One can easily check that $\mathfrak{A}_0$ and $\mathfrak{A}_1$ belong to K, whence $S(\mathfrak{A})$ belongs to $R(K)$. Theorem III of [5] guarantees that $S(\mathfrak{A})$ is a semilattice of groupoids which belong to K, and one may check that the semilattice decomposition of $S(\mathfrak{A})$ into $\mathfrak{A}_0$ and $\mathfrak{A}_1$ is the only semilattice decomposition of $S(\mathfrak{A})$ into groupoids which belong to K. Therefore $S(\mathfrak{A})$ cannot be a Płonka sum of groupoids which belong to K. This also provides an example of an equational class K where $R(K)$ properly contains the class of all algebras which are Płonka sums of algebras in K.

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Fixed points and nonexpansive retracts in locally convex spaces*

by

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Abstract. Locally convex topological vector spaces can be normed over a topological semifield. Using this norm, Banach operators and nonexpansive mappings are defined and several fixed point theorems are proven. Also, it is shown for strictly convex spaces that, under suitable conditions, the fixed point set of a nonexpansive map is a nonexpansive retract.

0. Introduction. The concept of a topological semifield was introduced by Antonovskii, Boltyanskii and Sarymsakov [1]. They observed that it is possible to define a semifield valued "norm" for certain topological vector spaces; in particular the class of Hausdorff locally convex spaces. The aim of the present paper is to prove fixed point theorems in this class of spaces for Banach operators and nonexpansive mappings. Also we show that for strictly convex spaces, under suitable conditions, the fixed point set of a nonexpansive mapping is a nonexpansive retract.

These results extend those of Bahtin [2], Cain and Nashed [5], Hicks and Kubicek [9], Chandler and Faulkner [6], Bruck [3], [4] and others.

Let Δ be a nonempty set and $R^{\Delta} = \prod_{\alpha\in\Delta} R_\alpha$ be the product of the real line with the product topology. Addition and multiplications in R^{Δ} are defined pointwise. A partial ordering is defined by the cone $R^{\Delta}_{+} = \{f\colon f(\alpha) \geqslant 0,\ \alpha\in\Delta\}$. A general introduction to the space R^{Δ} may be found in [1].

If E is a real locally convex space, whose topology is generated by a family $\{\varrho_\alpha\colon \alpha\in\Delta\}$ of continuous seminorms, then the function $\varrho\colon E\to R^{\Delta}_{+}$ defined by $[\varrho(x)](\alpha) = \varrho_\alpha(x)$, $x\in E$, $\alpha\in\Delta$, satisfies

(1) $$\varrho(x) \geqslant 0,$$

(2) $$\varrho(\lambda x) = |\lambda|\,\varrho(x),$$

(3) $$\varrho(x+y) \leqslant \varrho(x)+\varrho(y)$$

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where "$\leqslant$" denotes the natural order induced by R_+^{Δ}. In case E is a T_2 space, (1) becomes

$$(1)' \qquad \varrho(x) = 0 \text{ if and only if } x = 0,$$

and we see that ϱ satisfies the axioms of a norm. The topology t_ϱ generated by ϱ is the original topology, where a t_ϱ neighbourhood of x is of the form $\Omega(x, U) = \{y: \varrho(x-y) \in U\}$, where U is a neighbourhood of zero in R^{Δ}.

1. Banach operators. We prove fixed point theorems for Banach operators, which is a variant of a concept introduced by Cheney and Goldstein [7, Theorem 1]. These results extend those of Cain and Nashed [5] and of Taylor [14].

DEFINITION 1.1. Let K be a nonempty subset of E. A mapping T: $K \to K$ is said to be a *Banach operator* if there exists a constant k such that $0 \leqslant k < 1$ and for $x \in K$, $\varrho(T^2x - Tx) \leqslant k\varrho(Tx - x)$. T is said to be a *contraction* if there exists a constant k such that $0 \leqslant k < 1$ and for each $x, y \in K$, $\varrho(Tx - Ty) \leqslant k\varrho(x-y)$.

Remark 1.2. Every contraction mapping is a Banach operator (in fact, let $y = Tx$) but not conversely. This is easily seen by letting $E = R$ and K a nonempty subset of R. Define T: $K \to K$ by $T(x) = x^2$. Then T is a contraction on any closed interval $[a, b] \subset (\frac{1}{2}, -\frac{1}{2})$, but a Banach operator on any $[a, b] \subset (-1, 1)$.

Remark 1.3. A Banach operator need not be continuous nor need its fixed points be unique. For if $E = R$ and $K = [0, 1]$, define T: $K \to K$ as follows: $T(x) = 0$, $x \in [0, \frac{1}{2})$, $T(x) = \frac{3}{4}$, $x \in [\frac{1}{2}, 1]$. Then clearly T is a discontinuous Banach operator with fixed points 0 and $\frac{3}{4}$.

DEFINITION 1.4. A sequence $\{x_n\}$ is said to be a *Cauchy sequence* if for each neighbourhood U of 0 in R^{Δ}, there exists an integer M such that $\varrho(x_m - x_n) \in U$, for all $n, m \geqslant M$. We say that E is *sequentially complete* if every Cauchy sequence converges in E. E is said to be *quasicomplete* if closed bounded sets are complete.

LEMMA 1.5. *Let E be Hausdorff and sequentially complete. Let T: $E \to E$ be a continuous Banach operator, then T has a fixed point.*

Proof. Let $x_0 \in E$. It follows by induction that

$$\varrho(T^{n+1}x_0 - T^n x_0) \leqslant k^n \varrho(x_0 - Tx_0).$$

Let m and n be two positive integers such that $m > n$, then

$$\varrho(T^n x_0 - T^m x_0) \leqslant \sum_{i=1}^{m-n} k^{n+i-1} \varrho(x_0 - Tx_0).$$

$\sum_{i=1}^{\infty} k^i$ converges, since $0 \leqslant k < 1$, so, given $\varepsilon > 0$, we can find n such that for all $m \geqslant n$, $\sum_{i=0}^{m-n} k^{n+i} \varrho(x_0 - Tx_0) < \varepsilon$. Thus $\varrho(T^n x_0 - T^m x_0) < \varepsilon$ for all $m \geqslant n$ and $\{T^n x_0\}$ is a Cauchy sequence which converges to some point y. Using the continuity of T and Hausdorff property of E, we have that $Ty = y$.

From Remark 1.2 and Lemma 1.5 we have the following

COROLLARY 1.6. [5, Theorem 2.2]. *Suppose D is a sequentially complete subset of E and the mapping T: $D \to D$ is a contraction. Then T has a unique fixed point in D.*

DEFINITION 1.7. A subset K of E is *starshaped* provided there is a point $p \in K$ such that for each $x \in K$, the segment joining x to p is contained in K. Such a point p will be called a *star centre* of K.

Clearly if K is convex, then every point in K is a star centre of K.

THEOREM 1.8. *Let K be a compact subset of E and T: $K \to K$ be a continuous mapping. Suppose*

(i) *there exists $q \in K$ and a fixed sequence of positive real numbers k_n ($k_n < 1$) converging to 1, such that $(1-k_n)q + k_n T(x) \in K$ for each $x \in K$: further for each $x \in K$ and k_n, $\varrho\big(T((1-k_n)q + k_n T(x)) - Tx\big) \leqslant \varrho((1-k_n)q + k_n T(x) - x)$; or,*

(ii) *K is starshaped with reference to $q \in K$; further, there exists $\varepsilon > 0$ such that for all $x, y \in K$, $\varrho(x-y) < \varepsilon$ implies $\varrho(Tx - Ty) \leqslant \varrho(x-y)$.*

Then T has a fixed point.

Proof. Define the mapping T_n by $T_n(x) = (1-k_n)q + k_n T(x)$, $x \in K$. If condition (i) obtains it is easily checked that each T_n is a continuous Banach operator, which by Lemma 1.5 has a fixed point $y_n \in K$. By the compactness of K, there is a subnet $\{y_\alpha\}_{\alpha \in D}$ which converges to $y \in K$. The corresponding net $\{k_\alpha\}_{\alpha \in D}$ converges to 1. Now $y_\alpha = T_\alpha y_\alpha = (1-k_\alpha)q + k_\alpha T(y_\alpha)$; so by continuity of T, $Ty = y$.

If condition (ii) holds, the result follows from [12, Theorem 2].

Remark 1.9. If K is starshaped about q, then the result [14, Corollary 2.3] follows from Theorem 1.8. Hypothesis (i) above weakens the starshaped assumption as can be seen from the following example.

EXAMPLE 1.10. Let K be the set $\{(0, y): y \in [-1, 1]\} \cup \{(1-1/n, 0): n \in N\} \cup \{(1, 0)\}$ with the metric induced by the norm $\|(x, y)\| = |x| + |y|$. Define the map T: $K \to K$ as follows: $T(0, y) = (0, -y)$, $T(1-1/n, 0) = (0, 1-1/n)$, $T(1, 0) = (0, 1)$. We can apply Theorem 1.8 with condition (i) to T with the choice $q = (0, 0)$, $k_n = 1-1/n$, $n = 1, 2, \ldots$, so that the existence of a fixed point for T is ensured, though K is not starshaped.

2. Nonexpansive mappings. In this section we prove fixed point theorems for nonexpansive mappings. The results of Hicks and Kubicek [9] are extended. Also we obtain as corollaries of our results, results of Chandler and Faulkner [6].

DEFINITION 2.1. Let C be a nonempty subset of E. A mapping $T\colon C \to E$ satisfies the *conditional fixed point property* (CFPP) if either T has no fixed points, or T has a fixed point in every closed starshaped subset which it leaves invariant.

DEFINITION 2.2. Let C be a nonempty subset of E and let $T\colon C \to E$ be a mapping (not necessarily continuous). We say that T is *demicompact* if each bounded sequence $\{x_n\}$ in C such that $(I-T)(x_n)$ converges has a convergent subsequence.

DEFINITION 2.3. Let C be a subset of E and $T\colon C \to C$. T is called *nonexpansive* if $\varrho(Tx-Ty) \leqslant \varrho(x-y)$ for all $x, y \in C$.

The following theorem extends results of Hicks and Kubicek [9, Theorem 3].

THEOREM 2.4. *Suppose E is sequentially complete and C is a subset of E. If $T\colon C \to C$ is nonexpansive then T satisfies* (CFPP) *if any one of the following holds*:

(i) *$(I-T)(K)$ is closed whenever K is bounded, closed and starshaped subset of C*;

(ii) *T is demicompact and continuous*;

(iii) *C is weakly compact and T is affine.*

Proof. Suppose K is a closed bounded starshaped subset of C.

(i) Let p be the star centre of K. For each t, $0 < t < 1$, define $T_t(x) = tT(x)+(1-t)p$, for $x \in K$. Since K is starshaped, each T_t clearly maps K into itself. Also T_t is a contraction, in fact $\varrho\big(T_t(x)-T_t(y)\big) = \varrho\big(tT(x)-tT(y)\big) \leqslant t\varrho(x-y)$ for all $x, y \in K$. Since K is complete, it follows that T_t has a unique fixed point x_t in K. Now $(I-T)(x_t) = x_t - Tx_t = x_t - (1/t)\big(T_t x_t - (1-t)p\big) = (1-1/t)(x_t - p)$ which clearly tends to zero in E as $t \to 1$, since K is bounded. Since $(I-T)(K)$ is closed, there exists a $x \in K$ such that $x - Tx = 0$, so x is a fixed point of T.

(ii) We show that $(I-T)(K)$ is closed and the result follows from (i). Let $\{(I-T)x_\alpha\colon \alpha \in D\}$ be a net in $(I-T)(K)$ such that $(I-T)(x_\alpha) \to y$. Since T is demicompact, $\{x_\alpha\}$ contains a convergent subnet which we also denote by $\{x_\alpha\}$. Since K is closed we have $x_\alpha \to x$, where x is some point of K. By the continuity of T it follows that $(I-T)(x_\alpha) \to (I-T)x$. Thus $(I-T)x = y$.

(iii) The proof is contained in [9, Theorem 3 (iii)].

DEFINITION 2.5. Let K be a nonempty subset of E and let $F\colon K \to K$ be a family of functions. F is said to be *equicontinuous* on K if for each $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that for all $x, y \in K$, if $\varrho(x-y) < \delta$ then $\varrho(Tx-Ty) < \varepsilon$ for all $T \in F$.

DEFINITION 2.6. A subset A of E is called a *retract* of E if there exists a continuous mapping $r\colon E \to A$ such that $r(a) = a$ for all $a \in A$. A subset A of E is a *nonexpansive retract* of E if either $A = \emptyset$ or there is a retraction of E onto A which is nonexpansive.

LEMMA 2.7. *Let K be a nonempty compact subset of E. Let $T\colon K \to K$ be a nonexpansive mapping. Then there exists a nonexpansive retraction $r\colon K \to A$, where $A = \bigcap\{T^n\colon n \geqslant 1\}$.*

Proof. The family $\{T^n\}$ is equicontinuous; for, given any $\varepsilon > 0$, choose $\delta = \varepsilon$ then $\varrho(x-y) < \delta$ implies $\varrho(T^n x - T^n y) \leqslant \varrho(x-y) < \varepsilon$ for all $n \geqslant 1$. Now, the retraction r guaranteed by [15, Theorem 2] is the pointwise limit of the mappings of the form T^n. Thus for any $x, y \in K$ and any $\varepsilon > 0$ there exists an n such that $\varrho(rx - T^n x) < \varepsilon$ and $\varrho(T^n y - ry) < \varepsilon$. Hence

$$\varrho(rx-ry) \leqslant \varrho(rx-T^n x)+\varrho(T^n x-T^n y)+\varrho(T^n y-ry) \leqslant \varrho(x-y)+2\varepsilon.$$

Since ε was arbitrary this shows that $\varrho(rx-ry) \leqslant \varrho(x-y)$ and thus r is nonexpansive.

DEFINITION 2.8. E is said to be *strictly convex* if for $x, y \in E$, from $x \neq y$ and $\varrho(x) = \varrho(y)$ follows $\varrho\dfrac{(x+y)}{2} < \varrho(x)$, i.e., $\left[\varrho\left(\dfrac{x+y}{2}\right)\right](\alpha) < [\varrho(x)](\alpha)$ for some $\alpha \in \Delta$.

THEOREM 2.9. *Let E be strictly convex and C be a nonempty compact subset of it. Let $T\colon C \to C$ be a nonexpansive mapping. If there exists $n \geqslant 1$ such that $T^n(C) \cap \partial_1 C = \emptyset$, then T has a fixed point in C, here $\partial_1 C$ is the boundary of C in the closed convex hull of C.*

Proof. Let $D = \bigcap_{n=1}^{\infty} T^n(C)$. Then D is compact. By Lemma 2.7 there exists a nonexpansive retraction $r\colon C \to D$. We shall show that D is convex. Suppose not, then there exist x and y in D such that $[x, y] = \{\alpha x + (1-\alpha)y\colon 0 \leqslant \alpha \leqslant 1\}$ is not entirely contained in D. Let $\beta = \sup\{\gamma\colon \alpha x + (1-\alpha)y \in D,\ 0 \leqslant \alpha \leqslant \gamma\}$. Since D is compact, $z = \beta x + (1-\beta)y \in D$. Now if $[x, y]$ contains no points of $C \setminus D$, then z is a ∂_1-boundary point of C in $D \subseteq T^n(C)$ for all n, contradicting the hypothesis. Hence there exists a point $w \in [x, y] \cap \{C \setminus D\}$. But then we will have $\varrho(rx-rw)+\varrho(rw-ry) \leqslant \varrho(x-w)+\varrho(w-y) = \varrho(x-y)$. Since $rx = x$ and $ry = y$, it follows that

$$\varrho(x-rw)+\varrho(rw-y) \leqslant \varrho(x-y). \tag{1}$$

Let us observe that $rw \neq w$, otherwise w will belong to D. The mapping r is nonexpansive, so rw is not collinear with x, y for if $w = ax+(1-a)y$ and $rw = bx+(1-b)y$ we will have $a \neq b$. Suppose $b < a < 1$, then $\dfrac{1-b}{1-a} > 1$.

Thus

$$\varrho(x-w) = \varrho\big(x-ax-(1-a)y\big) = (1-a)\varrho(x-y),$$
$$\varrho(rx-rw) = \varrho\big(x-bx-(1-b)y\big) = (1-b)\varrho(x-y).$$

Hence

$$\varrho(rx-rw) = (1-b)\varrho(x-y) = \frac{(1-b)}{1-a}(1-a)\varrho(x-y)$$
$$= \left(\frac{1-b}{1-a}\right)\varrho(x-w) > \varrho(x-w),$$

a contradiction.

Suppose $0 < a < b$, then $b/a > 1$. Thus $\varrho(y-w) = a\varrho(x-y)$ and $\varrho(ry-rw) = \varrho(y-rw) = b\varrho(x-y)$. Hence $\varrho(ry-rw) = b\varrho(x-y) = (b/a)a\varrho(x-y) = (b/a)\varrho(y-w) > \varrho(y-w)$, a contradiction. Thus x, y, rw are not collinear. But then $\varrho(x-rw)+\varrho(rw-y) > \varrho(x-y)$, contradicting (1). Thus D must be convex. Now $T\colon D \to D$, so T has a fixed point by Tychonoff's Theorem.

Definition 2.10. Let X be a topological space and $T\colon X \to X$ be a map. A subset M of X is said to be an *attractor* for compact sets under T if (i) M is nonempty compact and T-invariant and (ii) for any compact subset C of X and any open neighbourhood U of M there exists an integer N such that $T^n(C) \subseteq U$ for all $n \geqslant N$.

Theorem 2.11. *Let C be a complete starshaped subset of E. Let $T\colon C \to C$ be a nonexpansive mapping. If there exists $M \subseteq C$, an attractor for compact sets, then T has a fixed point.*

Proof. Let y be the star centre of C and L be the closed convex hull of M. M is compact, so $\overline{\text{co}}[M \cup \{y\}]$ is compact. But $L \subseteq \overline{\text{co}}[M \cup \{y\}]$ so L is compact. Let $D = \bigcup_{n=0}^{\infty} T^n(L)$, where $T^0(L) = L$. Clearly D is T-invariant. We shall show that D is totally bounded.

Let U be any neighbourhood of 0. Then there is an open symmetric neighbourhood V of 0 such that $V+V+V \subset U$. As $M+V$ is an open neighbourhood of M and M is an attractor for compact sets under T, there exists a positive integer N such that $T^n(L) \subset M+N$ for all $n \geqslant N$. Now for the compact set $M \cup \bigcup_{n=0}^{N-1} T^n(L)$, there is a finite subset P of C such that

$$M \cup \bigcup_{n=0}^{N-1} T^n(L) \subseteq P+V.$$

Thus

$$\bigcup_{n=0}^{\infty} T^n(L) = \bigcup_{n=0}^{N-1} T^n(L) \cup \bigcup_{n=N}^{\infty} T^n(L) \subseteq P+V \cup (M+N)$$
$$\subseteq (P+V) \subset (P+V+V) = P+V+V.$$

It follows that

$$D = \bigcup_{n=0}^{\infty} T^n(L) \subseteq P+V+V+V \subseteq P+U.$$

Furthermore, D being a closed subset of a complete set is complete and, hence, D is compact. Let $R = \bigcap_{n=0}^{\infty} T^n(D)$. Then R is nonempty and compact, $T(R) = T(\bigcap_{n=0}^{\infty} T^n(D)) = \bigcap_{n=0}^{\infty} T^{n+1}(D) = R$, so by definition of M, $R \subseteq M$. By Lemma 2.7 there exists a nonexpansive retraction $r\colon D \to R$. Note that the nonexpansive map $F = T_0 r$ maps L continuously into itself. Hence by Theorem 2.4 there exists a point $y_0 \in L$ such that $F(y_0) = y_0$. As $ry_0 \in R$ and R is T-invariant, the equation $y_0 = F(y_0) = T(ry_0)$ shows that $y_0 \in R$. Thus $y_0 = ry_0$, since r is an identity map on R. Thus $y_0 = T(ry_0) = Ty_0$. This completes the proof.

Corollary 2.12. *Let E be strictly convex and K be a nonempty compact subset of E. Let R and S be selfmappings of K such that R is continuous and S is nonexpansive and $T^n(K) \cap \partial_1 K = \emptyset$ for some $n \geqslant 1$. If $RS = SR$ then R and S have a common fixed point in K.*

Proof. Using strict convexity of E, continuity of S and Theorem 2.9 it follows that $F(S)$ is nonempty closed and convex subset of K. Since $RS = SR$ we have $R(S(F)) \subset S(F)$. Hence by Tychonoff's Theorem R has a fixed point in $F(S)$.

Corollary 2.13. *Let E be strictly convex and K be a compact subset of E. Let $\{T_\alpha\}$ be a family of commuting nonexpansive selfmappings of K. Suppose there is at least one $T \in \{T_\alpha\}$ for which there exists an $n \geqslant 1$ such that $T^n(K) \cap \partial_1 K = \emptyset$. Then the family $\{T_\alpha\}$ has a common fixed point in K.*

Proof. Recall that the fixed point set F of $T_\beta \in \{T_\alpha\}$ is a nonempty closed and convex subset of K. Thus F_β is compact and convex. The proof easily follows.

The following results of Chandler and Faulkner [6] ensue from the above theorems.

Corollary 2.14. *Let X be a compact subset of a strictly convex normed linear space E and $T\colon X \to X$ be nonexpansive. If there exists an $n \geqslant 1$ such that $T^n x \cap \partial_1 x = \emptyset$ then T has a fixed point in X.*

Corollary 2.15. *Let S be a closed, starshaped subset of a Banach space E and $T\colon S \to S$ nonexpansive. If there exists $M \subseteq S$, an attractor for compact sets, then T has a fixed point in S.*

3. Families of nonexpansive mappings. We prove the existence of common fixed points for a commutative family of nonexpansive mappings in Fréchet spaces. The well known result of Bahtin [2] foilow as an immediate corollary.

Lemma 3.1. [13, Lemma 2.2]. *Let M be a nonempty compact set of E. If for some $\alpha \in \Delta$*

(1) $$d_\alpha = \sup\{\varrho_\alpha(x-y)\colon x, y \in M\} > 0$$

then there is a u in the convex hull of M such that

$$r = \sup\{\varrho_\alpha(x-u):\ x \in M\} < d_\alpha.$$

LEMMA 3.2. *Let D be a bounded closed subset of E. Let T: $D \to D$ be a continuous demicompact mapping. Then $F(T)$ is sequentially compact in D.*

Proof. Let $\{x_n\}$ be a sequence in $F(T)$, i.e. $x_n - T(x_n) = 0$ for each n. Since T is demicompact and D is bounded there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x$ for some $x \in D$. By the continuity of T, $Tx = x$.

THEOREM 3.3. *Let C be a nonempty, closed bounded convex subset of a Fréchet space E and F be a commutative family of nonexpansive self mappings of C with nonempty fixed points set. Suppose there is at least one mapping in the family which is demicompact. Then the family F has a common fixed point.*

Proof. Let $D = \{B \subseteq C:\ B$ is nonempty, closed, convex and $T(B) \subseteq B$ for each $T \in F\}$. Since C belongs to D, D is nonempty. Define a partial order $<$ on D by $D_1 < D_2$ if and only if $D_2 \subseteq D_1$. We shall show that every chain in D has a lower bound. Let $\{B_\alpha:\ \alpha \in \Delta\}$ be a chain in D. Let $A = \bigcap \{B_\alpha:\ \alpha \in \Delta\}$.

Let $T_0 \in F$ be a demicompact mapping and let $F_\alpha = \{x \in B:\ T_0(x) = x\}$. It follows from Lemma 3.2 that F_α is compact and, hence, $\bigcap\{F_\alpha:\ \alpha \in \Delta\}$ is nonempty subset of A. It is clear that $A \in D$. Therefore by Zorn's Lemma, there exists a minimal nonempty closed and convex subset $B_0 \subseteq C$ such that $T(B_0) \subseteq B_0$ for each $T \in F$.

Let $G = \{x \in B_0:\ T_0(x) = x\}$. Then G is nonempty compact subset of B_0. Obviously $Tx = T(T_0 x) = T_0(Tx)$ for any $x \in G$ and $T \subseteq F$. It follows that $T(G) \in G$ for each $T \in F$ and $x \in G$. Let $H = \{K \subseteq C:\ K$ is nonempty, compact and $T(K) \subseteq K$ for each $T \in F\}$. Then H is nonempty, since G belongs to H. By Zorn's Lemma, there is a minimal nonempty compact set $M \subseteq C$ such that $T(M) \in M$ for each $T \in F$. Clearly $M \subseteq B_0$ and the minimality of M in H implies that $T(M) = M$ for each $T \in F$.

We claim that M consists of exactly one element. Suppose not, then since E is Hausdorff by Lemma 3.1, there is $\alpha \in \Delta$ and $u \in \mathrm{co}(M)$ such that $r = \sup\{\varrho_\alpha(u-x):\ x \in M\} < d_\alpha$. Since B_0 is convex and $M \subseteq B_0$, it follows that $u \in B_0$. For each $x \in M$, define

(2) $$N(x) = \{z \in E:\ \varrho_\alpha(z-x) \leqslant r\}.$$

Then $N(x)$ is convex and $u \in N(x)$ for each $x \in M$. Let $N = \bigcap_{x \in M} N(x)$ and $P = N \cap B_0$. Then P being the intersection of two convex sets, is convex. We shall show that $T(P) \subseteq P$ for each $T \in F$. Since $T(B_0) \subseteq B_0$ for each $T \in F$, it suffices to show that $T(N) \subseteq N$ for each $T \in F$. Let $z \in N$ and $T \in F$. By the definition of N we have

(3) $$\varrho_\alpha(z-x) \leqslant r$$

for each $x \in M$. Since $T(M) = M$, there is $y \in M$ such that $Ty = x$, hence $\varrho_\alpha(Tz-x) \leqslant \varrho_\alpha(z-y) \leqslant r$ for each $x \in M$. Thus $T(z) \in N(x)$ for each $x \in M$, that is $T(P) \subseteq P$ for each $T \in F$. Thus $P \in D$, and by (2) and the minimality of B_0 in D, we have $P = B_0$. Now ϱ_α being continuous and M compact, there are elements x and y in M such that $\varrho_\alpha(x-y) = d_\alpha$. This equality implies $y \notin N(x)$ and consequently $y \notin P$. However, since $M \subseteq B_0$ it follows that $y \in B_0$. This contradicts (3). Thus $M = \{x\}$ for some $x \in C$ and, hence, $Tx = x$ for each $T \in F$.

DEFINITION 3.4. Let $\mathscr{F}$ be an absorbing base for E of closed, circled neighbourhoods of θ. For each subset A of E let $\alpha(A) = \{U \in \mathscr{F}$: for each $V \in \mathscr{F}$ there exists a finite number of sets $S_1, \ldots, S_n$ with $A = \bigcup_{i=1}^{n} S_i$ and $S_i - S_i \subseteq U + V$, $i = 1, 2, \ldots, n\}$, and let $\beta(A) = \{U \in \mathscr{F}$: for each $V \in \mathscr{F}$ there exists a finite subset F of E with $A \subseteq F + V + U\}$. If $K \subseteq E$, then T: $K \to E$ is *α-condensing* if T is continuous and $\alpha(T(A)) \not\subseteq \alpha(A)$ for each bounded but not totally bounded set $A \subseteq K$; T is called *β-condensing* if it is continuous and $\beta(T(A)) \not\subseteq \beta(A)$ for each subset $A \subseteq K$. Note that $\beta(A)$ is also equal to the set $\{U \in \mathscr{F}$: for each $V \in \mathscr{F}$ there exists a totally bounded set $W \subseteq E$ with $A \subseteq U + V + W\}$.

LEMMA 3.5. *Let K be nonempty quasicomplete subset of E and let T: $K \to E$ be either α-condensing or β-condensing. Then T is demicompact.*

Proof. Let $\{x_n\}$ be a bounded sequence in K such that $x_n - Tx_n$ converges in E. To prove that T is demicompact, it suffices to show that $\{x_n\}$ is totally bounded. Thus, in the case T is α-condensing it suffices to show that $\alpha(\{Tx_n\}) \subseteq \alpha(\{x_n\})$. Let $U \in \alpha(\{Tx_n\})$ and $V \in \mathscr{F}$. Choose $W \in \mathscr{F}$ such that $W + W \subseteq V$. Then there are subsets $S_1, \ldots, S_n$ of $\{Tx_n\}$ such that $\{Tx_n\} \subseteq \bigcup_{i=1}^{n} S_i$ and $S_i - S_i \subseteq U + W$, $i = 1, 2, \ldots, n$. Since $x_n - Tx_n$ converges to some $y \in E$, there exists n_0 such that if $m \geqslant n_0$ then $x_m - Tx_m \in y + W$. Thus for $m \geqslant n_0$, $x_m \in T(x_m) + y + W \subseteq S_i + y + W$ for some i. Let $R_i = S_i + y + W$, $i = 1, 2, \ldots, n$ and $R_i = \{x_{i-n}\}$ for $i = n+1, \ldots, n_0 + n$. Then $\{x_n\} \subseteq \bigcup_{i=1}^{n+n_0} R_i$ and $R_i - R_i \subseteq U + V$ for all i. Therefore, $V \in \alpha\{x_n\}$ and $\{x_n\}$ is totally bounded. The proof when T is β-condensing is similar.

The converse of Lemma 3.5 is not true as can be seen from the following examples.

EXAMPLE 3.6. Let $E = R$ with the usual norm and $C = [0, 1]$. Define T: $C \to C$ as $T(x) = x/2$, $0 \leqslant x < 1$, $T(0) = 1$. Then T is neither α-condensing nor β-condensing due to lack of continuity. However, it follows from Bolzano–Weierstrass theorem that T is demicompact.

EXAMPLE 3.7. Let $B = \{e_1, e_2, \ldots, e_n, \ldots\}$ be the usual orthonormal basis for E_2. Define T: $B \to B$ by $T(e_i) = e_{i+1}$. Then T is continuous but neither α-condensing nor β-condensing. However, T is demicompact. Indeed, if $\{e_i\}_{i=1}^{\infty}$ is a bounded sequence in B such that $e_i - Te_i$ converges, then $\{e_i\}_{i=1}^{\infty}$ must be finite.

COROLLARY 3.8. *Let K be a nonempty, complete bounded, convex subset of a Fréchet space E. Let $F\colon K \to K$ be a family of commuting nonexpansive mappings. Suppose there is at least one mapping in F which is either α-condensing or β-condensing, then the family F has a common fixed point.*

Proof. We only need to show that the fixed points set of F is nonempty, which is a consequence of [11, Theorem 1].

COROLLARY 3.9. [2, Theorem 1]. *Let E be a real Banach space and K be a nonempty, bounded, closed and convex subset of it. Let F be a commuting family of nonexpansive mappings of K into itself. Let there be at least one mapping in F which is condensing; then the family F has a common fixed point.*

Proof. An appeal of [8, Theorem 3] guarantees the nonemptiness of the fixed point sets.

4. Nonexpansive retracts. In this section we show that for E strictly convex, the fixed point set of a nonexpansive mapping is a nonexpansive retract. Results of Bruck [3], [4] are extended to more general spaces.

For the proof of our next results we need to recall the following.

LEMMA 4.1. [10, Theorem 8]. *Let E be strictly convex, $g, h \in E$ with $\varrho(g) \leqslant \varrho(h)$. Suppose $0 < t < 1$ and $\varrho((1-t)h + tg) = \varrho(h)$. Then $g = h$.*

THEOREM 4.2. *Let E be strictly convex and K be a nonempty closed and convex subset of E. Let S be a compact (in the topology of weak pointwise convergence) and convex semigroup of nonexpansive mappings of K into itself such that for each $r, s \in S$*

$$\text{(I)} \qquad \overline{\mathrm{co}}\, R(r) \cap \overline{\mathrm{co}}\, R(s) \neq \emptyset,$$

where $R(r)$ denotes the range of R. Then $F(S)$, the fixed point set of S, is a nonexpansive retract of K.

Proof. Define a partial ordering on S by setting $f < g$ if $\varrho(fx - fy) \leqslant \varrho(gx - gy)$ for all $x, y \in K$ with inequality holding for at least one pair x, y and $f \leqslant g$ to mean $f < g$ or $f = g$. As in the proof of Lemma 3 in [9] there exists a minimal element r in $(S, \leqslant)$ and each $s \in S$ acts as an isometry on $R(r)$:

$$\text{(1)} \qquad \varrho\big(sr(x) - sr(y)\big) = \varrho\big(r(x) - r(y)\big).$$

If r is minimal in $(S, \leqslant)$ and $s \in S$, then $\tfrac{1}{2}sr + \tfrac{1}{2}r \in S$ and

$$\text{(2)} \qquad \begin{aligned} \varrho\big(\tfrac{1}{2}(sr+\tfrac{1}{2}r)(x) - \tfrac{1}{2}(sr+\tfrac{1}{2}r)(y)\big) &= \varrho\big(\tfrac{1}{2}(sr(x)-sr(y)) + \tfrac{1}{2}(r(x)-r(y))\big) \\ &\leqslant \tfrac{1}{2}\varrho\big(sr(x)-sr(y)\big) + \tfrac{1}{2}\varrho\big(r(x)-r(y)\big) \\ &\leqslant \varrho\big(r(x)-r(y)\big). \end{aligned}$$

Equality must hold throughout (2), since r is minimal. Using the strict convexity of E and equation (1) we must have $sr(x) - sr(y) = r(x) - r(y)$ for all $s \in S$ and $x, y \in K$. In fact, let $g = sr(x) - sr(y)$, $h = r(x) - r(y)$ and $t = \tfrac{1}{2}$, in Lemma 4.1.

Thus if r is minimal in S then each $s \in S$ acts as a translation on $R(r)$. In particular, r acts as a translation on $R(r)$. But since $R(r)$ is bounded and r-invariant, this means r acts as the identity on $R(r)$. Thus r is a nonexpansive retraction of K onto $R(r)$.

Let r_1, r_2 be minimal in $(S, \leqslant)$. We claim that $R(r_1) = R(r_2)$. Indeed, we have already shown that r_1 acts as a translation by some vector x on $R(r_2)$ and acts as an identity on $R(r_1)$. The continuity of r_1, r_2 and strict convexity of E imply that $R(r_1)$ and $R(r_2)$, being fixed point sets of nonexpansive mappings r_1 and r_2, are closed and convex. Thus, by condition (I), we have $R(r_1) \cap R(r_2) \neq \emptyset$. Thus $x = 0$, i.e. r_1 acts as the identity on $R(r_2)$, so that $R(r_2) \subset R(r_1)$. By symmetry $R(r_1) \subset R(r_2)$.

Now we claim that if r is minimal in $(S, \leqslant)$ then $R(r) = F(S)$. Obviously $F(S) \subset R(r)$. It remains to show that $R(r) \subset F(S)$. Let $s \in S$, by equation (1) sr is also minimal in $(S, \leqslant)$. But we have shown that minimal elements of S are retractions of K onto $R(r)$. If $y \in R(r)$, then $r(y) = y$ and $sr(y) = y$, so $s(y) = y$. Since this is true for all $s \in S$, we have $R(r) \subset F(S)$. Hence $R(r) = F(S)$. $F(S)$ is nonempty, because obviously $R(r) \neq \emptyset$ and r is a nonexpansive retraction of K onto $F(S)$.

Consider the following conditions on a compact convex semigroup S of nonexpansive mappings:

(FP1) S has a common fixed point.

(FP2) For each $s_1, s_2 \in S$, s_1 and s_2 have a common fixed point.

(FP3) For each $s_1, s_2 \in S$, $R(s_1) \cap R(s_2) \neq \emptyset$.

(FP4) For each $s_1, s_2 \in S$, $\mathrm{dist}\big(R(s_1), R(s_2)\big) = 0$.

We note that (I) does not imply (FP1), (FP2), (FP3) or (FP4) if E is not strictly convex, even if K is compact.

EXAMPLE 4.3. Let $E = R^2$ with the supremum norm and let K be the square, $K = \{(x, y)\colon \|x\| \leqslant 1, \|y\| \leqslant 1\}$. For $0 \leqslant t \leqslant 1$, define $f_t(x, y) = (|y| - t, y)$. Let $S = \{f_t\colon 0 \leqslant t \leqslant 1\}$. S is convex semigroup. Indeed,

$$f_t f_s(x, y) = f_t[f_s(x, y)] = f_t[(|y| - s, y)] = (|y| - t, y) = f_t(x, y).$$

$$\begin{aligned} f_{\lambda t + (1-\lambda)s}(x, y) &= \big(|y| - \lambda t - (1-\lambda)s, y\big) \\ &= \big(\lambda|y| - \lambda t + (1-\lambda)|y| - (1-\lambda)s, \lambda(y) + (1-\lambda)y\big) \\ &= \lambda\big(|y| - t, y\big) + (1-\lambda)\big(|y| - s, y\big) = \lambda f_t(x, y) + (1-\lambda) f_s(x, y). \end{aligned}$$

Thus S is a compact convex semigroup and each f_t in S is nonexpansive. (I) is satisfied because the range of f_t is the intersection of the graph of x

$=|y|-t$ and K, so $\{(0,0)\} = \bigcap \{\overline{\text{co}}\ R(f_t)\colon 0 \leqslant t \leqslant 1\}$. But none of the conditions (FP1), (FP2), (FP3) or (FP4) are satisfied.

THEOREM 4.4. *Let E be strictly convex and K be a weakly compact convex subset of E. If $T\colon K \to K$ is nonexpansive, then $F(T)$ is a nonexpansive retract of K. Moreover, the class of nonexpansive retracts of K is closed under arbitrary intersections.*

Proof. Let A be a nonempty subset of K. Let $M(A) = \{T\colon\ T\colon K \to K$ is nonexpansive and $A \subset F(T)\}$. Define an order on $M(A)$ by setting $T < T'$ if $\varrho(Tx-Ty) \leqslant \varrho(T'x-T'y)$ for all $x, y \in K$ with strict inequality holding for at least one pair (x, y). As usual $T \leqslant T'$ means $T < T'$ or $T = T'$. Then $\leqslant$ is a partial ordering on $M(A)$. Let $I(T) = \{T' \in M(A)\colon\ T' \leqslant T\}$. Then it follows from the proof of Lemma 3 in [9] that $I(T)$ is closed in $M(A)$. Thus $I(T)$ is weakly compact.

If $\mathscr{C}$ is a chain in $M(A)$, then $\{I(T)\colon\ T \in \mathscr{C}\}$ is a chain under inclusion. Since each $I(T)$ is weakly compact, there exists $T_0 \in \bigcap \{I(T)\colon\ T \in \mathscr{C}\}$; T_0 is a lower bound of $\mathscr{C}$. Thus, by application of Zorn's Lemma, we conclude that $M(A)$ has a minimal element. For $R \in M(A)$, let $S = \frac{1}{2}(I+R)$. Because E is strictly convex, if $\varrho(Su-Sw) = \varrho(u-w)$ then $Su-Sw = u-w$. In fact, letting $g = Su-Sw$, and $h = u-w$ and $t = \frac{1}{2}$, in Lemma 4.1 we have the required result. Also $F(R) = F(S)$.

Let T be minimal in $M(A)$, R be any function in $M(A)$ and let S be the function as above, then $ST \in M(A)$ while $ST \leqslant T$. By the minimality of T, therefore $ST = T$. Let $R(T)$ be the range of T. Therefore

(1) $$F(T) \subset R(T) \subset F(S)+F(R)$$

and in particular

(2) $$F(T) \subset F(R)$$

for $R \in \mathscr{C}$. Taking $R = T$ in (1) we see that $F(T) = R(T)$, so that T is a nonexpansive retraction onto $F(T)$. From (2) if T and R are minimal elements of $M(A)$ then $F(T) = F(R)$. This common set $F(T)$ is the smallest nonexpansive retract A_1 of K with $A_1 \supset A$. If R of (2) is any nonexpansive retraction with $F(R) = A_1$, it follows from (2) that $A \subset F(T) \subset A_1$.

Suppose $F(T) \neq \emptyset$. Let $G = F(T)$ and let S be the minimal element of $\mathscr{C}$. Taking $R = T$ in (2) we have $F(S) \subset F(T)$, while $F(\mathscr{C}) \subset F(R)$ since $R \in \mathscr{C}$. Therefore S is a nonexpansive retraction of K onto $F(R) = F(T)$.

Let $\{F_\alpha\}$ be a family of nonexpansive retracts of K. Let $P = \bigcap_\alpha F_\alpha$; we may assume $P \neq \emptyset$. We have already proved that if T is a minimal element of $\mathscr{C}$, then $P \subset F(T) \subset P_1$ for each nonexpansive retract P_1. In particular, $F(T) \subset F_\alpha$ for all α, so $P \subset F(T) \subset \bigcap_\alpha F_\alpha = P$, and T is the required nonexpansive retraction. As corollaries of Theorem 4.2 and Theorem 4.4 we have the following results of Bruck [4] and [3], respectively.

COROLLARY 4.5. *Let C be a nonempty closed convex subset of a strictly convex Banach space and let S be a semigroup of nonexpansive self mappings of C which is compact and convex in the topology of weak pointwise convergence. If S has the property that $\overline{\text{co}}\ R(s_1) \cap \overline{\text{co}}\ R(s_2) \neq \emptyset$ whenever $s_1, s_2 \in S$, then $F(S)$ is a nonexpansive retract of C.*

COROLLARY 4.6. *Let C be a closed bounded convex subset of a reflexive, strictly convex Banach space X. If $T\colon C \to C$ is nonexpansive, then $F(T)$, the fixed point set of T, is a nonexpansive retract of C. The class of nonexpansive retracts of C is closed under arbitrary intersection.*

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