(h+1)$. Since $a'(h+1) = M = b'(h+1)$, using $\{a'(i)\}_{i=1}^{h+1}$ as the L-winding sequence and $\{b'(i)\}_{i=1}^{h+1}$ as the R-winding sequence, it follows that B) is true for $h+1$. This involves a contradiction so that for Case 1 part B) is true

Case 2. Suppose $u(h+1) \neq v(h+1)$.

From part A) of this lemma, it follows that $f(P_{a(h+1)}) = f(P_{b(h+1)})$; therefore, by Lemma I, $g(Q_{v(h+1)}) = g(Q_{u(h+1)})$. Thus, define $a'(h+1) = v(h+1)$ and $b'(h+1) = u(h+1)$. Using $\{a'(i)\}_{i=1}^{h+1}$ and $\{b'(i)\}_{i=1}^{h+1}$ as the L- and R-winding sequences, respectively it follows that B) is true for this case for $h+1$. This involves a contradiction so that for Case 2 part B) is true. Thus, part B) follows. This completes the proof of this lemma.

Lemma III. *Under the hypothesis of Lemma* II *if* a) $\varnothing_j(P_m)$ *is in* I'_c, $1 < c \leqslant n$, *then the sequence* $\{g(Q_i)\}_{i=1}^{n}$ *has property* $W-U$ *with left-finishing point* $g(Q_1)$.

Proof. By hypothesis $\varnothing_j(P_1)$ is in I'_1. It follows by letting $h = k$ in part B) of Lemma II that the sequence $\{g(Q_i)\}_{i=1}^{n}$ has generalized property $W-U$ with L-finishing point $g(Q_{v(k)})$, R-finishing point $g(Q_{u(k)})$, starting point $g(Q_{s'})$, L-winding sequence $\{a'(i)\}_{i=1}^{k+1}$, and R-winding sequence $\{b'(i)\}_{i=1}^{k+1}$, as defined in the conclusion of part B) for $h = k$.

It follows from part A) of Lemma II that $\varnothing(t_1) = P_{a(k)} = P_1 = 0$. Since $\varnothing_j(P_{a(k)}) = \varnothing_j(P_1)$ is in $I'_{v(k)}$, $Q_{v(k)}$ is in $I'_{v(k)}$, $\varnothing_j(P_1)$ is in I'_1, and Q_1 is in I'_1, it follows that $I'_{v(k)} = I'_1$ and $Q_{v(k)} = Q_1$. Thus, the sequence $\{g(Q_i)\}_{i=1}^{n}$ has generalized property $W-U$ with L-finishing point $g(Q_1)$, R-finishing point $g(Q_{u(k)}) = g(Q_c)$, L-winding sequence $\{a'(i)\}_{i=1}^{k+1}$, and R-winding sequence $\{b'(i)\}_{i=1}^{k+1}$. Since $1 = v(k) \neq u(k) = c$, there is a positive integer E, $1 \leqslant E \leqslant k-1$, such that E is the last positive integer L such that $a'(L) = b'(L)$ and $L < k$. Consider the subsequences $\{a'(i)\}_{i=E+1}^{k+1}$ and $\{b'(i)\}_{i=E+1}^{k+1}$ of the L- and R-winding sequences $\{a'(i)\}_{i=1}^{k+1}$ and $\{b'(i)\}_{i=1}^{k+1}$, respectively. Define $H = (k+1)-(E+1)+1 = k-E+1$. For each i, $1 \leqslant i \leqslant H$, define $a''(i) = a'(E+i)$ and $b''(i) = b'(E+i)$. We now prove that the sequences $\{a''(i)\}_{i=1}^{H}$ and $\{b''(i)\}_{i=1}^{H}$ define left- and right-winding sequences for $\{g(Q_i)\}_{i=1}^{n}$ to have property $W-U$ with left-finishing point $g(Q_1) = g(Q_{a'(k+1)}) = g(Q_{a''(H)})$, right-finishing point $g(Q_c) = g(Q_{b'(k+1)}) = g(Q_{a''(H)})$ and starting point $g(Q_{a'(E)})$.

First consider 1) of the definition of property $W-U$. Now $a''(1) = a'(E+1) = a'(E)-1$ for if $a'(E+1) \neq a'(E)-1$ a contradiction to the definition of E exists. Similarly, $b''(1) = b'(E+1) = b'(E)+1$. The rest of 1) is clearly satisfied.

Now 2) and 3) of the definition of property $W-U$ are satisfied because these conditions exist in generalized property $W-U$ and $\{a'(i)\}_{i=1}^{k+1}$ and $\{b'(i)\}_{i=1}^{k+1}$ determine generalized property $W-U$ on $\{g(Q_i)\}_{i=1}^{n}$.

Now consider 4) of the definition of property $W-U$. By way of contradiction suppose there is a positive integer j, $1 \leqslant j \leqslant H$, such that $a''(j) > b''(j)$. Now $Q_1 = Q_{a'(k)} = Q_{a''(H)}$. Thus, there exists a positive integer $E' > E$ such that $a'(E') = b'(E')$ and $E' < k+1$. But this contradicts the definition of E. Thus, 4) is satisfied, and Lemma III follows.

COROLLARY. *Suppose the following changes are made to the hypothesis of Lemma* III:

a) *Hypothesis* 3) *of Lemma* II *is replaced by*: 3′), *there is a finite subset* $\{P_i\}_{i=1}^{m}$ *of* X_j *such that* $m \geqslant 3$, $P_1 < P_2 < \ldots < P_m$, $f(P_i) \neq f(P_{i+1})$ *for* $1 \leqslant i \leqslant m$, *either if* x *is in* $[P_1, P_m]$ *then* $f(x) \geqslant f(P_1)$ *or if* x *is in* $[P_1, P_m]$ *then* $f(x) \leqslant f(P_1)$, *and* $\bar{f}^{-1} \circ \bar{f}(\{P_i\}_{i=1}^{m}) = \{P_i\}_{i=1}^{m}$ *where* $\bar{f}$ *is the restriction of* f *to* $[P_1, P_m]$.

b) *Hypothesis* 4) *of Lemma* II *is replaced by*: 4′), *there is a positive integer* a, $1 \leqslant a \leqslant m$ *such that* $\varnothing_j(P_1)$ *is in* I'_a *and if* $1 \leqslant i \leqslant m$, $\varnothing_j(P_i)$ *is in* $\bigcup_{i=a}^{n} I'_i$.

c) *In the statement of Lemma* III, "$1 \leqslant c \leqslant n$", *is replaced by* "$a \leqslant c \leqslant n$".

Then the sequence $\{g(Q_i)\}_{i=a}^{n}$ *has property* $W-U$ *with left-finishing point* $g(Q_a)$.

3. The Main Theorem. In this section we give Theorem 1 that states sufficient conditions for a continuum expressed in terms of inverse expansions in finite trees not to be weakly chainable.

THEOREM 1. *Suppose*

1) $Y = \overset{\lim}{\leftarrow}\{Y_n, g_n^m, I\}$ *where each* Y_n *is a finite tree and each* g_n^m *is a mapping from* Y_m *onto* Y_n.

2) *There is a constant number* $\delta > 0$ *such that if* L *is a positive integer there is a sequence of points* $\{G_i\}_{i=1}^{a}$, $a \varepsilon I^+$, $a > 2$, *lying in* Y_L, *two points* $O_L = G_1$ *and* $C_L = G_2$ *lying in an arc* V_L *lying in* Y_L *such that no point of* $\{G_i\}_{i=3}^{a}$ *lies in* V_L *and* $\min\{d(G_i, G_j): i \neq j,\ 2 \leqslant i, j \leqslant a\} = \delta$.

3) *If* L *is a positive integer there is a positive integer* n, $n > L$, *such that there is a sequence* $\{Q_i\}_{i=1}^{b}$, $b \varepsilon I^+$, $b \geqslant n$, *of points lying in an arc* W_n *lying in* Y_n *and a finite increasing subsequence* $\{m(i)\}_{i=1}^{n}$ *of* $\{1, 2, \ldots, b\}$ *such that* $Q_1 < Q_2 < \ldots < Q_b$ *with respect to some order in* W_n *and if* j *is a positive integer*, $1 \leqslant j \leqslant n$, $S^j = \{g_L^n(Q_{m(j)}), g_L^n(Q_{m(j)+1}), \ldots, g_L^n(Q_b)\}$ *then*

1′) S^j *does not have property* $W-U$ *with left-finishing point* $Q_{m(j)}$ *the first point in* S^j,

2′) *for each* i, $1 \leqslant i \leqslant n$, $g(Q_{m(i)}) = C_L$,

3′) $g_{L|W_n}^n(\{Q_i\}_{i=1}^{b}) = \{G_i\}_{i=1}^{a} - \{O_L\} = \{G_i\}_{i=1}^{a} - \{G_1\}$,
$(g_{L|W_n}^n)^{-1} \circ g_{L|W_n}^n \times (\{Q_i\}_{i=1}^{b}) = \{Q_i\}_{i=1}^{b}$, *and*

4′) *between each* Q_i *and* Q_{i+1}, $1 \leqslant i \leqslant n$, *there is a point* X_i *such that* $g_L^n(X_i) = O_L = G_1$,

5′) *between each two points in* $\{Q_i\}_{i=1}^{b}$ *thrown by* g_L^n *to* $C_L = G_2$ *there is at least one point in* $\{Q_i\}_{i=1}^{b}$ *mapped by* g_L^n *to some point in* $\{G_i\}_{i=3}^{a}$.

Then Y *is not weakly-chainable.*

Proof. By way of contradiction suppose that there is a mapping of X onto Y where $X = \overset{\lim}{\leftarrow}\{X_n, f_n^m, I\}$ is the limit of an inverse system on intervals. It follows by Theorem 1 of [3] that there is an infinite diagram **D** of the form:

$$\begin{tikzcd}
X_1 \arrow[d, "\varnothing_1"'] & X_2 \arrow[l, "f_1^2"'] \arrow[d, "\varnothing_2"] & \ldots \arrow[l, "f_2^3"'] & X_k \arrow[l] \arrow[d, "\varnothing_k"'] & X_{k+1} \arrow[l, "f_k^{k+1}"'] \arrow[d, "\varnothing_{k+1}"'] & \ldots \\
Y_1 & Y_2 \arrow[l, "g_1^2"] & \ldots \arrow[l, "g_2^3"] & Y_k \arrow[l] & Y_{k+1} \arrow[l, "g_k^{k+1}"] & \ldots
\end{tikzcd}$$

and if $\varepsilon > 0$ there is a positive integer n such that if m is a positive integer and $m > n$ then the subdiagram

$$\begin{tikzcd}
X_n \arrow[d, "\varnothing_n"'] & X_m \arrow[l, "f_n^m"'] \arrow[d, "\varnothing_m"] \\
Y_n & Y_m \arrow[l, "g_n^m"]
\end{tikzcd}$$

is ε-commutative.

Suppose ε is a positive number and $\varepsilon < \frac{1}{2}\min\{d(G_i, G_j): 1 \leqslant i, j \leqslant a\}$, and L is the first positive integer such that the subdiagram D'_m

$$\begin{tikzcd}
X_L \arrow[d, "\varnothing_L"'] & X_m \arrow[l, "f_L^m"'] \arrow[d, "\varnothing_m"] \\
Y_L & Y_m \arrow[l, "g_L^m"]
\end{tikzcd} \quad \text{of } D$$

is $\varepsilon/16$-commutative, $m > L$.

The following statement follows directly from the fact that $\varnothing_L$ is uniformly continuous.

I) There is a positive integer N such that there are only N closed intervals each lying in X_L such that

1) The interiors of any two of them are mutually exclusive, and

2) The $\varnothing_L$-image of any one of them is the subarc $\overline{G_1'G_2'}$ of V_L where G_1' is in V_L, G_2' is in V_L, $d(G_1, G_1') = \frac{1}{4}\varepsilon$, and $d(G_2, G_2') = \frac{1}{4}\varepsilon$. Also denote $O_L' = G_1'$ and $C_L' = G_2'$.

Define n to be the least positive integer greater than or equal to $N+L$ such that there is a sequence $\{Q_i\}_{i=1}^{b}$, $b\varepsilon I^+$, $b \geqslant n$, of points lying in an arc W_n lying in Y_n satisfying all the conditions given in 3) in the hypothesis of Theorem 1.

There are points Z_1 and Z_2 in X_n such that $\varnothing_n(Z_1) = Q_1$ and $\varnothing_n(Z_2) = Q_b$, $\varnothing_n([Z_1, Z_2]) = \overline{Q_1Q_b}$ and $Z_1 < Z_2$ in $[0, 1]$. The following statement follows by induction.

II) There exists a finite sequence $\{P_i\}_{i=1}^{k}$, $k\varepsilon I^+$, of points lying in $\overline{Z_1Z_2}$ such that $k \geqslant 3$ and

1) $P_1 = Z_1$, $P_k = Z_2$.

2) If $1 \leqslant i \leqslant k-1$, and $\varnothing_n(P_i) = Q_j$, $1 \leqslant j \leqslant k$, then P_{i+1} is the first point P in $\overline{Z_1Z_2}$ after P_i such that $\varnothing_n(P) = \varnothing_{j+1}$ or $\varnothing_n(P) = Q_{j-1}$.

Since g_L^n is uniformly continuous and $\{Q_i\}_{i=1}^{b}$ is finite:

1) for each i in $\{1, 2, \ldots, a\}$ there is a closed interval I_i of length $\frac{1}{8}\varepsilon$ lying in Y_L and containing G_i where $I_i \cap I_j = \varnothing$, for $i \neq j$, $1 \leqslant i, j \leqslant a$, and

2) there exists a set $\{I_i'\}_{i=1}^{b}$ of mutually exclusive closed intervals lying in W_n containing $\{Q_i\}_{i=1}^{b}$, respectively, such that if e is in $\{1, 2, \ldots, b\}$ and $g_h^n(Q_e) = G_u$ for u in $\{1, 2, \ldots, a\}$, then $g_h^n[I_e']$ is a subset of I_u, and furthermore if v is in $\{1, 2, \ldots, a\}$, x is in W_n and $g_L^n(x)$ is in I_v, $1 \leqslant v \leqslant a$, then there is w in $\{1, 2, \ldots, b\}$ such that x is in I_w'.

For each j in I^+, $1 \leqslant j \leqslant n$, define $c(j)$ to be the last positive integer i in $\{1, 2, \ldots, k\}$ such that $\varnothing_n(P_i) = Q_{m(j)}$, and define e_j to be the last point x in $\overline{Z_1Z_2}$, $x \geqslant P_{c(j)}$, such that $f_L^n(x) = f_L^n(P_{c(j)})$ and $\varnothing_n(x)$ is in $I_{m(j)}'$.

We now prove the following.

III) If j is in $\{1, 2, \ldots, n\}$ and y is in (e_j, Z_2) then $f_L^n(y) \neq f_L^n(P_{c(j)})$. Suppose that j is a positive integer in $\{1, 2, \ldots, n\}$ such that III) is not true for j.

First observe that if y is in $(e_j, Z_2]$ and $f_L^n(y) = f_L^n(P_{c(j)})$ then there is a positive integer g in $\{1, 2, \ldots, b\}$ such that $\varnothing_n(y)$ is in I_g' and $g > m(j)$. An argument is now given for the above statement. Suppose that y is in $(e_j, Z_2]$ and $f_L^n(y) = f_L^n(P_{c(j)})$. Now $d[\varnothing_L \circ f_L^n(y) = \varnothing_L \circ f_L^n(P_{c(j)}), C_L] < \varepsilon/16$ since $\varnothing_n(P_{c(j)}) = Q_{m(j)}$. Thus, $\varnothing_n(y)$ is in some I_g' where $g_L^n(I_g') \subset I_2$, $1 \leqslant g \leqslant b$. Now $g \neq m(j)$ for otherwise there is a contradiction involved with the definition of e_j. Suppose then that $g < m(j)$. However, since $\varnothing_n(P_k) = Q_b$ and by 5') between each two points in $\{Q_i\}_{i=1}^{b}$ thrown by g_L^n to C_L there are at least two points in $\{Q_i\}_{i=1}^{b}$ mapped by g_L^n to some point in $\{G_i\}_{i=3}^{a}$, it follows that there is a point $R > e_j$ such that $\varnothing_n(R) = Q_{m(j)}$. This involves a contradiction with the definition of $P_{c(j)}$. Thus, it follows that $g > m(j)$.

Now let F be the first point y in $(e_j, Z_2]$ such that $f_L^n(y) = f_L^n(P_{c(j)})$. It follows from the observation above that there is a positive integer v in $\{1, 2, \ldots, b\}$ such that $\varnothing_n(F)$ is in I_v' and $v > m(j)$.

Since if y is in $(e_j, F]$ then $f_L^n(y) \neq f_L^n(P_{c(j)})$, as the case to consider, suppose that if y is in $(e_j, F]$ then $f_L^n(y) > f_L^n(P_{c(j)})$. Let H be the greatest point in $f_L^n([e_j, F])$. It is noted that there are at least two points from the set $\{P_i\}_{i=1}^{k}$ lying in $(e_j, F]$ since by 3) of the hypothesis of this lemma $\{g(Q_{m(j)+\alpha})\}_{\alpha=0}^{2}$ does not have property $W-U$ with left-finishing point $g(Q_{m(j)})$ which implies that $g(Q_{m(j)}) \neq g(Q_{m(j)+2})$.

Define $\{P_i\}_{i=h(1)}^{h(2)}$, $h(1)$, $h(2)$ in I^+, to be the set of all points in $\{P_i\}_{i=1}^{k}$ lying in the open interval (e_j, F), $1 \leqslant h(1), h(2) \leqslant k$.

Let $\{R_i\}_{i=1}^{t}$, t in I^+, denote the set $\{P_i\}_{i=h(1)}^{h(2)} \cup \{e_j, F\}$ where $R_1 = e_j$, $R_t = F$, and $R_1 < R_2 < \ldots < R_t$, $t > 4$. Also note that $f_L^n(R_i) \neq f_L^n(R_{i+1})$, $1 \leqslant i \leqslant t-1$.

Define U to be the set to which x belongs if and only if there is a positive integer i in $\{1, 2, \ldots, t-1\}$ and a positive integer j in $\{1, 2, \ldots, t-1\}$ where $f_L^n(R_j)$ is in the open interval determined by $f_L^n(R_i)$ and $f_L^n(R_{i+1})$ and x is the first point y in (R_i, R_{i+1}) such that $f_L^n(y) = f_L^n(R_j)$, or x is in the set $\{R_i\}_{i=1}^{t}$.

Let $U = \{U_i\}_{i=1}^{g}$, $g\varepsilon I^+$, where $U_1 < U_2 < \ldots < U_g$, $U_1 = R_1$ and $U_g = R_t$. Define

$$f\colon [e_j, F] \to [f_L^n(e_j), H] \text{ by } f(x) = \frac{f_L^n(U_{i+1}) - f_L^n(U_i)}{U_{i+1} - U_i} \cdot (x - U_i) + f_L^n(U_i),$$

for $U_i \leqslant x \leqslant U_{i+1}$, $i = 1, 2, \ldots, g-1$. We now show that the hypothesis of the corollary of Lemma III) in section 2 is satisfied.

1) The subdiagram D′, the set of intervals $\{I_i\}_{i=1}^{a}$, the set of points $\{G_i\}_{i=1}^{a}$ and the map f satisfy the conditions of paragraph 1 of section 2.

2) The set $\{Q_i\}_{i=1}^{b}$ satisfies 1) of the hypothesis of Lemma II of section 2.

3) The set $\{I_i'\}_{i=1}^{b}$ satisfies 2) of the hypothesis of Lemma II of section 2.

4) The function f and the set $\{U_i\}_{i=1}^{g}$ satisfy condition a) of the corollary to Lemma III of section 2. A proof of 4) follows. The case under consideration and the definition of f imply that if x is in $[e_j, F]$ then $f(x) \geqslant f(e_j)$. Also $f(U_i) \neq f(U_{i+1})$, $1 \leqslant i \leqslant g$. Since f agrees with f_L^n on $\{U_i\}_{i=1}^{g}$, the diagram

$$\begin{array}{ccc} \bar{X}_L & \xleftarrow{f} & \bar{X}_n \\ \varnothing_L \downarrow & & \downarrow \varnothing_n \\ Y_L & \xleftarrow[g_L^n]{} & Y_n \end{array}$$

is $\varepsilon/16$-commutative on $\{U_i\}_{i=1}^{g}$ where $\bar{X}_L = [f(P_{c(j)}), H]$ and $\bar{X}_n = [e_j, F]$. It remains to be shown that $f^{-1} \circ f\{U_i\}_{i=1}^{g} = \{U_i\}_{i=1}^{g}$. Thus, suppose that there is a point x in X_n such that $x \notin \{U_i\}_{i=1}^{g}$ and $f(x)$ is in $\{f(U_i)\}_{i=1}^{g}$. There is a positive integer i, $1 \leqslant i \leqslant t$, such that $R_i < x < R_{i+1}$. It follows from the definitions of $\{U_i\}_{i=1}^{g}$ and f that either if $f(R_i) < f(R_{i+1})$ then f is strictly increasing on $[R_i, R_{i+1}]$ or if $f(R_i) > f(R_{i+1})$ then f is strictly decreasing on $[R_i, R_{i+1}]$. In order to show that this last statement is true, suppose that $f(R_i) < f(R_{i+1})$ and f is not strictly increasing on $[R_i, R_{i+1}]$. Then there are positive integers p and q, $1 \leqslant p, q \leqslant g$, such that $R_i < U_p < U_q < R_{i+1}$ and $f(U_p) > f(U_q)$. However, then there is a point $y < U_p$ that is the first point P in (R_i, R_{i+1}) such that $f_L^n(P) = f_L^n(U_p) = f_L^n(R_j)$ for some j, $1 \leqslant j \leqslant t$.

The existence of y involves a contradiction with U_q being a member of the set $\{U_i\}_{i=1}^{g}$. Since f is either strictly increasing or decreasing on $[R_i, R_{i+1}]$, f is one-to-one on $[R_i, R_{i+1}]$. There is a positive integer v and a positive integer m, $1 \leqslant v \leqslant t$, $1 \leqslant m \leqslant g$, such that $R_i < U_m < R_{i+1}$, $f(x) = f(R_v) = f(U_m)$ and $f(U_m)$ is in the open interval determined by $f(R_i)$ and $f(R_{i+1})$. It follows from the definition of $\{U_i\}_{i=1}^{g}$ that $U_m < x$. This involves a contradiction since f is one-to-one on $[R_i, R_{i+1}]$. Thus, $f^{-1} \circ f\{U_i\}_{i=1}^{g} = \{U_i\}_{i=1}^{g}$ and 4) is true.

5) Condition b) of the corollary to Lemma III of section 2 is satisfied with respect to $\{U_i\}_{i=1}^{g}$ since for each j, $1 \leqslant j \leqslant g$, $f(U_j) = f_L^n(R_u)$ for some u, $1 \leqslant u \leqslant t$, implies that $\varnothing_n(U_j)$ is in some I'_h, $1 \leqslant h \leqslant b$, and since $\varnothing(U_1) = \varnothing_n(R_1)$ is in $I'_{m(j)}$ and $\varnothing_n(R_t) = \varnothing_n(U_g)$ is in I'_v, $v > m(j)$.

6) Condition c) of the corollary to Lemma III of section 2 is satisfied since $m(j) < v < b$.

Since all of the conditions to the corollary to Lemma III of Section 2 are satisfied, it follows by that corollary that $\{Q_{m(j)}, Q_{m(j)+1}, \ldots, Q_b\}$ has property $W-U$ with left-finishing point $Q_{m(j)}$. This involves a contradiction with 3.1′) in the hypothesis of this Lemma. Thus, III above is true.

The following statement follows directly from III.

IV) If j and u are in I^+, $1 \leqslant j, u \leqslant n$ and $j < u$ then $f_L^n(P_{c(j)}) \neq f_L^n(P_{c(u)})$.

We now prove the following statement.

V) Suppose that j and u are in I^+, $1 \leqslant j, u \leqslant n$ and $j < u$, then there is a point O_1 in $(P_{c(j)}, P_{c(u)})$ such that $g_L^n \circ \varnothing_n(O_1) = O_L$ and $f_L^n(O_1)$ is in the open interval determined by $f_L^n(P_{c(j)})$ and $f_L^n(P_{c(u)})$. As the case to consider,

suppose $f_L^n(P_{c(j)}) < f_L^n(P_{c(u)})$. Consider e_j as defined above. From 3.4′) in the hypothesis of this theorem it follows that there is a point O_2 that is the first point P in $\overline{Q_{m(j)+1}, Q_{m(u)}}$ and $g_L^n(P) = O_L$. There is a point O_1 that is the first point P in $(e_j, P_{c(u)})$ such that $\varnothing_n(P) = O_2$. It follows by III, that $f_L^n(O_1) > f_L^n(P_{c(j)})$.

If $f_L^n(P_{c(u)}) < f_L^n(O_1)$, it follows that there is a point C_1 in (e_j, O_1) such that $\varnothing_n(C_1)$ is I'_t, $m(j) < t$, t in I^+, $g_L^n(Q_t) = C_L$. Thus, there is a point O_4 in (e_j, O_1) such that $\varnothing_n(O_4) = O_2$. This involves a contradiction with the definition of O_1. Thus, V follows.

We now prove the following.

VI) Suppose that j and u are in I^+, $1 \leqslant j, u \leqslant n$, and $j \leqslant u$, and O_1 is defined as in V), then there exist two closed intervals I''_j and I''_u such that I''_j is a subset of the closed interval determined by $f_L^n(P_{c(j)})$ and $f_L^n(O_1)$, I''_u is a subset of the closed interval determined by $f_L^n(P_{c(u)})$ and $f_L^n(O_1)$, I''_j and I''_u have mutually exclusive interiors, $\varnothing_L(I''_j) = \overline{O'_n C'_n}$, and $\varnothing_L(I''_u) = \overline{O'_n C'_n}$. As the case to consider suppose that $f_L^n(P_{c(j)})$ is less than $f_L^n(P_{c(u)})$. It follows by the use of V) that $f_L^n(P_{c(j)}) < f_L^n(O_1) < f_L^n(P_{c(u)})$. Since D' is $\varepsilon/8$-commutative and $g_L^n \circ \varnothing_n(O_1) = O_L$, $\varnothing_L \circ f_L^n(O_1)$ is in $\overline{O_L O'_L}$. Since D' is $\varepsilon/8$-commutative $\varnothing_n(P_{c(j)})$ is in $I'_{m(j)}$ and $\varnothing_n(P_{c(u)})$ is in $I'_{m(u)}$, $\varnothing_L \circ f_L^n(P_{c(j)})$ and $\varnothing_L \circ f_L^n(P_{c(u)})$ are both in $\overline{C_L C'_L} \subset Y_L$. Thus, $\varnothing_L([f_L^n(P_{c(j)}), f_L^n(O_1)])$ and $\varnothing_L([f_L^n(P_{c(u)}), f_L^n(O_1)])$ both contain $\overline{O'_L C'_L}$. Thus, I_j and I_u as described in VI exists.

It follows from VI) that the set $\{f_L^n(P_i)\}_{i=1}^{n}$ can be renamed as the set $\{T_i\}_{i=1}^{n}$, where for each i in $\{1, 2, \ldots, n-1\}$, $T_i < T_{i+1}$. Suppose i is in $\{1, 2, \ldots, n-1\}$. It follows from VI) by letting $j = i$ and $u = i+1$ that there exist two closed intervals I''_i and I''_{i+1}, lying in $[T_i, T_{i+1}]$ such that I''_i contains T_i, I''_{i+1} contains T_{i+1}, $f(P_{c(j)}) = T_i$, and $f(P_{c(u)}) = T_{i+1}$, for j, u in $\{1, 2, \ldots, n\}$.

Furthermore, it follows from VI) that the interiors of I''_i and I''_{i+1} are mutually exclusive. Since $T_1 < T_2 < \ldots < T_n$, it follows that if j and u are in $\{1, 2, \ldots, n\}$ then the interiors of I''_j and T'_u are mutually exclusive. It also follows from VI) that $\varnothing_L(I''_j) = [O'_L C'_L]$, for j in $\{1, 2, \ldots, n\}$. Since $n > N$, the existence of $\{T'_i\}_{i=1}^{n}$ involves a contradiction with I) above. Thus, there is no mapping from X onto Y.

4. Results on combinatorial properties of finite sequences. In this section certain definitions and theorems are given concerning certain combinatorial properties of finite sequences that result from recursive definitions. The combinatorial property of most importance is called Property $W-U$. The recursive definitions of the finite sequences and the lemmas in this section concerning Property $W-U$ are used, in part, in section 5 to show that the example in [2] is not weakly chainable.

DEFINITION 1. Suppose $S = \{P_i\}_{i=1}^{n}$ is a finite sequence, then the *reverse* of S, denoted S^R, is the finite sequence $S^R = \{P_i^R\}_{i=1}^n$ where for each i, $1 \leqslant i \leqslant n$, $P_i^R = P_{n-i+1}$.

Definition 2. Suppose each of $P = \{P_i\}_{i=1}^n$ and $Q = \{Q_i\}_{i=1}^m$ is a finite sequence such that for each i, $1 \leqslant i \leqslant n$, and each j, $1 \leqslant j \leqslant m$, $P_i \neq P_{i+1}$ and $Q_j \neq Q_{j+1}$. Then the sequence $P \oplus Q$ is defined to be sequence $T = \{T_i\}_{i=1}^k$ such that the following is true:

1. $T_i = P_i$, $1 \leqslant i \leqslant n$.
2. If $P_n = Q_1$, then $T_n = Q_1$ and $T_{n+i} = Q_{i+1}$ for $i = 1, \ldots, m-1$ and $k = n+m-1$. (In this case T_n is said to be both the last point in P and the first in Q.)
3. If $P_n \neq Q_1$ then $T_{n+i} = Q_i$ for $i = 1, \ldots, m$ and $k = m+n$.

An example S of an m-system of recursive sequences defined below in Definition 3 is given in I) page 233 in section 5. It is shown in I) in section 5 how the example S relates to the example Y given in [2]. Properties about m-systems of recursive sequences developed in this section will be applied to Y in section 5 in order to show that Y is not weakly chainable.

Definition 3. Suppose m is a positive integer. To say that $S = \{S_t^m, v(i), P_r^j, k(i), c_i\}$ is an *m-system of recursive sequences* means there is a finite collection of finite sequences:

$$\begin{aligned} S_1^1 &= P_1^1, P_2^1, \ldots, P_{v(1)}^1, & S_{m+1}^1 &= (S_1^1)^R,\\ S_2^1 &= P_1^2, P_2^2, \ldots, P_{v(2)}^2, & S_{m+2}^1 &= (S_2^1)^R,\\ &\ldots\ldots & &\ldots\ldots\\ S_m^1 &= P_1^m, P_2^m, \ldots, P_{v(m)}^m, & S_{2m}^1 &= (S_m^1)^R, \end{aligned}$$

where for some point P_c and for each j, $1 \leqslant j \leqslant m$, $P_1^j = P_{v(j)}^j = P_c$, $v(i) \geqslant 4$, $(1 \leqslant i \leqslant m)$ such that if $1 \leqslant k \leqslant m$ then for each i, $P_i^k \neq P_{i+1}^k$, $1 \leqslant i \leqslant v(k)$, and there is a finite collection of finite sequences of positive integers:

$$\begin{aligned} c_1 &= c_1^1, c_2^1, \ldots, c_{k(1)}^1,\\ c_2 &= c_1^2, c_2^2, \ldots, c_{k(2)}^2,\\ &\ldots\ldots\\ c_m &= c_1^m, c_2^m, \ldots, c_{k(m)}^m \end{aligned}$$

such that each $c_L^k \varepsilon \{1, \ldots, 2m\}$ and if $n \varepsilon I^+$, $n \geqslant 2$, then

$$\begin{aligned} S_1^n &= \oplus \sum_{i=1}^{k(1)} S_{c_i^1}^{n-1}, & S_{m+1}^n &= (S_1^n)^R,\\ S_2^n &= \oplus \sum_{i=1}^{k(2)} S_{c_i^2}^{n-1}, & S_{m+2}^n &= (S_2^n)^R,\\ &\ldots\ldots & &\ldots\ldots\\ S_m^n &= \oplus \sum_{i=1}^{k(m)} S_{c_i^m}^{n-1}, & S_{2m}^n &= (S_m^n)^R, \end{aligned}$$

where each $k(t)$, $1 \leqslant t \leqslant m$, is a positive integer and for each t, $1 \leqslant t \leqslant m$, and each i, $1 \leqslant i \leqslant k(t)$, $S_{c_i^t}^{n-1} \neq S_{c_{i+1}^t}^{n-1}$. Furthermore, for each j, $1 \leqslant j \leqslant m$, $S_{c_i^j}^{n-1}$ is called the ith component sequence of S_j^n, and $(S_{c_{k(j)-(i+1)}^j}^{n-1})^R$ is called the ith component sequence of $(S_j^n)^R$. Also $(S_{c_{k(j)-(i+1)}^j}^{n-1})^R$ is denoted $S_{c_i^j+m}^{n-1}$. Finally, $\bar{S}_m^n$ denotes $\{S_i^j\colon i, j \varepsilon I^+,\ 1 \leqslant j \leqslant n,\ 1 \leqslant i \leqslant 2m\}$.

Before reading the lemmas which follow in this section the reader should recall the definitions of property $W-U$, property $W-U$ extended, and related definitions given in Definition 1 and Definition 2 of section 2.

Lemma I. *Suppose that v is a positive integer greater than two and $\{P_i\}_{i=1}^v$ is a finite sequence such that for each i, $1 \leqslant i \leqslant v$, $P_i \neq P_{i+1}$. Then $\{P_i\}_{i=1}^v$ has property $W-U$ with starting point P_k, left-finishing point P_a, and right-finishing point P_b, where $a \neq k \neq b$, if and only if P has property $W-U$ extended with starting point P_k, left-finishing point P_a, and right-finishing point P_b, $1 \leqslant a, b \leqslant v$, $1 < k < v$.*

Lemma II. *If $L = \{P_i\}_{i=1}^n$ is a finite sequence such that $P_1 = P_n$ then $S = L \oplus L^R$ has property $W-U$ extended with starting point the last point in L (with starting point the first point in L^R), left-finishing point the first point in S, and right-finishing point the last point in S.*

Definition 4. Suppose that $S = \{S_t^m, v(i), P_r^j, k(i), c_i\}$ is an m-system of recursives sequences, each of e, f, m and n is a positive integer, $n \geqslant 2$, $f \geqslant 3$, $e \geqslant 2$, $T = \oplus \sum_{i=1}^{e} L_i^n = \{t_i\}_{i=1}^h$, where each $L_i^n \varepsilon \bar{S}_m^n$, T has property $W-U$ ($W-U$ extended) with respect to the starting point t_w, left-winding sequence $\{a(i)\}_{i=1}^u$, and right-winding sequence $\{b(i)\}_{i=1}^u$, u, $w \varepsilon I^+$, $1 < w < u$, $u \geqslant 3$. To say that the *winding process* of $\{a(i)\}_{i=1}^u$ (respectively $\{b(i)\}_{i=1}^u$) winds to *a* point t_v in T, $1 \leqslant v \leqslant h$, with respect to the positive integer k, $1 \leqslant k \leqslant u$, means that $t_{a(k)} = t_v$ (respectively, $t_{b(k)} = t_v$). If the winding process of $\{a(i)\}_{i=1}^u$ (respectively, $\{b(i)\}_{i=1}^u$) winds to a point t_v in T, $1 \leqslant v \leqslant h$, with respect to the positive integer k, $1 \leqslant k \leqslant u$, then to say that the winding process of $\{a(i)\}_{i=1}^u$ (respectively $\{b(i)\}_{i=1}^u$) *winds away* from t_v means $k < u$. If the winding process of $\{a(i)\}_{i=1}^u$ (respectively $\{b(i)\}_{i=1}^u$) winds to a point t_v in T, $1 \leqslant v \leqslant h$, with respect to the positive integer k, $1 \leqslant k \leqslant u$, and winds away from t_v, then to say that the winding process of $\{a(i)\}_{i=1}^u$ (respectively $\{b(i)\}_{i=1}^u$) *winds back* to t_v with respect to the positive integer L means $L > k$, $L \leqslant u$, and $t_{a(L)} = t_v$ (respectively $t_{b(L)} = t_v$).

Furthermore, if g is a positive integer, $1 \leqslant g \leqslant e$, $L_g^n + L_{g+1}^n = (P_1, P_2, \ldots, P_r) + (Q_1, Q_2, \ldots, Q_z) = t_{n(1)}, t_{n(2)}, \ldots, t_{n(x)}, t_{n(x+1)}, \ldots, t_{n(w)}$ where each of r, z, x and w is a positive integer and $t_{n(x)} = P_r$ and $T_{n(x+1)} = Q_2$, then to say that the winding process of $\{a(i)\}_{i=1}^u$ (respectively, $\{b(i)\}_{i=1}^u$) *winds to* P_r or Q_1 with respect to the positive integer h, $1 \leqslant h \leqslant u$, (respectively winds away from P_r or Q_1) (respectively winds back to P_r or Q_1 with respect to the positive integer k, $1 \leqslant k \leqslant u$) means that it winds to $t_{n(x)}$ with respect to h (respectively winds away from $t_{n(x)}$) (respectively, winds back to $t_{n(x)}$ with respect to k); if $1 < j < r$, then to say that it *winds to* P_j with respect to k (respectively winds away from P_j) (respectively winds back to P_j with respect to k) means that it winds to $t_{n(j)}$ with respect to k (respectively, winds away from $t_{n(j)}$) (respectively winds back to $t_{n(j)}$ with respect to h). If $1 < j < z$ then to say it winds to Q_j with respect to k (respectively winds away from Q_j) (respectively winds back to Q_j with respect to h) means that it winds to $t_{n(j)+r}$ with respect to k (winds away from $t_{n(j)+r}$) (respectively winds back to $t_{n(j)+r}$ with respect to h). To say that the winding process of $\{a(i)\}_{i=1}^{u}$ (respectively $\{b(i)\}_{i=1}^{u}$) *winds into* L_g^n, $1 \leqslant g \leqslant e$, with respect to the positive integer j, $1 \leqslant j \leqslant u$, means there is a point P_v in L_g^n such that it winds to P_v with respect to j. If $L_i^n \varepsilon \bar{S}_m^n$, $1 \leqslant g \leqslant e$, is such that the winding process of $\{a(i)\}_{i=1}^{u}$ (respectively $\{b(i)\}_{i=1}^{u}$) winds into L_i^n with respect to j, $1 \leqslant j \leqslant u$, then to say that the winding process of $\{a_i\}_{i=1}^{u}$ (respectively, $\{b_i\}_{i=1}^{u}$) *winds out of* L_i^n with respect to the positive integer $I > j$, $1 \leqslant I \leqslant u$, means the winding process of $\{a(i)\}_{i=1}^{u}$ (respectively $\{b(i)\}_{i=1}^{u}$) winds into $L_j^n \varepsilon \bar{S}_m^n$ with respect to I, to a point in L_j^n not in L_i^n, where $j \neq i$.

DEFINITION 5. Suppose $S = \{S_t^m, v(i), P_r^j, k(i), c_i\}$ is an m-system of recursive sequences, and $P = \oplus \sum_{i=1}^{t} L_i$ where each $L_i \varepsilon \bar{S}_m^n$, $t > 1$. To say that P has the *winding form* means that there are two sequences of positive integers $\{a(i)\}_{i=1}^{n}$ and $\{b(i)\}_{i=1}^{n}$, $n \varepsilon I^+$, $n > 1$, (called, respectively, the left-winding sequence and the right-winding sequence) and a positive integer j, $1 \leqslant j \leqslant n$, such that the following is true:

1. $a(1) = j$; $b(1) = j+1$; $L_{a(1)} = L_{b(1)}^R$.
2. $L_{a(i)} = L_{b(i)}$ or $L_{a(i)} = L_{b(i)}^R$, $1 \leqslant i \leqslant n$.
3. $a(i) = a(i+1)+k$; $b(i) = b(i+1)+k$, $k = 0$ or $k = 1$, $1 \leqslant i \leqslant n$.
4. If $L_{a(i)} = L_{b(i)}$ then either $b(i) = b(i-1)$ or $a(i) = a(i-1)$, $1 \leqslant i \leqslant n$.
5. If $L_{a(i)} = L_{b(i)}^R$ then $b(i) \neq b(i-1)$ and $a(i) \neq a(i-1)$, $1 \leqslant i \leqslant n$.
6. $a(i) < b(i)$, $1 \leqslant i \leqslant n$.
7. $L_i \neq L_i^R$, $1 \leqslant i \leqslant n$.

Furthermore, n is called a *winding form sequence cardinality of* P, and $L_{a(n)}$ and $L_{b(n)}$ are called respectively the *left-* and *right-finishing component sequences* for P with respect to $\{a(i)\}_{i=1}^{n}$ and $\{b(i)\}_{i=1}^{n}$, and $L_{a(1)}$ is called the *left-starting component sequence* and $L_{b(1)}$ is called the *right-starting component sequence*. Furthermore, to say that P has the *complete winding form* means there is a positive integer $g \leqslant t$ such that $P' = \oplus \sum_{i=1}^{g} L_i$ has the winding form with $\{a(i)\}_{i=1}^{n}$ and $\{b(i)\}_{i=1}^{n}$ as, respectively, the left- and right-winding sequences with $L_1 = L_{a(n)}$ (L_1 is called the left-finishing component sequence of the complete winding form with respect to $\{a(i)\}_{i=1}^{n}$ and $\{b(i)\}_{i=1}^{n}$, or $P' = \oplus \sum_{i=g}^{t} L_i$ has the winding form with respect to $\{a'(i)\}_{i=1}^{h}$ and $\{b'(i)\}^{h}$ as respectively, the left- and right-winding sequence with $L_{(t)} = L_{b'(h)}$. (L_t is called the right-finishing component sequence of the complete winding form with respect to $\{a'(i)\}_{i=1}^{h}$ and $\{b'(i)\}_{i=1}^{h}$).

Furthermore, if P has winding form with, respectively, $\{a(i)\}_{i=1}^{n}$ and $\{b(i)\}_{i=1}^{n}$ as the left- and right-winding sequences then P has the *return winding form* if the left-ending component sequence is $L_{a(1)}$ and the right-ending component sequence is not $L_{b(1)}$, or if the right-ending component sequence is $L_{b(1)}$ and the left-ending component sequence is not $L_{a(1)}$.

DEFINITION 6. Suppose that $S = \{S_t^m, v(i), P_r^j, k(i), c_i\}$ is an m-system of recursive sequences. The following properties are defined.

1. To say that S has *property* $W-1$ with respect to the positive integer n means that no single member component sequence P of $\bar{S}_m^n$ has property $W-U$ extended with the left-finishing point the first point of P or with the right-finishing point the last point of P.

2. To say that S has *property* $W-2$ with respect to the positive integer n means that if each of L_n^1 and L_n^2 is a member of $\bar{S}_m^n$ and $P = L_n^1 \oplus L_n^2$ then P has property $W-U$ extended with either the left-finishing point the first point in P or with the right-finishing point the last point in P if and only if $L_n^1 = (L_n^2)^R$ and the *starting point* is the *last point* in L_n^1.

3. To say that S has *property* $W-3$ with respect to the positive integer n means that if the sequence P is defined as in (1.) or as in (2.), $P = P_1, P_2, \ldots, P_k, \ldots, P_g$, $g > 3$, and P has property $W-U$ extended with left-winding sequence $\{a(i)\}_{i=1}^{t}$, right-winding sequence $\{b(i)\}_{i=1}^{t}$, and starting point P_k, $1 < k < t$, then if for some i, $1 < i < t$, $a(i) = k$ then $b(i) = k$, or if for some j, $1 < k < t$, $b(j) = k$ then $a(j) = k$.

4. To say that S has *property* $W-4$ with respect to the positive integer n means that if $P = L_n^1 \oplus L_n^2$ where each of L_n^1 and L_n^2 is a member of $\bar{S}_m^n$ and P has property $W-U$ extended with the first point in P the left-finishing point (respectively, the last point in P the right-finishing point) then the last point in P is the right-finishing point (respectively the first point in P is the left-finishing point).

LEMMA III. *Suppose that* $S = \{S_t^m, v(i), P_r^j, k(i), c_i\}$ *is an* m-*system of recursive sequences and* $L_n \varepsilon \bar{S}_m^n$, $n > 1$. *Then* $L_n \oplus L_n^R$ *has the complete winding form with respect to* $n-1$ *with left-finishing component sequence the first component sequence in* L_n, *with the right-finishing component sequence the last component sequence in* L_n^R, *with left-starting component sequence the last component sequence in* L_n, *and with right-starting component sequence the first component sequence in* L_n^R.

LEMMA IV. *Suppose that* $S = \{S_t^m, v(i), P_r^j, k(i), c_i\}$ *is an* m-*system of recursive sequences such that*:

1. S has properties $W-1$, $W-2$, $W-3$, and $W-4$ with respect to $n=1$.

2. If n is a positive integer, $n>1$, and $L_n \varepsilon \bar{S}^n_m$ then L_n does not have the complete winding form with respect to the positive integer $n-1$.

3. If n is a positive integer, $n>1$, and each of L^1_n and L^2_n belongs to $\bar{S}^n_m$, then $L^1_n \oplus L^2_n$ has the complete winding form with respect to the positive integer $n-1$ if and only if $L^1_n = (L^2_n)^R$, and the left-finishing component sequence is the first component sequence in L^1_n and the right-finishing component sequence is the last component sequence in L^2_n, the left-starting component sequence is the last component sequence L^1_n and the right-starting component is the first component sequence in L^2_n.

4. If $n \varepsilon I^+$, $n>1$, and each of L_n, L^1_n and L^2_n belongs to $\bar{S}^n_m$, then L_n and $L^1_n \oplus L^2_n$ do not have the return winding form with respect to the positive integer $n-1$.

Then if k is a positive integer then S has properties $W-1$, $W-2$, $W-3$, and $W-4$ with respect to the positive integer k.

Proof. The proof is given by induction on k. The truth of Lemma IV for $k=1$ follows directly from the part of the hypothesis given in 1. By way of contradiction, let $h+1$ be the least positive integer for which Lemma IV is not true.

First suppose that S does not have property $W-1$ with respect to $h+1$. Thus, by supposition, there is $S^{h+1} \varepsilon \bar{S}^{h+1}_m$ with $S^{h+1} = S^h_1 + S^h_2 + \ldots + S^h_v$ where v is a positive integer, $1 \leqslant v \leqslant 2m$, each S^h_j is in $\bar{S}^h_m$, $1 \leqslant j \leqslant v$, $S^{h+1} = \{P_i\}^R_{i=1}$, and S^{h+1} has property $W-U$ extended with left-finishing point P_1 the first point in S^{h+1} (case 1), or with right-finishing point P_R the last point in S^{h+1} (case 2). Consider case 1. Let $\{a(i)\}^u_{i=1}$ and $\{b(i)\}^u_{i=1}$ be, respectively, the left- and right-winding sequences for property $W-U$ extended on S^{h+1} with starting point P_s, $1 \leqslant s \leqslant R$, as required under case 1. Let g be the first positive integer n, $1 \leqslant n \leqslant v$, such that P_s is some point in S^h_n. As subcase 1 of case 1, suppose that P_s is not the first or last point in S^h_g. Subcase 2 of case 1 is the supposition that P_s is the first or last point in S^h_g. Under subcase 1, there is a subsequence $\{a(i)\}^e_{i=1}$, $e<u$, of $\{a(i)\}^u_{i=1}$, and there is a subsequence $\{b(i)\}^e_{i=1}$, $e<u$, of $\{b(i)\}^u_{i=1}$ such that either $P_{a(e)}$ is the first point in S^h_g or such that $P_{b(e)}$ is the last point in S^h_g since S^{h+1} has property $W-U$ with left-finishing point P_1, and, thus, the winding process of either $\{a(i)\}^u_{i=1}$ or $\{b(i)\}^u_{i=1}$ winds out of S^h_g. But then a single member component sequence of $\bar{S}^h_m$, namely S^h_g, has property $W-U$ extended with either the left-finishing point the first point in the sequence or with right-finishing point the last point in the sequence. In either case there is a contradiction to the induction hypothesis concerning property $W-1$.

Now suppose subcase 2 of case 1 where P_s is the first or last point in S^h_g. Then $S^{h+1} = S^h_1 + \ldots + S^h_g + S^h_{g+1} + \ldots + S^h_v$. Consider the case under subcase 2 where P_s is only the first point in S^h_g. Since S^{h+1} can not have property $W-U$ extended with starting point the first point in S^{h+1}, g can not be 1; but, $g=1$ is the only possibility that P_s can be only the first point in S^h_g since if $g>1$ and P_s is the first point in S^h_g it is also the last point in S^h_{g-1}. Thus consider the case where P_s is the last point in S^h_g. It follows S^h_{g+1} exists since P_s can not be the last point in S^{h+1}.

Since S^{h+1} has property $W-U$ extended with left-finishing point P_1, there is a first positive integer F such that $P_{a(F)}$ is the first point in S^h_g. By the induction hypothesis concerning property $W-4$, if the winding process of $\{a(i)\}^u_{i=1}$ (respectively, $\{b(i)\}^u_{i=1}$) winds to the first (respectively the last) point in S^h_g (respectively in S^h_{g+1}) then the winding process of $\{b(i)\}^u_{i=1}$ (respectively $\{a(i)\}^u_{i=1}$) winds to the last (respectively, the first) point in S^h_{g+1} (respectively, in S^h_g). Thus, the subsequences $\{a(i)\}^F_{i=1}$ and $\{b(i)\}^F_{i=1}$ of $\{a(i)\}^u_{i=1}$ and $\{b(i)\}^u_{i=1}$, respectively, as left- and right-winding sequences, describe property $W-U$ extended on $S_1 = S^h_g + S^h_{g+1}$ with left-finishing point $P_{a(F)}$ the first point in S_1 and right-finishing point $P_{b(F)}$ the last point in S_1. It then follows by the induction hypothesis concerning property $W-2$ that $S^h_g = (S^h_{g+1})^R$. Thus, $S^h_g \oplus S^h_{g+1}$ has the winding form with a winding form sequence cardinality of one, with respect to the left-winding sequence $a'(1)=1$ and the right-winding sequence $b'(1)=1$.

S_1 is a subsum T in S^{h+1} with the following properties:

1) For some positive integer $k \geqslant 2$, T is a subsum of k-consecutive component sequences in S^{h+1}.

2) T has property $W-U$ extended with left-finishing point the first or last point in some component sequence in T, and with right-finishing point the first or last point is some component sequence in T.

3) T has the winding form.

4) The winding process of property $W-U$ extended on T winds into each of the component sequences in T.

The definition of C which follows describes the collection of all subsums T in S^{h+1} which satisfy the properties 1), 2), 3) and 4) given above.

Define C to be the set of all tuples $(\bar{S}, G, E, L, M, N, H, \{a(i)\}^H_{i=1}, \{b(i)\}^H_{i=1}, \{a'(i)\}^N_{i=1}, \{b'(i)\}^N_{i=1}$, where:

1) $\bar{S}$ is of the form $\bar{S} = S^h_{g-L} + S^h_{g-(L-1)} + \ldots + S^h_{g-1} + S^h_g + S^h_{g+1} + \ldots + S^h_{g+M}$ where each of g, L, H, and M is a positive integer, $1 \leqslant g-L \leqslant g-1 < g < g+1 \leqslant g+h \leqslant k_v$.

2) $\bar{S}$ has property $W-U$ extended such that two subsequences $\{a(i)\}^H_{i=1}$ and $\{b(i)\}^H_{i=1}$ of $\{a(i)\}^u_{i=1}$ and $\{b(i)\}^u_{i=1}$, respectively, are the left- and right-winding sequences, respectively, where $H \leqslant u$.

3) E and G are positive integers, $g-L \leqslant E < G \leqslant g+M$, such that $P_{a(H)}$ is the first or last point in S^h_E and $P_{b(H)}$ is the first or last point in S^h_G,

4) $\bar{S}$ has the winding form with left- and right-winding sequences $\{a'(i)\}^N_{i=1}$ and $\{b'(i)\}^N_{i=1}$, respectively, with *left-ending component sequence* S^h_E and *right-ending component sequence* S^h_G. (N is the winding form sequence cardinality of $\bar{S}$.)

5) If S_i^h is one of the sequences in the sum forming $\bar{S}$ then there is a positive integer $\bar{H}$, $1 \leqslant \bar{H} \leqslant H$, such that $P_{a(\bar{H})} \varepsilon S_i^h$ or $P_{b(\bar{H})} \varepsilon S_i^h$.

Define c_1 to be the tuple

$$(S_1, G_1, E_1, L_1, M_1, N_1, H_1, \{a(i)\}_{i=1}^{H_1}, \{b(i)\}_{i=1}^{H_1}, \{a'(i)\}_{i=1}^{N_1}, \{b'(i)\}_{i=1}^{N_1})$$

where $S_1 = S_g^h + S_{g+1}^h$, $G_1 = g$, $E_1 = g+1$, $L_1 = g$, $M_1 = g+1$, $N_1 = 1$, $H_1 = F$, and $a'(i)$, $b'(i)$, $a(i)$, $b(i)$ are defined as above.

C is a set since $c_1 \varepsilon C$. Let H_2 be the largest positive integer x such that x is the seventh term for some tuple in C. H_2 exists since if x is a seventh term for some tuple in C then $x \leqslant u$. Let C_2 be an element in C such that H_2 is the seventh term in C_2. Let

$$C_2 = (S_2, G_2, E_2, L_2, M_2, N_2, H_2, \{a(i)\}_{i=1}^{H_2}, \{b(i)\}_{i=1}^{H_2}, \{a'(i)\}_{i=1}^{N_2}, \{b'(i)\}_{i=1}^{N_2})$$

with

$$S_2 = S_{g-L_2}^h + \ldots + S_{E_2-1}^h + S_{E_2}^h + S_{E_2+1}^h + \ldots$$
$$\ldots + S_{G_2-1}^h + S_{G_2}^h + S_{G+1}^h + \ldots + S_{g+M_2}^h.$$

Now $E_2 \neq 1$ and $G_2 \neq k_v$ because otherwise S^{h+1} has the complete winding form. Thus $E_2 - 1$ and $G_2 + 1$ exist.

There are four cases to be considered implied from 3) in the definition of C. These cases are:

i) $P_{a(H_2)}$ is the first point in $S_{E_2}^h$ and $P_{b(H_2)}$ is the first point in $S_{G_2}^h$,

ii) $P_{a(H_2)}$ is the last point in $S_{E_2}^h$ and $P_{b(H_2)}$ is the last point in $S_{G_2}^h$,

iii) $P_{a(H_2)}$ is the last point in $S_{E_2}^h$ and $P_{b(H_2)}$ is the first point in $S_{F_2}^h$ and

iv) $P_{a(H_2)}$ is the first point in $S_{E_2}^h$ and $P_{b(H_2)}$ is the last point in $S_{G_2}^h$. A similar result follows in all four cases.

Thus, only iv) is considered. Assume iv). Since S^{h+1} has property $W-U$ with left-finishing point P_1, the winding process of $\{a(i)\}_{i=1}^u$ winds into either $S_{E_2-1}^h$ or $S_{E_2}^h$, initially, with respect to the positive integer H_2+1, and similary the winding process of $\{b(i)\}_{i=1}^u$ winds into either $S_{G_2}^h$ or $S_{G_2+1}^h$ with respect of H_2+1. We now prove the following statement. If $D < u$ is a positive integer greater than H_2 such that for no positive integer m, $H_2 < m < D$, $P_{a(m)}$ is the first point in $S_{E_2-1}^h$ or the last point in $S_{E_2}^h$ and if $P_{a(D)}$ is the first point in $S_{E_2}^h$, then $P_{b(D)}$ is the last point in $S_{G_2}^h$. Consider the case where the winding process of $\{a(i)\}_{i=1}^u$ winds into $S_{E_2-1}^h$ with respect to some integer J, $H_2 < J < u$, without winding to the first point in $S_{E_2-1}^h$ and winds back to the first point in $S_{E_2}^h$ with respect to the positive integer J', and the winding process of $\{b(i)\}_{i=1}^u$ winds into $S_{G_2}^h$ with respect to H_2+1. Since both $S_{E_2-1}^h$ and $(S_{G_2}^h)^R$ begin and end with P_c, the sequence $\{a''(i)\}_{i=1}^{J'-(H_2+1)}$ and $\{b''(i)\}_{i=1}^{J'-(H_2+1)}$ where $a''(i) = a(i)+H_2$ and $b''(i) = b(i)+H_2$, $1 \leqslant i \leqslant J'-(H_2+1)$, determine, respectively, left- and right-winding sequence for $(S_{E_2-1}^h) \oplus (S_{G_2}^h)^R$ to have property $W-U$ extended with starting point the last point in $S_{E_2-1}^h$. Then by applying the induction hypothesis concerning property $W-3$ to $S_{E_2-1}^h \oplus (S_{G_2}^h)^R$, it follows that the winding process of $\{b(i)\}_{i=1}^u$ must wind into $S_{G_2}^h$ with respect to J without winding to the first point in $S_{G_2}^h$ and then must wind to (back to) the last point in $S_{G_2}^h$ so that $P_{b(D)} = P_{b''(J-(H_2+1))}$ is the last point in $S_{G_2}^h$. By considering the other cases involving $S_{E_2-1}^h \oplus S_{G_2+1}^h$, $S_{E_2}^h \oplus (S_{G_2}^h)^R$, and $S_{E_2}^h \oplus S_{G_2+1}^h$, a similar result is seen so that our statement follows. Thus, for some positive integer $H_3 \leqslant u$ the winding process of $\{a(i)\}_{i=1}^u$ must wind to either the first point of $S_{E_2-1}^h$ or the last point of $S_{E_2}^h$ and correspondingly the winding process of $\{b(i)\}_{i=1}^u$ must, by use of the induction hypothesis concerning property $W-4$, winds either to the first point of $S_{G_2}^h$ or to or the last point in $S_{G_2+1}^h$. First say the first point in $S_{E_2-1}^h$ and the last point in $S_{G_2+1}^h$. But then by the induction hypothesis concerning property $W-2$, $S_{E_2-1}^h = (S_{G_2+1}^h)^R$. But then there is an element c_3 in C that has H_3 as its seventh term and $H_3 > H_2$ which involves a contradiction.

Presently under these cases the first term in c_3, namely S_3, would be

$$S_3 = \oplus \sum_{I=L_2-1}^{M_2+1} S_{g-I}^h, \quad \text{or} \quad S_3 = \oplus \sum_{I=L_2}^{M_2+1} S_{g-I}^h,$$

or

$$S_3 = \oplus \sum_{I=L_2-1}^{M_2} S_{g-I}^h, \quad \text{or} \quad S_3 = \oplus \sum_{I=L_2}^{M_2} S_{g-I}^h$$

depending on the relationship of $g - L_2$ to E_2 and correspondingly G_2 to M_2. Also $\{a'(i)\}_{i=1}^{H_2}$ and $\{b'(i)\}_{i=1}^{H_2}$ are extended by one element namely $a'_{(H_2+1)} = E_2 - 1$ and $b'_{(H_2+1)} = G_2 + 1$, respectively, to form new left- and right-*complete* winding sequences for S_3. Property 5) of the definition of the winding form is satisfied with respect to S_3 for here $S_{E_2-1}^h = L'_{a(N_2+1)}$, $S_{G_2+1}^h = L'_{b(N_2+1)}$, with $L'_{a(N_2+1)} = (L'_{b(N_2+1)})^R$, and $b'_{(N_2)} = G_2$, $a'_{(N_2)} = E_2$, $b'_{(N_2+1)} = G_2+1$, and $a'_{(N_2+1)} = E_2 - 1$; so that as property 5) of the definition of the complete winding form requires, $b'(N_2) \neq b'(N_2+1)$ and $a'(N_2) \neq a'(N_2+1)$. It is noted that the other properties of the definition of the winding form are satisfied with respect to S_3. Now another case to consider is that the winding process of $\{a(i)\}_{i=1}^u$ winds (as before) to the first point of $S_{E_2-1}^h$, but the winding process of $\{b(i)\}_{i=1}^u$ winds to the first point in $S_{G_2}^h$. It then follows by the induction hypothesis concerning property $W-2$ and considering (similarly as before) $S_{E_2-1}^h \oplus (S_{G_2}^h)^R$ as having property $W-U$ extended as specified in the induction hypothesis concerning property $W-2$ that $S_{E_2-1}^h = [(S_{G_2}^h)^R]^R = S_{G_2}^h$.

It is seen that property 4) of the definition of the winding form is satisfied (as well as the others needed) and that once again there is a contradiction concerning H_2. The other cases concerning $S_{E_2-1}^h$, $S_{E_2}^h$, $S_{G_2}^h$, and $S_{G_2+1}^h$ produce a similar contradiction about H_2. Thus there is a contradiction involved in assuming $E_2 = 1$ or $E_2 \neq 1$. A similar contradiction is reached under subcase 2 where P_s is the last point in S_g^h.

A similar series of contradictions are reached if case 2 is assumed where P_r is the right-finishing point. Thus S has property $W-1$ with respect to $h+1$.

Next, suppose that S does not have property $W-2$ with respect to $h+1$. By Lemma I, if Q is a finite point sequence, then $Q+Q^R = \{q_i\}_{i=1}^{n_1}$ has property $W-U$ extended with left-finishing point q_1, right-finishing point q_{n_1}, and starting point the last point in Q; thus, it follows that there exist S_1^{h+1}, S_2^{h+1} in S_m^{h+1}, such that (case 1) $S_1^{h+1} = (S_2^{h+1})^R$, $S_2 = S_1^{h+1}+S_2^{h+1}$, and S_2 has property $W-U$ extended with left-finishing point the first point in S_2 or with right-finishing point the last point in S_2, and with starting point P_w a point that is not the last point in S_1^{h+1}, or such that (case 2) $S_1^{h+1} \neq (S_2^{h+1})^R$, $S_2 = S_1^{h+1}+S_2^{h+1}$, and S_2 has property $W-U$ extended with left-finishing point the first point in S_2 or with the right-finishing point the last point in S_2 and with starting point P_w. Under case 1, for example, if the left-finishing point of S_2 is the first point in S_2 and P_w belongs to S_1^{h+1}, then the winding process of one of the winding sequences associated with S_2 must wind out of S_1^{h+1} either on the left or right; thus, S_1^{h+1} does not have property $W-1$. However, this involves a contradiction to the result the S has property $W-1$ with respect to $h+1$. Now assume case 2. Let $S_2 = S_1^{h+1}+ +S_2^{h+1} = \oplus \sum_{i=1}^{e} L_i^h$ where $e \varepsilon I^+$ and each $L_i^h \varepsilon \bar{S}_m^h$. It follows in a similar manner as in the argument above showing that S has property $W-1$ with respect to $h+1$ that the starting point P_w can not be a point in some L_i^h that is not a first or last point in L_i^h. Furthermore, if P_w is a first or last point in some L_i^h, $1 \leqslant i \leqslant e$, then also argued as before it follows that S_2 has the complete winding form with L_i^h as the left-finishing component sequence, or with L_e^h as the right-finishing component sequence. However, this involves a contradiction with 3) in the hypothesis of Lemma IV. Thus, S has property $W-2$ with respect to $h+1$. Next suppose that S does not have property $W-3$ with respect to $h+1$. Then there is $S_1^{h+1} \varepsilon \bar{S}_m^{h+1}$ (case 1) or there exist S_2^{h+1}, $S_3^{h+1} \varepsilon \bar{S}_m^{h+1}$ (case 2) such that either $P = S_1^{h+1}$ or $P = S_2^{h+1}+S_3^{h+1}$ has property $W-U$ extended with starting point P_k, left-winding sequence $\{a(i)\}_{i=1}^{t}$, right-winding sequence $\{b(i)\}_{i=1}^{t}$, and such that for some positive integer m', $1 < m' < t$, $a_{(m')} = k$ and $b_{(m')} \neq k$, or for some positive integer n', $1 < n' < t$, $b_{(n')} = k$ and $a_{(n')} \neq k$. Consider subcase 1 of case 1 where $P = S_1^{h+1} = P_1, \ldots, P_k, \ldots, P_r = S_1^h+S_2^h+\ldots+S_{g-1}^h+S_g^h+S_{g+1}^h+S_{g+2}^h+\ldots+S_v^h$ where each $S_1^h \varepsilon \bar{S}_m^h$, and there is a first positive integer i_1, $1 < i_1 < t$, such that $a(i_1) = k$ and $b(i_1) \neq k$. Subcase 2 of case 1 is identical to subcase 1 of case 1 except there is a first positive integer i_2, $1 < i_2 < t$, such that $b(i_2) = k$ and $a(i_2) \neq k$. Let g be the first positive integer n such that P_k belongs to S_n^h, $1 < g < v$. It follows by the same argument in showing that S has property $W-1$ that P_k is the last point in S_g^h. If the winding process of $\{a(i)\}_{i=1}^{t}$ does not wind out of S_g^h with respect to some positive integer $\bar{v}$ greater than m where P_m is the first point in S_g^h, then by the use of the induction hypothesis concerning property $W-4$ the winding process of $\{b(i)\}_{i=1}^{t}$ could not wind out of S_{g+1}^h, and by the use of the induction hypothesis concerning property $W-3$, it would follow that $a(i_1) = b(i_1) = k$. Thus contradicting $a(i_1) = k$ and $b(i_1) \neq k$. Thus, the winding process of $\{a(i)\}_{i=1}^{k}$ does wind out of S_g^h and it follows in a similar manner that was argued in showing that L has property $W-1$ that $S_g^h = (S_{g+1}^h)^R$. Thus, it can be argued as was done in showing S has property $W-1$ that there is a set C of tuples of the form

$$(\bar{S}, G, E, L, M, N, H, \{a(i)\}_{i=1}^{H}, \{b(i)\}_{i=1}^{H}, \{a'(i)\}_{i=1}^{N}, \{b'(i)\}_{i=1}^{N})$$

defined exactly as in that argument.

Suppose that there is a $c_4 \varepsilon C$ such that

$$c_4 = (S_4, G_4, E_4, L_4, M_4, N_4, H_4, \{a(i)\}_{i=1}^{H_4}, \{b(i)\}_{i=1}^{H_4}, \{a'(i)\}_{i=1}^{N_4}, \{b'(i)\}_{i=1}^{N_4})$$

and $a(H_4) = a(i_1) = k$ and $b(H_4) = b(i_1) \neq k$. Since i_1 is the first positive integer n, $1 < n < t$, such that $a(n) = k$ and $b(n) \neq k$ and the starting point is P_k, it follows that $P_{a(i_1)} = P_{a(H_4)}$ is the last point in $S_g^h = S_{E_4}^h$. Thus, S_4 has the winding form with left-ending component sequence $S_{E_4}^h = S_g^h$ and right-ending component sequence $S_{G_4}^h \neq S_{g+1}^h$. It then follows by definition that S_1^{h+1} has the return winding form which contradicts 4) of the hypothesis of Lemma IV. Thus, there is no such $c_4 \varepsilon C$.

Thus, there is a greater positive integer T, $1 < T < i_1$ such that T is th seventh term for some $c_T \varepsilon C$. Let

$$c_T = (S_T, G_T, E_T, L_T, M_T, N_T, H_T, \{a(i)\}_{i=1}^{H_T=T}, \{b(i)\}_{i=1}^{H_T=T}, \{a'(i)\}_{i=1}^{N_T}, \{b'(i)\}_{i=1}^{N_T})$$

be such a tuple in C. But the winding process of $\{a(i)\}_{i=1}^{t}$ and $\{b(i)\}_{i=1}^{t}$ must exist with respect to positive integers greater than T and for a positive integer at least as great as i_1. Thus by a similar argument as was given in showing that S has property $W-1$, it follows that there is element $c' \varepsilon C$ whose seventh term T' is at least one more than T. Since $T \neq i_1$, this involves a contradiction with the definition of T. Thus, the assumptions of subcase 1 of case 1 have lead to a contradiction. A similar contradiction follows under subcase 2 of case 1. Furthermore, a similar series of contradictions follows under case 2 since by the hypothesis 4) of Lemma IV, no sum of 2 component sequences of $\bar{S}_m^{h+1}$ can have the return winding form with respect to h. Thus, S has property $W-3$ with respect to $h+1$.

Now suppose that S does not have property $W-4$ with respect to $h+1$. Then there exist S_y^{h+1}, S_z^{h+1} in $\bar{S}_m^{h+1}$, $1 < y, z < 2m$, such that $S_5 = S_y^{h+1}+S_z^{h+1} = \{P_i\}_{i=1}^{d}$, $d \varepsilon I^+$, such that S_5 has property $W-U$ extended with starting point P_k, left-winding sequence $\{a(i)\}_{i=1}^{t}$, right-winding sequence $\{b(i)\}_{i=1}^{t}$ and such that for some first positive integer $j_1 \leqslant t$, $P_1 = P_{a(j_1)}$ and $P_d \neq P_{b(j_1)}$ case 1), or such that for some first positive integer $j_2 \leqslant t$,

$P_{b(j_2)} = P_d$ and $P_1 \neq P_{a(j_2)}$ case 2). Since S has property $W-2$ with respect to $h+1$, it follows that $S_y^{h+1} = (S_z^{h+1})^R$ and P_k is the last point in S_y^{h+1}. Assume case 1.

Once again it can be argued as was done in showing that S has properties $W-1$, $W-2$, and $W-3$ that there is a set of C of tuples defined exactly as in those arguments where S_g^h (defined as before) is the last component sequence in the sum of component sequences from $\bar{S}_m^h$ forming S_y^{h+1}.

Suppose that there is $c_6 \varepsilon C$ such that

$$c_6 = (S_6, G_6, E_6, L_6, M_6, N_6, H_6, \{a(i)\}_{i=1}^{H_6}, \{b(i)\}_{i=1}^{H_6}, \{a'(i)\}_{i=1}^{N_6}, \{b'(i)\}_{i=1}^{N_6})$$

and $a(H_6) = 1 = a(j_1)$ with $b_{(H_6)} = b(aj_1) \neq d$.

Thus, S_6 has the winding form with left-ending component sequence $S_{E_6}^h$ the first component sequence in the sum of component sequences from $\bar{S}_m^h$ forming S_y^{h+1} and right-ending component sequence some component sequence in the sum of component sequences from $\bar{S}_m^h$ forming $S_5 = S_y^{h+1} + S_z^{h+1}$ that is not the last component sequence in that sum. However, this contradicts 3) in the hypothesis of this lemma since under these circumstances the right-finishing component sequence must be the last component sequence in the above sum. (according to 3 of the hypothesis) Thus, no such c_6 exist.

Thus, there is a greatest positive integer J, $1 < J < j_1$ such that J is the seventh term for some $c_J \varepsilon C$. But the winding process of $\{a(i)\}_{i=1}^{t}$ and $\{b(i)\}_{i=1}^{t}$ must exist with respect to positive integers greater than J and for a positive integer at least as great as j_1. Thus, by a similar argument as was given in showing that S has property $W-1$, it follows that there is an element $c_{J'} \varepsilon C$ whose seventh term J' is at least one more than J. Since $J' \neq j_1$, this involves a contradiction with the definition of J. Case 2 leads to a similar contradiction. Thus S has property $W-4$ with respect to $h+1$. Thus, Lemma IV follows.

LEMMA V. *Suppose that* $S = \{S_i^m, v_i, P_r^j, k_i, c_i\}$ *is an m-system of recursive sequences satisfying* 1), 2), 3), *and* 4) *of the hypothesis of Lemma* IV. 5) *Suppose that h is a positive integer,* $1 \leqslant h \leqslant 2m$, *and n is a positive integer greater than one. Then if* S_h^n *in* $\bar{S}_m^n$ *has the form:*

$$S_h^n = \oplus \sum_{i=1}^{K_{h-1}} S_{c_i^h}^{n-1} + S_h^{n-1} = \{P_i^n\}_{i=1}^{H(n)},$$

where each $S_{c_i^h}^{n-1} \varepsilon \bar{S}_m^{n-1}$, $1 \leqslant i \leqslant K_h$, *and* $H(n) \varepsilon I^+$, *there is a finite increasing subsequence* $I^n = \{m_n(i)\}_{i=1}^{n}$ *of* $\{1, 2, \ldots, H(n)\}$ *such that if j is a positive integer,* $1 \leqslant j \leqslant n$, $S_j = \{P_{m_n(j)}^n, P_{m_n(j)+1}^n, P_{m_n(j)+2}^n, \ldots, P_{H(n)}^n\}$ *does not have property* $W-U$ *extended with left-finishing point* $P_{m_n(j)}^n$, *the first point in* S_j.

Proof. The proof is given by induction on n. Suppose $n = 2$. Then

$$S_h^2 = \oplus \sum_{i=1}^{k_h-1} S_{c_i^h}^1 + S_h^1 = \{P_i^2\}_{i=1}^{H(2)}, \qquad H(2) \varepsilon I^+.$$

Define $\{m_2(i)\}_{i=1}^{2}$ by $m_2(1) = 1$ and $m_2(2) = k$, $1 < k < H(2)$, where k is the positive integer such that P_k^2 is the first point in S_h^1. Since S has property $W-1$ with respect to the positive integer one, $S_h^1 = \{P_{m_2(2)}^2 = P_k^2, P_{m_2(2)+1}^2, \ldots, P_{H(2)}^2\}$ does not have property $W-U$ extended with left-finishing point $P_{m_2(2)}^2 = P_k^2$. Since S has property $W-1$ with respect to the positive integer two, $S_h^2 = \{P_{m_2(1)}^2 = P_1^2, P_{m_2(1)+1}^2, \ldots, P_{H(2)}^2\}$ does not have property $W-U$ extended with left-finishing point $P_{m_2(1)}^2 = P_1$. Thus, Lemma V is true for $n = 2$.

By way of contradiction, let $L+1$ be the least positive integer for which Lemma V is not true. Consider

$$S_h^{L+1} = \oplus \sum_{i=1}^{k_h-1} S_{c_i^h}^L + S_h^L = \{P_i^{L+1}\}_{i=1}^{H(L+1)},$$

$$H(L+1) \varepsilon I^+, \quad I^L = \{m_L(i)\}_{i=1}^{L} \quad \text{and} \quad S_h^L = \{P_i^L\}_{i=1}^{H(L)}, \quad H(L) \varepsilon I^+.$$

Let v be the positive integer, $1 < v < H(L+1)$, such that $\{P_i^{L+1}\}_{i=v}^{H(L+1)} = \{P_i^L\}_{i=1}^{H(L)}$. Since S has property $W-1$ with respect to $L+1$, $S_h^{L+1} = \{P_i^{L+1}\}_{i=1}^{H(L+1)}$ does not have property $W-U$ extended with left-finishing point P_1^{L+1}.

Define $I^{L+1} = \{m_{L+1}(i)\}_{i=1}^{L+1}$ by

1) $m_{L+1}(1) = 1$,

2) $m_{L+1}(j) = m^L(j-1) + v - 1$; $j = 2, 3, \ldots, L+1$.

Since for each j, $2 \leqslant j \leqslant L+1$, $\{P_i^{L+1}\}_{i=m_{L+1}(j)}^{H(L+1)} = \{P_i^L\}_{i=m_L(j-1)}^{H(L)}$, and $\{P_i^L\}_{i=m_L(j-1)}^{H(L)}$ does not have property $W-U$ extended with left-finishing point $P_{m_L(j-1)}^L$, it follows that $\{P_i\}_{i=m_{L+1}(j)}^{H(L+1)}$, does not have property $W-U$ extended with left-finishing point $P_{m_{L+1}(j)}^{L+1}$. Thus, Lemma V is true for $L+1$, and this involves a contradiction to the definition of $L+1$. Thus, Lemma V is true for all positive integers.

COROLLARY. *The statement of Lemma* V *remains true if "property* $W-U$ *extended" is replaced with "property* $W-U$".

Proof. Suppose the hypothesis of Lemma V. It is claimed then if $n \varepsilon I^+$, $n > 1$ then the sequence I^n required to satisfy the conclusion of Lemma V is the same sequence required to satisfy the conclusion of the corollary to Lemma V. By way of contradiction to this claim, suppose that there is a positive integer n, $n > 1$, and a positive integer j, $1 \leqslant j \leqslant n$, with $I^n = \{m_n(i)\}_{i=1}^{n}$ and $S_j = \{P_{m_n(i)}^n\}_{i=j}^{H(n)}$ defined as in the conclusion of Lemma V, such that S_j has property $W-U$ with left-finishing point $P_{m_n(j)}^n$, the first point in S_j.

Since $P^n_{m_n(j)}$ is the first point in S_j, $P^n_{m_n(j)}$ is not the starting point. Since $P^n_{m_n(j)}$ is not the starting point and since by Lemma V S_j does not have property $W-U$ extended with left-finishing point $P^n_{m_n(j)}$, it follows by Lemma I that S_j does not have property $W-U$ with starting point $P^n_{m_n(j)}$. This involves a contradiction. Thus, the above claim is true and the corollary follows.

5. The Main Example. In this section it is shown that the example given in [2] is not weakly chainable. Throughout this section the example of [2] is given by $Y = \overset{\lim}{\leftarrow}\{Y_n, g^m_n\}$ where each Y_n is the simple triod $T = \{(r, \theta)\colon 0 \leqslant r \leqslant 1 \text{ and } \theta = 0,\ \theta \leqslant \frac{1}{2}\pi \text{ or } \theta = \pi\}$ (in polar coordinates in the plane), and each $g^{n+1}_n\colon T \to T$ (referred to as the mapping $f\colon T \to T$ in [2]) is defined as follows:

$$g^{n+1}_n(x, \tfrac{1}{2}\pi) = \begin{cases} (1-4x, \pi) & \text{if } 0 \leqslant x \leqslant \frac{1}{4}, \\ (4x-1, \frac{1}{2}\pi) & \text{if } \frac{1}{4} \leqslant x \leqslant \frac{1}{2}, \\ (3-4x, \frac{1}{2}\pi) & \text{if } \frac{1}{2} \leqslant x \leqslant \frac{3}{4}, \\ (4x-3, 0) & \text{if } \frac{3}{4} \leqslant x \leqslant 1, \end{cases}$$

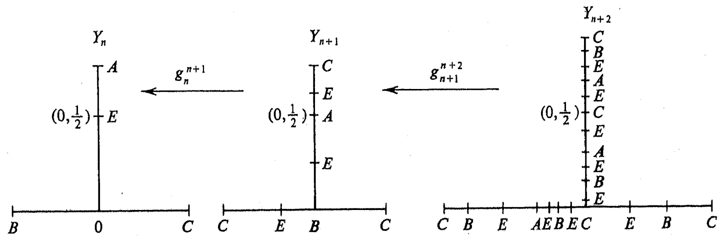
$$g^{n+1}_n(x, \pi) = \begin{cases} (1-3x, \pi) & \text{if } 0 \leqslant x \leqslant \frac{1}{3}, \\ (3x-1, \frac{1}{2}\pi) & \text{if } \frac{1}{3} \leqslant x \leqslant \frac{1}{2}, \\ (2-3x, \frac{1}{2}\pi) & \text{if } \frac{1}{2} \leqslant x \leqslant \frac{2}{3}, \\ (3x-2, 0) & \text{if } \frac{2}{3} \leqslant x \leqslant 1, \end{cases}$$

$$g^{n+1}_n(x, 0) = \begin{cases} (1-2x, \pi) & \text{if } 0 \leqslant x \leqslant \frac{1}{2}, \\ (2x-1, 0) & \text{if } \frac{1}{2} \leqslant x \leqslant 1. \end{cases}$$

Denote by 0 the point $(0, 0) = (0, \frac{1}{2}\pi) = (0, \pi)$, by A the point $(1, \frac{1}{2}\pi)$, by B the point $(1, \pi)$, by C the point $(1, 0)$, and by E the point $(\frac{1}{2}, \frac{1}{2}\pi)$.

Suppose n is a positive integer. The following statements follow as a consequence of the nature of the mapping $g^{n+2}_n\colon Y_{n+2} \to Y_n$. The arc $\overline{CO} \subset Y^{n+2}_n$ contains a sequence $\{W_i\}^4_{i=1}$ of points, such that $W_i < W_{i+1}$, $1 \leqslant i \leqslant 4$, in the order $\overline{CO}$, $W_1 = C$, $W_4 = 0$, and $\{g^{n+2}_n(W_i)\}^4_{i=1}$ is the sequence C, B, E, C. The arc $\overline{OB} \subset Y^{n+2}_n$ contains a sequence $\{R_i\}^8_{i=1}$ of points such that $R_i < R_{i+1}$, $1 \leqslant i \leqslant 8$, in the order $\overline{OB}$, $R_1 = 0$, $R_8 = B$, and $\{g^{n+2}_n(R_i)\}^8_{i=1}$ is the sequence C, E, B, E, A, E, B, C. The arc $\overline{OE} \subset Y^{n+2}_n$ contains a sequence $\{H_i\}^7_{i=1}$ of points such that $H_i < H_{i+1}$, $1 \leqslant i \leqslant 7$, in the order $\overline{OE}$, $H_1 = 0$, $H_7 = E$, and $\{g^{n+2}_n(H_i)\}^7_{i=1}$ is the sequence C, E, B, E, A, E, C.

The arc $\overline{EA} \subset Y^{n+2}_n$ contains a sequence $\{V_i\}^6_{i=1}$ of points such that $V_i < V_{i+1}$, $1 \leqslant i \leqslant 6$, in the order $\overline{EA}$, $V_1 = E$, $V_6 = A$, and $\{g^{n=2}_n(V_i)\}^6_{i=1}$ is the sequence C, E, A, E, B, C (see Fig. 1).



Points labeled C, B, E or A in Y_{n+1} or Y_{n+2} denote points thrown to C, B, E, or A in Y_n respectively, by g^{n+1}_n for Y_{n+1} and by g^{n+2}_n for Y_{n+2}.

We now state and indicate the proof of the following.

I) Suppose $S = \{S^m_i, V_i, P^j_r, K_i, C_i\}$ is the m-system of recursive sequences defined by

$$S^1_1 = P^1_1, P^1_2, P^1_3, P^1_4 = C, B, E, C,$$
$$S^1_2 = P^2_1, P^2_2, \ldots, P^2_8 = C, E, B, E, A, E, B, C,$$
$$S^1_3 = P^3_1, P^3_2, \ldots, P^3_7 = C, E, B, E, A, E, C,$$
$$S^1_4 = P^4_1, P^4_2, \ldots, P^4_6 = C, E, A, E, B, C,$$

and for each $n \geqslant 2$,

$$S^n_1 = S^{n-1}_1 + S^{n-1}_2, \quad S^n_2 = (S^{n-1}_2)^R + S^{n-1}_3 + (S^{n-1}_3)^R + (S^{n-1}_1)^R,$$

$$S^n_3 = (S^{n-1}_2)^R + S^{n-1}_3 + S^{n-1}_4 \quad \text{and} \quad S^n_4 = (S^{n-1}_4)^R + (S^{n-1}_3)^R + (S^{n-1}_1)^R.$$

Then

1) If $n \geqslant 3$, $(S^n_1)^R$ begins with the pattern $S^1_1 = C, B, E, C$.

2) If L and n are positive integers and $n \geqslant 2$:

a) $\overline{OC} \subset Y_{L+n}$ contains a sequence $W_1, W_2, \ldots, W_{n(1)}$, of points such that $W_i < W_{i+1}$, $1 < i < n(1)$, in the order $\overline{CO}$, $\{g^{L+n}_L(W_j)\}^{n(1)}_{j=1} = S^n_1$, and $\{W_i\}^{n(1)}_{i=1}$ contains the only points in $\overline{CO} \subset Y_{L+n}$ mapped by g^{L+n}_L into the set $\{A, B, C, E\} \subset Y_L$.

b) $OB \subset Y_{L+n}$ contains a sequence $\{R_i\}^{n(2)}_{i=1}$ of points such that $R_i < R_{i+1}$, $1 \leqslant k \leqslant n(2)$, in the order $\overline{OB}$, $\{g^{L+n}_n(R_j)\}^{n(2)}_{j=1} = S^n_2$, and $\{R_i\}^{n(2)}_{i+1}$ contains the only points in $\overline{OB}$ mapped by g^{L+n}_L into $\{A, B, C, E\} \subset Y_L$.

c) $\overline{OE} \subset Y_{L+n}$ contains a sequence of points $\{H_i\}^{n(3)}_{i=1}$ such that $H_i < H_{i+1}$, $1 \leqslant i \leqslant n(3)$, in the order $\overline{OE}$, $\{g^{L+n}_L(H_j)\}^{n(3)}_{j=1} = S^n_3$, and $\{H_j\}^{n(3)}_{j=1}$ contains the only point in $\overline{OE}$ mapped by g^{L+n}_L into $\{A, B, C, E\} \subset Y_L$.

d) $\overline{EA} \subset Y_{L+n}$ contains a sequence $\{V_i\}^{n(4)}_{i=1}$ of points such that $V_i < V_{i+1}$,

$1 \leqslant i < n(4)$, in the order of $\overline{EA}$, $\{g_L^{L+n}(V_j)\}_{j=1}^{n(4)} = S_4^n$, and $\{V_j\}_{j=1}^{n(4)}$ contains the only points in $\overline{EA}$ mapped by g_L^{L+n} into $\{A, B, C, E\} \subset Y_L$.

The proof of I) follows directly from the definitions of g_L^{L+n} and S using mathematical induction.

We now state and prove the following.

II) a) If n is a positive integer, $n \geqslant 2$, and $(S_1^n)^R = S_5^n = \{g_L^n(W_i)\}_{i=n(1)}^{1} = \{g_L^n(Q_i)\}_{i=1}^{n(1)}$, then there is a finite increasing subsequence $\{m(i)\}_{i=1}^{n}$ of $\{1, 2, \ldots, n(1)\}$ such that if j is a positive integer, $1 \leqslant j \leqslant n$, and $S^j = \{g_L^n(Q_{m(j)}), g_L^n(Q_{m(j)+1}), \ldots, g_L^n(Q_{n(1)})\}$, S^j does not have property $W-U$ with left-finishing point $g_L^n(Q_{m(j)})$ the first point in S^j.

b) If j is in $\{1, 2, \ldots, n\}$, $\{g_L^n(Q_{m(j)}), g_L^n(Q_{m(j)+1}), g_L^n(Q_{m(j)+2}), g_L^n(Q_{m(j)+3})\} = \{C, B, E, C\}$.

Now II) a) follows directly from the corollary to Lemma V of section 4 since S is an m-system of recursive sequences satisfying the hypothesis of that corollary. In particular, it is noted that condition 5) of the hypothesis to the corollary to Lemma V of section 4 is satisfied since $(S_1^n)^R = S_5^n = (S_1^{n-1} + S_2^{n-1})^R = (S_2^{n-1})^R + (S_1^{n-1})^R$.

Also, II) b) follows directly from 1) of I) above.

Theorem 1 of section 3 is now applied in order to show that Y is not weakly chainable. The definition of Y implies that 1) in the hypothesis of Theorem 1) is satisfied. By defining $G_1 = 0$, $G_2 = C$, $G_3 = B$, $G_4 = E$, and $G_5 = A$, $\delta = \min\{d(G_i, G_j)\colon i \neq j,\ 1 \leqslant i, j \leqslant 4\}$, and $V_L = \overline{OC}$ for each positive integer L, it is seem that 2) in the hypothesis of Theorem 1) is satisfied. Consider 3) in the hypothesis of Theorem 1). Suppose L and n are positive integers and $n > L+1$. It follows from 2) a) and II) a) that $\{Q_i\}_{i=1}^{n(1)}$ and $\overline{OC}$ in 2) a) and II) a) satisfy the hypothesis for $\{Q_i\}_{i=1}^{b}$ and W_n in 3) 1′) and 3) 2′) in the hypothesis of Theorem 1. It follows from properties of the mapping g_L^n that $\{Q_i\}_{i=1}^{n(1)}$ and $\overline{OC}$ in 2) a) and II) a) satisfy the hypothesis for $\{Q_i\}_{i=1}^{b}$ and W_n in 3) 3′), 3) 4′) and 3) 5′) in the hypothesis of Theorem 1. Thus all the conditions to Theorem 1) are satisfied and it follows that Y is not weakly-chainable.

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