On a fixed point theorem of Krasnoselskii and triangle contractive operators

by

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Abstract. The paper presents some variants to a fixed point theorem of Krasnoselskii for operators on a closed convex subset of a Banach space of the form $U+F$ where U is contractive and F is completely continuous. A study is made of triangle contractive operators in a Hilbert space. It is proved that a triangle contractive operator satisfying certain rather mild conditions is on each bounded set the uniform limit of a sequence of operators (T_n) with $T_n = U_n+F_n$ where U_n is contractive and F_n is completely continuous.

Finally, a fixed point theorem is proved for operators of the form $U+F$ where U is triangle contractive and F is completely continuous.

Introduction. Let X be a Banach space and let K be a bounded closed convex subset of X. A well-known theorem of Krasnoselskii [8] states that if U is a contraction of K (i.e. $\|Ux-Uy\| \leqslant k\|x-y\|$ for $0<k<1$) and F is a completely continuous operator on K such that

$$(*) \qquad Ux+Fy \in K \quad \text{for every } x, y \text{ in } K$$

then $U+F$ has a fixed point. Krasnoselskii's theorem has been extended by Nashed and Wong [9] to the case U is a φ-contraction and to the case U is bounded linear and such that U^p is a φ-contraction for some $p>1$.

Our aim in this paper is to present some variants to Krasnoselskii's theorem and to its generalizations by Nashed and Wong (loc. cit.). In our version, K will be the closure of a bounded open convex subset of X, and the condition $(*)$ will be replaced by the following weaker one

$$(**) \qquad Ux+Fx \in K \quad \text{for each } x \text{ in } K.$$

Extensions to the case of unbounded domains will also be considered.

The concept of a triangle contractive operator on a Hilbert space, a noteworthy extension of the concept of a contraction, was introduced and studied by Daykin and Dugdale [6] (cf. also Rhoades [10], [11], Daykin [5] and Ang and Hoa [1]). Roughly speaking, an operator on a Hilbert space is said to be *triangle contractive* if it decreases areas of triangles in some appropriate manner (cf. Section 2 for a precise definition). The concept of a triangle contractive operator (a TC operator for short) is no doubt an attractive one, geometrically. We shall show that a TC operator satisfying certain rather mild conditions is on each bounded set the uniform limit of a sequence of operators (T_n) with $T_n = U_n + F_n$ where U_n is contractive and F_n is completely continuous (i.e. is continuous and maps bounded sets into compact sets). In fact, we shall study operators of the form $U+F$ where U is TC and F is completely continuous, and prove a fixed point theorem for such operators.

We shall consider operators that are quasibounded in the following sense (Granas [7]):

$$\limsup_{\|x\|\to\infty} \|Tx\|/\|x\| < \infty.$$

If T is a quasibounded operator, we put

$$|T| = \limsup_{\|x\|\to\infty} \|Tx\|/\|x\|.$$

Then $|T|$ is called the *quasinorm of* T. Note that if T is bounded linear, then T is quasibounded and $|T|$ is precisely equal to the norm of T as a bounded linear operator.

The remainder of the paper is divided into two sections. Section 1 is devoted to some fixed point theorems of the Krasnoselskii type. Section 2 is devoted to a study of operators of the form $U+F$ where U is TC and F is completely continuous. We shall prove a fixed point theorem for such operators.

Section 1. Fixed point theorems of the Krasnoselskii type. Throughout this section, X denotes a Banach space, G denotes a domain (open connected set) of X and $\mathrm{cl}(G)$ its closure.

DEFINITION 1.1. Let φ be a continuous real-valued function on the positive real numbers such that

$$0 < \varphi(r) < r \quad \text{for} \quad r > 0.$$

A mapping

$$U\colon \mathrm{cl}(G) \to X$$

is said to be a *φ-contraction* (Boyd and Wong [2]) if

$$\|Ux - Uy\| \leqslant \varphi(\|x-y\|) \quad \text{for every } x, y \text{ in } \mathrm{cl}(G).$$

We shall commence with the following theorem.

THEOREM 1. *Let G be a convex open set in X and let $0 \in G$. Let*

$$U\colon \mathrm{cl}(G) \to X$$

be either a φ-contraction or the restriction to $\mathrm{cl}(G)$ of a bounded linear operator U' on X such that $(U')^p$ is a φ-contraction for some $p \geqslant 1$. Let

$$F\colon \mathrm{cl}(G) \to X$$

be a completely continuous operator. Put

$$T = U + F$$

and suppose T maps $\mathrm{cl}(G)$ into itself. Then the following holds:

(i) *If G is bounded, then T has a fixed point.*

(ii) *If G is unbounded and if $|T| < 1$, then T has a fixed point.*

Remark. This theorem is to be compared with Theorem 4 of Browder–Nussbaum [4].

For the proof of Theorem 1, we shall use properties of the Browder–Nussbaum degree [4] as follows. Let G be a domain in X, let H, F be mappings of $\mathrm{cl}(G)$ into X satisfying the following conditions:

a) For each fixed v in $\mathrm{cl}(G)$, the mapping

$$S_v\colon \mathrm{cl}(G) \to X$$

defined by $S_v u = Hu + Fv$ is a homeomorphism of G onto an open subset G_v of X, mapping $\mathrm{cl}(G)$ homeomorphically onto $\mathrm{cl}(G_v)$.

b) The mapping $v \to S_v$ is a locally compact mapping of $\mathrm{cl}(G)$ into the space of homeomorphisms of $\mathrm{cl}(G)$ into X with the topology of uniform convergence on $\mathrm{cl}(G)$.

Let $Tu = Hu + Fu$ for u in $\mathrm{cl}(G)$. Suppose $T^{-1}(0)$ is a compact subset of G. Then, $\deg(T, G, 0)$ is defined. (In fact, the Browder–Nussbaum degree is defined for more general operators, but this simplified version is all that we shall need).

The following proposition is implicitly contained in the Browder–Nussbaum paper (loc. cit.).

PROPOSITION 1.1. (i) *If $\deg(T, G, 0) \neq 0$, then there exists an x in G such that $Tx = 0$.*

(ii) *Let A, B be continuous mappings of $\mathrm{cl}(G)\times[0, 1]$ into X such that $A(., t)$ and $B(., t)$ are continuous uniformly with respect to t in $[0, 1]$, and for each $0 \leqslant t \leqslant 1$, the map $A_t(.) \equiv A(., t)$ is a homeomorphism of G onto an open set G_t of X, taking $\mathrm{cl}(G)$ homeomorphically onto $\mathrm{cl}(G_t)$, and the map $B_t(.) \equiv B(., t)$ is a completely continuous operator of $\mathrm{cl}(G)$ into X. Suppose that for each $0 \leqslant t \leqslant 1$, the pair A_t, B_t satisfies condition* b) *above. Suppose further that for each t,*

$$(A_t + B_t)^{-1}(0) \cap \partial G = \emptyset$$

(where ∂G is the boundary of G) and that the set of the (x, t)'s for which $A_t x + B_t x = 0$ is bounded in $\mathrm{cl}(G)\times[0, 1]$. *Then*

$$\deg(A_0+B_0, G, 0) = \deg(A_1+B_1, G, 0).$$

We now turn to the

Proof of Theorem 1. We first consider part (i) of the theorem, beginning with the case U is a φ-contraction. In order to be able to use properties of the Browder–Nussbaum degree, we shall prove that for each $0 \leqslant t \leqslant 1$, the map $H_t = I - tU$ is a homeomorphism of G onto an open subset of X, taking $\mathrm{cl}(G)$ homeomorphism of G onto an open subset of X, taking $\mathrm{cl}(G)$ homeomorphically onto $\mathrm{cl}(H_t(G))$. We have

$$\|x-y\| - \varphi(\|x-y\|) \leqslant \|H_t(x) - H_t(y)\| \leqslant \|x-y\| + \varphi(\|x-y\|)$$

which shows that H_t is a homeomorphism of $\mathrm{cl}(G)$ onto a *closed* subset of X. We shall show next that $H_t(G)$ is an open subset of X. Let $x_0 \in G$, and let $r > 0$ be such that the closed ball $B'(x_0, r)$ is contained in G. Put $\varrho = \sup\{\varphi(s)\colon 0 \leqslant s \leqslant r\}$. Then $\varrho < r$. For $\|v\| < r - \varrho$, define the map V on the closed ball $B'(0, r)$ as follows:

$$Vh = tU(x_0+h) - y_0 + v$$

where $y_0 = tU(x_0)$. We shall show that V takes $B'(0, r)$ into itself. Indeed,

$$\begin{aligned}\|Vh\| &\leqslant \|tU(x_0+h) - tU(x_0)\| + \|v\| \leqslant t\varphi(\|h\|) + \|v\| \\ &\leqslant \varrho + r - \varrho = r.\end{aligned}$$

Since it is clear that V is a φ-contraction, V has a fixed point h (say) by a theorem of Boyd and Wong (loc. cit.), i.e.,

$$h = tU(x_0+h) - y_0 + v$$

or

$$x_0 + h - tU(x_0+h) = x_0 - y_0 + v.$$

We have proved that the open ball $B(x_0 - y_0, r - \varrho)$ is contained in the image of $B'(x_0, r)$ under H_t. It follows that G has an open image under H_t as claimed.

If $I - (U+F)$ does not vanish on the boundary ∂G of G, then, since $0 \in G$ and since G is convex, $I - t(U+F)$ does not vanish on ∂G for $0 \leqslant t \leqslant 1$. Consider the homotopy $I - t(U+F)$, $0 \leqslant t \leqslant 1$. Since G is bounded, Proposition 1.1 applies, and we have

$$\deg(I-(U+F), G, 0) = \deg(I, G, 0) = 1.$$

Hence $U+F$ has a fixed point in G.

The case U is the restriction to $\mathrm{cl}(G)$ of a bounded linear operator U' such that $(U')^p$ is a φ-contraction for some $p \geqslant 1$, is handled in a similar way. This proves part (i).

Consider now part (ii). Since $|T| < 1$, there exists for each k with $|T| < k < 1$, an $r_1 > 0$ such that

$$\|Tx\| \leqslant k\|x\| \quad \text{for all } x \text{ in } \mathrm{cl}(G) \text{ with } \|x\| > r_1.$$

Now there exists an $r_2 > r_1$ such that

$$T[B'(0, r_1) \cap \mathrm{cl}(G)] \subset B'(0, r_2).$$

Put

$$K_1 = \mathrm{cl}(G) \cap B'(0, r_2).$$

Then T maps K_1 into itself. Hence T has by part (i) above a fixed point in K_1 and hence in $\mathrm{cl}(G)$. ∎

Corollary 1. *Let U be a φ-contraction on X, and let F be a completely continuous operator on X. Suppose*

$$|U+F| < 1.$$

Then $R(I-U-F) = X$, where R denotes the range of a map.

This follows from Theorem 1, part (ii), for $G = X$. Indeed, if y is any point of X, then the operator $U+F+y$ satisfies

$$|U+F+y| < 1.$$

Hence, by Theorem 1, part (ii), $U+F+y$ has a fixed point x (say), which clearly satisfies $x - (Ux+Fx) = y$. ∎

Remark. Corollary 1 above contains as special cases a result of Granas (loc. cit.) which corresponds to $U = 0$, and a result of Nashed and Wong (Theorem 3, loc. cit.) where U is a contraction of coefficient $0 < \gamma < 1$ and F is completely continuous and quasibounded with $|F| < 1-\gamma$.

Corollary 2. *Let U be a bounded linear operator on X such that some iterate U^p, $p \geqslant 1$, is a φ-contraction. Suppose F is completely continuous on X. Suppose $|U+F| < 1$. Then*

$$R(I-U-F) = X$$

where R denotes the range of a map.

This follows from Theorem 1, part (ii), for $G = X$, in the same way that Corollary 1 follows from the theorem.

Remark. Corollary 2 above is a counterpart of a result of Nashed and Wong (Theorem 4, loc. cit.) in which U is bounded linear with U^p a contraction of coefficient $0 < \gamma < 1$, and F is completely continuous with $|F| < 1-\gamma$.

A well-known extension of the Schauder fixed point theorem of F. E. Browder [3] states that if F is completely continuous on X such that for some n, $F^n(X)$ is bounded, then F has a fixed point. We propose to consider operators of the form $U+F$ where U is a φ-contraction and F is an asymptotically linear, completely continuous operator such that for some n, F^n is quasibounded. More precisely, we have

THEOREM 2. *Let G be an unbounded convex open set in X, and let $0 \in G$. Let T be a map of $\mathrm{cl}(G)$ into itself of the form $U+F$, where U is a φ-contraction, F is completely continuous such that*

(i) $|U| = 0$ *and* (ii) $\lim_{\|x\| \to \infty} \|Fx - Bx\|/\|x\| = 0$

where $B \neq 0$ is a bounded linear operator on X. If for some $n \geqslant 1$, F^n is defined and satisfies $|F^n| < 1$, then T has a fixed point.

Remark. If $B = 0$, then condition (ii) of Theorem 2 implies $|F| = 0$. Thus, the corresponding problem for $B = 0$ is covered by Theorem 1.

For the proof of Theorem 2, we need some lemmas.

LEMMA 1.1. *Let U, F satisfy the conditions of Theorem 2. Then, there exist a k_0 in $[0, 1]$ and an $r_2 > 0$ such that for every $r \geqslant r_2$*

$$\|(U+F)^n(x)\| \leqslant k_0 r \quad \text{and} \quad \|B^n x\| \leqslant k_0 r$$

for every x in $\mathrm{cl}(G)$ such that $\|x\| \leqslant r$.

Proof. Put $W = U + (F-B)$ and $Y = F - B$. Then

$$T = U+F = B+W \quad \text{and} \quad F = B+Y.$$

We claim that

$$(U+F)^m = (B+W)^m = \sum_{i=0}^{m-1} B^i W (B+W)^{m-i-1} + B^m \tag{1}$$

and

$$F^m = \sum_{i=0}^{m-1} B^i Y (B+Y)^{m-i-1} + B^m. \tag{2}$$

Indeed, identity (1) holds for $m = 1$. If it holds for m, then

$$(B+W)^{m+1} = (B+W)(B+W)^m = B(B+W)^m + W(B+W)^m.$$

Using the linearity of B, one verifies that (1) holds for $m+1$. Thus, by induction, it holds for every m. Identity (2) is proved by induction in exactly the same manner. From (1) and (2) one deduces

$$\|(U+F)^n x\| \leqslant \|F^n x\| + \sum_{i=1}^{n-1} \|B^i W (B+W)^{n-i-1} x\| + \sum_{i=1}^{n-1} \|B^i Y (B+Y)^{n-i-1} x\|$$

and

$$\|B^n x\| \leqslant \|F^n x\| + \sum_{i=1}^{n-1} \|B^i Y (B+Y)^{n-i-1} x\|$$

for x in $\mathrm{cl}(G)$.

By the conditions of Theorem 2, for each k with $|F^n| < k < 1$, and each c with

$$0 < c < \tfrac{1}{2}(1-k) \sum_{i=1}^{n-1} |B|^i (|B|+c+1)^{n-i-1}$$

there exists an $r_1 > 0$ such that for each x in $\mathrm{cl}(G)$ with $\|x\| \geqslant r_1$ the following holds

$$\|F^n x\| \leqslant k\|x\|, \quad \|Yx\| \leqslant c\|x\| \quad \text{and} \quad \|Wx\| \leqslant c\|x\|.$$

Let $r_2 > r_1$ be such that Yx and $Wx \in B'(0, cr_2)$ for each x in $\mathrm{cl}(G) \cap B'(0, r_1)$ and $(U+F)x \in B'(0, r_2)$ for each x in $\mathrm{cl}(G) \cap B'(0, r_1)$ (here as elsewhere $B'(0, r)$ denotes the closed ball of center 0 and radius r). For each $r \geqslant r_2$ and x in $\mathrm{cl}(G)$ such that $\|x\| \leqslant r$, we have

$$\|(B+W)^i x\| \leqslant (|B|+c+1)^i r. \tag{3}$$

Indeed, this holds for $i = 1$. If it holds for $i \leqslant n-1$, then

$$\begin{aligned}\|(B+W)^{i+1} x\| &\leqslant \|B(B+W)^i x\| + \|W(B+W)^i x\| \\ &\leqslant |B|(|B|+c+1)^i r + \|W(B+W)^i x\|.\end{aligned}$$

If $\|(B+W)^i x\| \leqslant r_1$, then

$$\|(B+W)^{i+1} x\| = \|(U+F)(B+W)^i x\| \leqslant r_2 \leqslant r \leqslant (|B|+c+1)^{i+1} r.$$

If $\|(B+W)^i x\| \geqslant r_1$, then

$$\|W(B+W)^i x\| \leqslant c\|x\| \leqslant cr.$$

Thus

$$\|(B+W)^{i+1} x\| \leqslant [|B|(|B|+c+1)^i + c] r \leqslant (|B|+c+1)^{i+1} r$$

which completes the induction process and (3) is proved.

In a similar way one shows that

$$\|(B+Y)^i x\| \leqslant (|B|+c+1)^i r \quad \text{for each } x \text{ in } \mathrm{cl}(G) \text{ with } \|x\| \leqslant r.$$

Furthermore, if $\|(B+W)^i x\| \leqslant r_1$, then

$$\|W(B+W)^i x\| \leqslant cr_2 \leqslant cr \leqslant c(1+|B|+c)^i r.$$

If $\|(B+W)^i x\| \geqslant r_1$, then

$$\|W(B+W)^i x\| \leqslant c\|(B+W)^i x\| \leqslant c(|B|+c+1)^i r.$$

In a similar way one has

$$\|Y(B+Y)^i x\| \leqslant c(|B|+c+1)^i r.$$

Hence

$$\|(U+F)^n x\| \leqslant kr + 2c \sum_{i=1}^{n-1} |B|^i (|B|+c+1)^{n-i-1} r.$$

Put

$$k_0 = k + 2c \sum_{i=1}^{n-1} |B|^i (|B|+c+1)^{n-i-1}.$$

Then $k_0 < 1$, and

$$\|(U+F)^n x\| \leqslant k_0 r \quad \text{and} \quad \|B^n x\| \leqslant k_0 r$$

for each x in $\mathrm{cl}(G)$ such that $\|x\| \leqslant r$ and $r \geqslant r_2$. ∎

COROLLARY OF LEMMA 1.1. *If U, F satisfy the conditions of Theorem 2, then for $T = U+F$, the set $(I-T)^{-1}(0)$ is compact.*

Proof. By Lemma 1.1,

$$(I-T)^{-1}(0) \subset B'(0, r_2).$$

Hence, the set is compact. ∎

LEMMA 1.2. *Let U, F satisfy the conditions of Theorem 2. Let*

$$A_t = (I - tT)^{-1}(0).$$

Then A_t is compact for each $0 \leqslant t \leqslant 1$, and there exists $r_3 > r_2$ such that $A_t \subset B'(0, r_3)$ for each $0 \leqslant t \leqslant 1$.

Proof. The case $t = 0$ is trivial. The case $t = 1$ follows from the corollary of Lemma 1.1. Hence, we shall consider $0 < t < 1$ only. Let

$$0 < a < (1-k_0) / \sum_{i=0}^{n-1} |B|^i.$$

Then there exists $r_3 > r_2$ such that for each x in $\mathrm{cl}(G)$ with $\|x\| > r_3$, one has $\|Wx\| \leqslant a\|x\|$. We shall show that

$$A_t \subset B'(0, r_3).$$

Indeed, if for some $0 < t < 1$, there exists x in A_t with $\|x\| \geqslant r_3$ then $Tx = x/t$, i.e.,

$$(B+W)x = (U+F) = x/t.$$

It follows that

$$Bx = x/t - Wx.$$

By the linearity of B, one has

$$B^n x = x/t^n - Wx/t^{n-1} - BWx/t^{n-2} - \ldots - B^{n-1}Wx = (x - tWx - t^2 BWx - \ldots - t^n B^{n-1} Wx)/t^n.$$

Then

$$\|B^n x\| \geqslant (\|x\|/t^n)(1 - a(1 + |B| + |B|^2 + \ldots + |B|^{n-1})).$$

Hence

$$\|B^n x\| \geqslant k_0 \|x\|/t^n \quad \text{for some } 0 < t < 1 \text{ and } \|x\| > r_3 > r_2.$$

This contradicts Lemma 1.1. Hence, we have

$$A_t \subset B'(0, r_3) \quad \text{for each } 0 \leqslant t \leqslant 1$$

as claimed. ∎

We are now ready for

Proof of Theorem 2. Suppose $(I-T)^{-1}(0) \cap \partial G = \emptyset$. Consider the homotopy

$$S_t = tT\colon \mathrm{cl}(G) \times [0, 1] \to \mathrm{cl}(G).$$

By Lemma 1.2, the set $(I-T)^{-1}(0)$ is compact, and the set of the (x, t)'s for which $x - S_t x = 0$ is bounded. Since $\mathrm{cl}(G)$ is convex with $0 \in G$, and since T takes $\mathrm{cl}(G)$ into itself, one has

$$(I - S_t)^{-1}(0) \cap \partial G = \emptyset \quad \text{for} \quad 0 \leqslant t \leqslant 1.$$

By Proposition 1.1

$$\deg(I-T, G, 0) = \deg(I, G, 0) = 1.$$

Hence T has a fixed point in G. ∎

Section 2. Triangle contractive maps and Krasnoselskii operators. Throughout this section, H will denote a real Hilbert space. Let $0 < \alpha < 1$. An operator U on H is said to be *α-triangle contractive* if for each x, y, z in H, the following holds: either

(i) $\|Ux - Uy\| \leqslant \alpha\|x-y\|$ and $\|Uy - Uz\| \leqslant \alpha\|y-z\|$ and $\|Uz - Ux\| \leqslant \alpha\|z-x\|$ or

(ii) $\Delta(Ux, Uy, Uz) \leqslant \alpha\Delta(x, y, z)$

where $\Delta(x, y, z)$ is the area of the triangle x, y, z (cf. Daykin–Dugdale [6] where the concept was first defined). We shall use the abbreviation α-TC for α-triangle contractive. If there exists an $0 < \alpha < 1$ for which U is α-TC, we say that U is TC (abbreviation for triangle contractive). Throughout this section, α will stand for a positive number strictly smaller than 1.

If U is a TC operator which maps H into a line, then we say that U is *trivial*. Our aim in this section is to prove a number of properties of TC operators, and to establish a fixed point theorem for operators of the form $U+F$ where U is TC and F is completely continuous.

THEOREM 3. *Let U be TC nontrivial. Then U is Lipschitzian on each bounded subset of H, i.e., for each bounded subset D of H, there exists an $a_D \geqslant 0$ such that*

$$\|Ux-Uy\| \leqslant a_D \|x-y\| \quad \textit{for all } x, y \textit{ in } D.$$

Proof. Suppose this is not the case. Then, there exists a bounded set D such that for each $n \geqslant 1$, there exists x_n, y_n in D with

$$\|Ux_n - Uy_n\| > n\|x_n - y_n\|.$$

Let U be α-TC; the above inequality implies that for each x in H

$$\Delta(Ux, Ux_n, Uy_n) \leqslant \alpha\Delta(x, x_n, y_n)$$

or (from the definition of the area of a triangle)

$$\pi(Ux, L(Ux_n, Uy_n))\|Ux_n - Uy_n\| \leqslant \alpha\pi(x, L(x_n, y_n))\|x_n - y_n\|$$

where $L(u, v)$ is the line through u and v, and $\pi(x, L(x_n, y_n))$ is the distance from x to $L(x_n, y_n)$. One readily deduces that

$$\pi(Ux, L(Ux_n, Uy_n)) \leqslant (\alpha/n)\pi(x, L(x_n, y_n)).$$

Since x_n, y_n are in D and since D is bounded, there exists an $M > 0$ such that

$$\pi(x, L(x_n, y_n)) \leqslant \|x - x_n\| \leqslant \|x\| + M \quad \text{for each } n.$$

This implies

$$\pi(Ux, L(Ux_n, Uy_n)) \leqslant (\alpha/n)(\|x\| + M).$$

Similarly, one has for y, z in H

$$\pi(Uy, L(Ux_n, Uy_n)) \leqslant (\alpha/n)(\|y\| + M),$$
$$\pi(Uz, L(Ux_n, Uy_n)) \leqslant (\alpha/n)(\|z\| + M).$$

Hence for each $\varepsilon > 0$, there exists $n_0 \geqslant 1$ such that for all $n \geqslant n_0$

$$\pi(Ux, L(Ux_n, Uy_n)) > \varepsilon,$$
$$\pi(Uy, L(Ux_n, Uy_n)) < \varepsilon,$$
$$\pi(Uz, L(Ux_n, Uy_n)) < \varepsilon.$$

The line $L(Ux_n, Uy_n)$ thus has a nonvoid intersection with the open balls $B(Ux, \varepsilon)$, $B(Uy, \varepsilon)$ and $B(Uz, \varepsilon)$. Since $\varepsilon > 0$ is arbitrary, it follows that Ux, Uy, Uz are collinear. Thus $U(H)$ is part of a line, i.e., U is trivial, a contradiction. Hence U is Lipschitzian on D as desired. ■

Remark. Daykin and Dugdale (loc. cit.) have proved that if U is discontinuous, then U is trivial. Thus in the preceding theorem (and in all that follows) U is continuous (from the nontriviality hypothesis).

COROLLARY OF THEOREM 3. *If U is TC nontrivial, then U maps each bounded set into a bounded set.*

This is immediate from the theorem.

The following theorem proves a crucial property for a large class of TC operators.

THEOREM 4. *Let U be α-TC nontrivial such that $|U| > \alpha$. Then there exists a sequence of operators (U_n) on H with the following properties*

$$U_n = V_n + F_n$$

where

(i) *F_n is completely continuous with $F_n(H)$ contained in a line,*

(ii) *for each $r > 0$, there exists $n_r > 1$ such that for all $n \geqslant n_r$, V_n restricted to the closed ball $B'(0, r)$ is a contraction with coefficient δ_0 not depending on $n \geqslant n_r$,*

(iii) *U_n converges to U uniformly on $B'(0, r)$ for each $r > 0$.*

Proof. We first remark that there exist a $\gamma > \alpha$ and a sequence (x_n), $x_n \neq 0$ for each n, in H such that $\|x_n\| \to \infty$ and $\|Ux_n\| \geqslant \gamma\|x_n\|$ for each n. This follows readily from the condition that $|U| > \alpha$. This being the case, we define a sequence of operators as follows. Let H_n be the homogeneous hyperplane orthogonal to x_n, let H'_n be the homogeneous hyperplane orthogonal to Ux_n. Let P_n be the orthogonal projection onto H_n, and let P'_n be the orthogonal projection onto H'_n. Then define the sequence of operators (V_n) by

$$V_n = P'_n UP$$

and the sequence of operators (F_n) by

$$F_n x = (e_n, Ux)e_n \quad \text{for} \quad x \text{ in } H,\ e_n = Ux_n/\|Ux_n\|.$$

Here $(.,.)$ denotes the inner product.

Clearly, $F_n(H)$ is part of a line and $U(B'(0, r))$ is bounded. Hence F_n is completely continuous. The remainder of the proof is split into a number of steps as follows.

Step 1. *For each $r > 0$, there exists $n_1(r)$ such that for all $n \geqslant n_1(r)$ U takes $L(y, x_n)$ into $L(Uy, Ux_n)$ for each y in $B'(0, 2r)$.*

(Here L denotes the line passing through two given points. It is understood that n is sufficiently large so that x_n is distinct from y, Ux_n distinct from Uy for all y in $B'(0, 2r)$.)

Proof of Step 1. By the corollary to Theorem 3, there exists an $R > 0$ such that U takes $B'(0, 2r)$ into $B'(0, R)$. Then, for each x in $B'(0, 2r)$ one has

$$\|Ux_n - Ux\| \geqslant \|Ux_n\| - \|Ux\| \geqslant \gamma\|x_n\| - \|Ux\|$$
$$\geqslant \gamma\|x_n - x\| - \|x\| - \|Ux\|$$
$$\geqslant \gamma'\|x_n - x\| - \gamma\|x\| - \|Ux\| + (\gamma - \gamma')\|x_n - x\|$$

where $\alpha < \gamma' < \gamma$. Since $\|x_n\| \to \infty$, there exists $n_1(r)$ such that for each $n \geqslant n_1(r)$

$$0 < (\gamma-\gamma')\|x_n - x\| - 2\gamma r - R \leqslant (\gamma-\gamma')\|x_n - x\| - \gamma\|x\| - \|Ux\|.$$

It follows that

$$\|Ux_n - Ux\| \geqslant \gamma'\|x_n - x\| > \alpha\|x_n - x\|.$$

Since U is α-TC, one has for each x, y in $B'(0, 2r)$

$$\Delta(Ux, Uy, Ux_n) \leqslant \alpha\Delta(x, y, x_n)$$

or (from the definition of the area of a triangle)

$$\pi(Ux, L(Uy, Ux_n))\|Uy - Ux_n\| \leqslant \alpha\pi(x, L(y, x_n))\|y - x_n\|.$$

One deduces that

$$\gamma'\|x_n - y\|\pi(Ux, L(Uy, Ux_n)) \leqslant \pi(Ux, L(Uy, Ux_n))\|Uy - Ux_n\| \leqslant \alpha\pi(x, L(y, x_n))\|y - x_n\|.$$

Hence

(1) $$\pi(Ux, L(Uy, Ux_n)) \leqslant \delta\pi(x, L(y, x_n))$$

with $\delta = \alpha/\gamma' < 1$. Thus for each x in $L(y, x_n)$, one has $Ux \in L(Uy, Ux_n)$, and, hence, U takes $L(y, x_n)$ into $L(Uy, Ux_n)$ for each y in $B'(0, 2r)$ and for each $n \geqslant n_1(r)$. This completes Step 1.

Step 2. *There exists $n_r > n_1(r)$ such that for each $n \geqslant n_r$, V_n restricted to $B'(0, 2r)$ is a contraction.*

For x in $B'(0, 2r)$, let D_x be the line through x, of direction Ux_n. Let V_n be defined as above. Then, for each x, y in $B'(0, 2r)$

(1') $$\|V_n x - V_n y\| = \|P'_n U P_n x - P'_n U P_n y\| = \pi(UP_n x, D_{UP_n y})$$

because P'_n is the orthogonal projection onto H'_n. We claim that there exists $n_r > n_1(r)$ such that for each $n \geqslant n_r$ one has

$$\|V_n x - V_n y\| \leqslant \delta_0\|P_n x - P_n y\| \quad \text{for all } x, y \text{ in } B'(0, 2r).$$

(Here $\delta < \delta_0 < 1$ where, we recall, δ was defined to be α/γ'.) If $\|UP_n x - UP_n y\| = 0$, then $\pi(UP_n x, D_{UP_n y}) = 0$ and hence, trivially

$$\|V_n x - V_n y\| \leqslant \delta_0\|P_n x - P_n y\|.$$

Now, let

$$\|UP_n x - UP_n y\| > 0.$$

Put

$$u = [UP_n x - UP_n y]/\|UP_n x - UP_n y\|, \quad v_n = [Ux_n - UP_n y]/\|Ux_n - UP_n y\|,$$
$$e_n = Ux_n/\|Ux_n\|.$$

Note that by the property of (x_n), $Ux_n \neq 0$, and hence, e_n is defined. Note also that for large n, $Ux_n \neq UP_n y$ for all y in $B'(0, 2r)$. We have then

$$|\sin\langle u, e_n\rangle| = \left(1-(u, e_n)^2\right)^{1/2},$$
$$|\sin\langle u, v_n\rangle| = \left(1-(u, v_n)^2\right)^{1/2}$$

where $\langle .,. \rangle$ denotes the angle between two vectors.

One has

(2) $$\pi(UP_n x, D_{UP_n y}) = \|UP_n x - UP_n y\|\,|\sin\langle u, e_n\rangle| \leqslant k\|P_n x - P_n y\|\,|\sin\langle u, e_n\rangle|$$

where k is a constant such that

$$\|Ux - Uy\| \leqslant k\|x - y\| \quad \text{for all } x, y \text{ in } B'(0, 2r).$$

One has furthermore

(3) $$\pi(UP_n x, L(UP_n y, Ux_n)) = \|UP_n x - UP_n y\|\,|\sin\langle u, v_n\rangle|.$$

Now, it is clear that

$$\sin^2\langle u, v_n\rangle - \sin^2\langle u, e_n\rangle = (u, e_n)^2 - (u, v_n)^2 = [(u, e_n) - (u, v_n)][(u, e_n) + (u, v_n)]$$

from which it follows that

(4) $$|\sin^2\langle u, v_n\rangle - \sin^2\langle u, e_n\rangle| = |(u, e_n - v_n)(u, e_n + v_n)| \leqslant 2\|e_n - v_n\|.$$

We claim that $\|v_n - e_n\| \to 0$ uniformly with respect to x, y in $B'(0, 2r)$. Indeed

$$v_n - e_n = [Ux_n - UP_n y]/\|Ux_n - UP_n y\| - Ux_n/\|Ux_n\|$$

and hence,

$$\|v_n - e_n\| \leqslant \|Ux_n\|(\|Ux_n - UP_n y\|^{-1} - \|Ux_n\|^{-1}) + \|UP_n y\|/\|Ux_n - UP_n y\|.$$

Now, by the first part of the proof of Step 1, we have $\|UP_n y\| \leqslant R$ for y in $B'(0, 2r)$. Since $\|Ux_n\| \to \infty$, we have

$$\|v_n - e_n\| \to 0 \text{ as claimed.}$$

Let $\varepsilon_n = 2\|v_n - e_n\|$. There exists an $n_r \geqslant 1$ such that for all $n \geqslant n_r$

(4') $$\varepsilon_n < (1/2)(\delta/k)^2 \quad \text{and} \quad (\delta/k)^2/((\delta/k)^2 - \varepsilon_n) \leqslant (\delta_0/\delta)^2 \quad \text{with} \quad \delta < \delta_0 < 1$$

for all x, y in $B'(0, 2r)$ (note ε_n is function of x and y). As a result, for $n \geqslant n_r$, one has:

(i) For all x, y in $B'(0, 2r)$ with $\sin^2\langle u, e_n\rangle \leqslant \delta^2/k^2$

$$\pi(UP_n x, D_{UP_n y}) \leqslant \delta\|P_n x - P_n y\| \quad \text{(in view of (2))}$$

and thus (by (1′))

$$\|V_n x - V_n y\| \leqslant \delta_0 \|P_n x - P_n y\|.$$

(ii) For x, y in $B'(0, 2r)$ such that $\sin^2\langle u, e_n\rangle > \delta^2/k^2$

$$\sin^2\langle u, v_n\rangle > \tfrac{1}{2}(\delta/k)^2 \quad \text{(by (4) and (4′))}.$$

From (2) and (3), one has then

$$\pi(UP_n x, D_{UP_n y}) = \pi(UP_n x, L(UP_n y, Ux_n))|\sin\langle u, e_n\rangle|/|\sin\langle u, v_n\rangle| \equiv \text{RHS}.$$

Now

(5) $$\text{RHS} \leqslant \pi(UP_n x, L(UP_n y, Ux_n))|\sin\langle u, e_n\rangle|/(\sin^2\langle u, e_n\rangle - \varepsilon_n)^{1/2}.$$

Since the function

$$x \to x/(x^2 - a^2)^{1/2}$$

is decreasing for $x > |a|$, we have then

$$|\sin\langle u, e_n\rangle|/(\sin^2\langle u, e_n\rangle - \varepsilon_n)^{1/2} \leqslant \delta k^{-1}/((\delta/k)^2 - \varepsilon_n)^{1/2} \leqslant \delta_0/\delta$$

from which it follows that

$$\pi(UP_n x, D_{UP_n y}) \leqslant \delta_0 \pi(P_n x, L(P_n y, x_n)).$$

Now

$$\pi(P_n x, L(P_n y, x_n)) \leqslant \|P_n x - P_n y\|.$$

Hence, in view of (1′)

$$\|V_n x - V_n y\| \leqslant \delta_0 \|P_n x - P_n y\|.$$

We have just shown that for $n \geqslant n_r$

$$\|V_n x - V_n y\| \leqslant \delta_0 \|P_n x - P_n y\| \leqslant \delta_0 \|x - y\| \quad \text{for all } x, y \text{ in } B'(0, 2r).$$

This completes Step 2.

Step 3. *(U_n) converges to U uniformly on $B'(0, r)$.*

We shall show that for each $\varepsilon > 0$, there exists $n_\varepsilon \geqslant 1$ such that for every $n \geqslant n_\varepsilon$

$$\|Ux - U_n x\| < \varepsilon \quad \text{for all } x \text{ in } B'(0, r).$$

For x in $B'(0, r)$, let x' be the intersection point of $L(x_n, x)$ with H_n (x' may well not belong to $B'(0, r)$, but for all sufficiently large n, x' is in $B'(0, 2r)$, and this was the reason why in Step 1 and Step 2, we had to consider U and V_n on $B'(0, 2r)$ rather than on $B'(0, r)$). Now, by Step 1, one has $Ux \in L(Ux_n, Ux')$ for each x in $B'(0, r)$. One has

$$U_n x = P_n' UP_n x + (e_n, Ux) e_n$$

and thus, $U_n x$ is the projection of $Ux \in L(Ux_n, Ux')$ into $D_{UP_n x}$ (the line through $UP_n x$, of direction e_n).

By Theorem 3

$$\|Ux' - UP_n x\| \leqslant k\|x' - P_n x\|.$$

(Note that both x' and $P_n x$ are in $B'(0, 2r)$.) If $x' = 0$ then $P_n x = x' = 0$, which implies

$$\|x' - P_n x\| = 0 \quad \text{and hence} \quad \|Ux' - UP_n x\| = 0.$$

If $x' \neq 0$, then, by considering the similar triangles x', x_n, 0 and x', x, $P_n x$, one has

$$\|x' - P_n x\|/\|x'\| = \|x - P_n x\|/\|x_n\| \leqslant 2r/\|x_n\|.$$

(Note that x, $P_n x$ are in $B'(0, r)$.)

One then deduces

$$\|x' - P_n x\| \leqslant 2r\|x'\|/\|x_n\| \leqslant 4r^2/\|x_n\|.$$

Hence, for all x in $B'(0, r)$

$$\|Ux' - UP_n x\| \leqslant 4r^2 k/\|x_n\|.$$

Since $\|x_n\| \to \infty$ for $n \to \infty$, there exists $n_1 \geqslant 1$ such that for all $n \geqslant n_1$

$$4r^2 k/\|x_n\| < \tfrac{1}{4}\varepsilon,$$

which implies

$$\|Ux' - UP_n x\| < \tfrac{1}{4}\varepsilon \quad \text{or} \quad Ux' \in B(UP_n x, \tfrac{1}{4}\varepsilon) \quad \text{for} \quad n \geqslant n_1.$$

Since Ux is in $L(Ux_n, Ux')$ and since $U_n x$ is the projection of Ux into $D_{UP_n x}$, one has

$$\|U_n x - Ux\| = \pi(Ux, D_{UP_n x}) \quad (= \text{distance from } Ux \text{ to } D_{UP_n x}).$$

If $\|Ux - UP_n x\| = 0$, then $\pi(Ux, D_{UP_n x}) = 0$ and hence $\|U_n x - Ux\| = 0$. Now, let $\|Ux - UP_n x\| > 0$. Put

$$u = (Ux - UP_n x)/\|Ux - UP_n x\|, \quad v_n = (Ux_n - UP_n x)/\|Ux_n - UP_n x\|.$$

Then $\|v_n - e_n\| \to 0$ uniformly on $B'(0, r)$ for $n \to \infty$. Let $\varepsilon_n = 2\|v_n - e_n\|$ and note, for further use, that by inequality (4) in Step 2,

$$\sin^2\langle u, v_n\rangle \geqslant \sin^2\langle u, e_n\rangle - \varepsilon_n.$$

Since, by what precedes, (ε_n) converges to 0 uniformly on $B'(0, r)$, there exists $n_2 > n_1$ such that for all $n \geqslant n_2$

$$\varepsilon_n < \varepsilon^2/2(rk)^2 \quad \text{and} \quad (\varepsilon/rk)/((\varepsilon/rk)^2 - \varepsilon_n)^{1/2} < 2 \quad \text{for all } x \text{ in } B'(0, r).$$

Our aim is to prove there exists $n_3 > n_2$, such that for all $n \geqslant n_3$

$$(*) \qquad \|U_n x - Ux\| \leqslant \varepsilon \quad \text{for each } x \text{ in } B'(0, r).$$

We distinguish two cases.

Case 1. *x in $B'(0, r)$ is such that $|\sin\langle u, e_n\rangle| \leqslant \varepsilon/kr$.*

In this case, we have

$$\|U_n x - Ux\| = \pi(Ux, D_{UP_n x}) = \|Ux - UP_n x\| \, |\sin\langle u, e_n\rangle| \leqslant k\|x - P_n x\| \, |\sin\langle u, e_n\rangle| \leqslant kr|\sin\langle u, e_n\rangle| \leqslant \varepsilon.$$

Thus $(*)$ holds in this case.

Case 2. *x in $B'(0, r)$ is such that $|\sin\langle u, e_n\rangle| > \varepsilon/rk$.*

In this case, since $n > n_2$, one has

$$|\sin\langle u, v_n\rangle| \geqslant (\sin^2\langle u, e_n\rangle - \varepsilon_n)^{1/2} \geqslant \varepsilon/2rk.$$

Now

$$\pi(Ux, L(UP_n x, Ux_n)) = \|Ux - UP_n x\| \, |\sin\langle u, v_n\rangle|.$$

Hence

$$\begin{aligned}\pi(Ux, D_{UP_n x}) &= \pi(Ux, L(UP_n x, Ux_n)) |\sin\langle u, e_n\rangle| / |\sin\langle u, v_n\rangle| \\ &\leqslant \pi(Ux, L(UP_n x, Ux_n)) |\sin\langle u, e_n\rangle| / (\sin^2\langle u, e_n\rangle - \varepsilon_n)^{1/2} \\ &\leqslant \pi(Ux, L(UP_n x, Ux_n)) (\varepsilon/kr) / ((\varepsilon/kr)^2 - \varepsilon_n)^{1/2} \\ &\leqslant 2\pi(Ux, L(UP_n x, Ux_n)).\end{aligned}$$

Recall that $\|Ux_n\| \geqslant \gamma\|x_n\| \to \infty$ for $n \to \infty$. Hence, there exists $n_3 > n_2$ such that for all $n \geqslant n_3$, $\|Ux_n\| > 4R$. Since Ux and Ux' are in $B'(0, R)$ and since Ux, Ux', Ux_n are collinear, one has

$$\pi(Ux, L(Ux_n, UP_n x)) \leqslant 2\pi(Ux', L(Ux_n, UP_n x)).$$

But

$$\pi(Ux', L(Ux_n, UP_n x)) \leqslant \|Ux' - UP_n x\| < \tfrac{1}{4}\varepsilon.$$

Hence

$$\pi(Ux, L(Ux_n, UP_n x)) < \tfrac{1}{2}\varepsilon.$$

It follows that

$$\|U_n x - Ux\| = \pi(Ux, D_{UP_n x}) \leqslant 2\pi(Ux, L(UP_n x, Ux_n)) \quad \text{for all } x \text{ in } B'(0, r)$$

for all $n \geqslant n_3$. Thus $(*)$ holds in this second case.

We have just proved that $U_n \to U$ uniformly on $B'(0, r)$. This completes Step 3 and the proof of the theorem.

COROLLARY 1. *Let U be as in Theorem 4. If, in addition, the sequence $e_n = Ux_n/\|Ux_n\|$ has a convergent subsequence, then*

$$U = V + F$$

where V is a contraction and F is completely continuous such that $F(H)$ is part of a line. In particular, if H is finite dimensional, then $U = V + F$ where V and F are as above.

Proof. We can assume that the sequence (e_n) itself converges to e (say). Recall

$$F_n x = (e_n, Ux) e_n.$$

Let $Fx = (e, Ux) e$. Then

$$\|F_n x - Fx\| \leqslant \|Ux\| \, \|e - e_n\| + |(e, Ux)| \, \|e - e_n\|.$$

Since U is bounded on $B'(0, r)$, one has $F_n \to F$ uniformly on $B'(0, r)$. Let $V_n = U_n - F_n$. Then $V_n \to U - F$ uniformly on $B'(0, r)$ (since $U_n \to U$ uniformly on $B'(0, r)$). Since for each $n \geqslant n_r$, V_n is a contraction with coefficient $\delta_0 < 1$ (not depending on $n \geqslant n_r$) V is a contraction with coefficient δ_0. ∎

COROLLARY 2. *Let U be as in Theorem 4. Suppose in addition that U is nontrivial. Let H_i be a finite dimensional subspace of H, let P_i be the orthogonal projection onto H_i. If $P_i U$ has no fixed point, then $U = V + F$ where V is a contraction and F is completely continuous with $F(H)$ contained in a line.*

Proof. Let U be α-TC. The proof consists of two steps.

Step 1. We shall prove that $P_i U$ is α-TC. Indeed, one has

$$\|P_i x - P_i y\| \leqslant \|x - y\| \quad \text{for all } x, y \text{ in } H.$$

We claim that for all x, y, z in H

$$\Delta(P_i x, P_i y, P_i z) \leqslant \Delta(x, y, z).$$

Indeed,

$$\Delta(P_i x, P_i y, P_i z) = \tfrac{1}{2}\pi(P_i x, L(P_i y, P_i z)) \|P_i x - P_i z\|.$$

Let x' be the orthogonal projection of x into $L(z, y)$. Then

$$\pi(x, L(y, z)) = \|x - x'\|.$$

Since $P_i x'$ is in $L(P_i y, P_i z)$, one has

$$\pi(P_i x, L(P_i y, P_i z)) \leqslant \|P_i x - P_i x'\| \leqslant \|x - x'\|.$$

It follows that

$$\Delta(P_i x, P_i y, P_i z) \leqslant \tfrac{1}{2}\pi(x, L(y, z)) \|y - z\| = \Delta(x, y, z)$$

as claimed. Thus, P_i decreases both distances and areas of triangles, and hence, $P_i U$ is α-TC if U is α-TC.

Step 2. Note from the hypothesis that $P_i U$ has no fixed point. Since H_i is finite dimensional, $P_i U$ has a fixed line L (say [1]). It follows that $P_i U|L$ has no fixed point. Hence, there exists a sequence (x_n) in L such that $\|x_n\| \to \infty$ and $\|P_i U x_n\| \geqslant \|x_n\|$ for each n. This implies

$$\|Ux_n\| \geqslant \|x_n\| \quad \text{for each } n.$$

Let x_0 be any point of L. Then, for all sufficiently large n, one has

$$\|Ux_n - Ux_0\| > \alpha \|x_n - x_0\|.$$

It follows that $U(L)$ is part of a line L' (say); in particular Ux_n is in L' for each n. Hence

$$e_n = Ux_n/\|Ux_n\| \to e' \quad \text{where } e' \text{ is a direction vector of } L'.$$

Thus, by the preceding corollary, $U = V+F$ where V is a contraction and F is completely continuous such that $F(H)$ is part of a line. ■

We end up this paper with the following

THEOREM 5. *Let U be an α-TC operator of H with $|U| > \alpha$. Let F be a completely continuous operator on H such that $|U+F| < 1$. Then*

$$R[I-(U+F)] = H$$

where R denotes the range of a map.

For the proof, we need some lemmas.

LEMMA 2.1. *Let U be a nontrivial TC operator on H. Let K be a closed bounded subset of H. If $0 \notin (I-U)(K)$, then there exists $a > 0$ such that*

$$\|(I-U)x\| \geqslant a \quad \textit{for every } x \textit{ in } K.$$

Proof. Let U be α-TC. Suppose by contradiction that there exists a sequence (x_n) in K with $\|Ux_n - x_n\| \to 0$. Then, since $0 \notin (I-U)(K)$, (x_n) has no convergent subsequence. Hence, there exist a subsequence also denoted (x_n) (by a change of notation) and a $d > 0$ such that $\|x_m - x_n\| \geqslant d$ for every $n \neq m$. Since

$$\|(Ux_n - x_n) - (Ux_m - x_m)\| \to 0 \quad \text{for} \quad m, n \to \infty$$

there exists for each β with $\alpha < \beta < 1$, an n_0 such that for every $m, n \geqslant n_0$, $m \neq n$, one has

$$\|Ux_n - Ux_m\| \geqslant \beta \|x_n - x_m\| > \alpha \|x_n - x_m\|.$$

Since U is α-TC, one has

$$(1) \quad \Delta(Ux_n, Ux_m, Ux_k) \leqslant \alpha \Delta(x_n, x_m, x_k) \quad \text{for all } m, n, k \geqslant n_0 \text{ with } n \neq m.$$

Since in bounded sets, the area is uniformly continuous in the three variables jointly, one has

$$(2) \quad |\Delta(Ux_n, Ux_m, Ux_k) - \Delta(x_n, x_m, x_k)| \to 0 \quad \text{for} \quad m, n, k \to \infty$$

(this is true since $\|Ux_n - x_n\| \to 0$ for $n \to \infty$).

We claim that $\Delta(x_n, x_m, x_k) \to 0$ for $n, m, k \to \infty$. Indeed, if this is not the case, then there exists a subsequence

$$\Delta(x_{n_i}, x_{m_i}, x_{k_i}) \geqslant b > 0 \quad \text{for some } b.$$

We can assume, by another change of notation, that

$$\Delta(x_n, x_m, x_k) \geqslant b > 0.$$

By (2), there exists for each $\alpha < \beta' < 1$, an $n_1 > n_0$ such that

$$\Delta(Ux_n, Ux_m, Ux_k) \geqslant \beta' \Delta(x_n, x_m, x_k) > \alpha \Delta(x_n, x_m, x_k) \quad \text{for all } m, n, k \geqslant n_1.$$

But this contradicts (1). Hence, we have

$$\Delta(x_n, x_m, x_k) \to 0 \quad \text{for} \quad n, m, k \to \infty$$

as claimed. Now

$$\Delta(x_n, x_m, x_k) = \tfrac{1}{2}\pi(x_n, L(x_m, x_k))\|x_m - x_k\|$$

where

$$\|x_m - x_k\| \geqslant d > 0 \quad \text{for} \quad m \neq k.$$

Hence,

$$\pi(x_n, L(x_m, x_k)) \to 0 \quad \text{for} \quad n, m, k \to \infty.$$

Thus, there exists $n_2 \geqslant 1$ such that for $n, m, k \geqslant n_2$, we have

$$\pi(x_n, L(x_m, x_k)) < \tfrac{1}{8}d.$$

Fix $m_0, k_0 \geqslant n_2$, and call a_n the orthogonal projection of x_n into $L(x_{m_0}, x_{k_0})$. Then

$$\begin{aligned} \|a_n - a_m\| &= \|a_n - x_m + x_m - x_n + x_n - a_m\| \\ &\geqslant \|x_n - x_m\| - \|a_n - x_n\| - \|a_m - x_m\| \\ &\geqslant d - \tfrac{1}{8}d - \tfrac{1}{8}d \geqslant \tfrac{3}{4}d > 0 \quad \text{for all } n \neq m,\ n, m \geqslant n_2. \end{aligned}$$

Since $a_n \in L(x_{m_0}, x_{k_0})$, it follows that the set $\{a_n : n \geqslant n_2\}$ is not bounded, a contradiction. We conclude that there exists $a > 0$ such that

$$\|Ux - x\| \geqslant a \quad \text{for each } x \text{ in } K. ■$$

COROLLARY OF LEMMA 2.1. *Let U be as in Lemma 2.1. Let K be a closed* *bounded set in H. Let F be a completely continuous operator on H. Suppose* $0 \notin [I-(U+F)](K)$. *Then, there exists an* $a > 0$ *such that*

$$\|Ux+Fx-x\| \geqslant a \quad \textit{for each } x \textit{ in } K.$$

Proof. Suppose by contradiction that there exists a sequence (x_n) in K such that

$$\|x_n - Ux_n - Fx_n\| \to 0 \quad \text{for} \quad n \to \infty.$$

Then (x_n) has no convergent subsequence, as is easily seen. Now, $F(K)$ is relatively compact; hence, there exists a subsequence (Fx_{n_k}) converging to a z (say). By a change of notation, we can assume that $Fx_n \to z$. Then we have

$$\|x_n - Ux_n - z\| \leqslant \|x_n - Ux_n - Fx_n\| + \|Fx_n - z\| \to 0.$$

Consider the operator U_z on H defined by

$$U_z x = Ux + z.$$

Then, U_z is a nontrivial α-TC operator and

$$\lim_{n\to\infty} \|x_n - U_z x_n\| = 0.$$

As noted earlier, (x_n) has no convergent subsequence. The set $(x_n) \equiv A$ is then an infinite closed bounded set, and we can assume $x_n \neq x_m$ for $n \neq m$. By Lemma 2.1, U_z has a fixed point in A. The set A_U of the fixed points of U_z in A is infinite, as can be seen by repeating the argument, using the same Lemma 2.1. This implies, since U_z is TC, that A_U is part of a line L (say), and thus,

$$A_U \subset L \cap K.$$

Since $L \cap K$ is compact, A_U contains a convergent subsequence of (x_n), a contradiction. This proves that (x_n) has a subsequence (x_{n_k}) which converges to x_0 (say) in K. It is clear that

$$Fx_{n_k} \to Fx_0 = z.$$

Hence

$$\lim_{k\to\infty} \|x_{n_k} - Ux_{n_k} - Fx_{n_k}\| = \|x_0 - Ux_0 - Fx_0\| = 0,$$

i.e., $x_0 = (U+F)x_0$ which implies contradiction. Hence, there exists an $a > 0$ such that $\|Ux+Fx-x\| \geqslant a$ for all x in K as desired. ∎

LEMMA 2.2. *Let U be a nontrivial* TC *operator on H. Let F be a completely continuous operator on H. Suppose* $|U+F| < 1$. *Then, there exist* $r_0 > 0$ *and* $0 < \delta < 1$ *such that*

$$(U+F)x \in B'(0, \delta r_0) \quad \textit{for} \quad \|x\| \leqslant r_0.$$

Proof. Since $|U+F| < 1$, there exists for each δ with $|U+F| < \delta < 1$, an $r > 0$ such that

$$\|Ux+Fx\| \leqslant \delta\|x\| \quad \text{for all } \|x\| \geqslant r.$$

Since U is nontrivial TC, $U(B'(0, r))$ is bounded by the corollary of Theorem 3. Since F is completely continuous, $F(B'(0, r))$ is bounded. Hence, there exists an $r_0 > r$ such that

$$Ux+Fx \in B'(0, \delta r_0) \quad \text{for each } x \text{ in } B'(0, r).$$

What precedes shows that $Ux+Fx$ is in $B'(0, \delta r_0)$ for each x in $B'(0, r_0)$. ∎

We now turn to

Proof of Theorem 5. It is sufficient to prove that $U+F$ has a fixed point. With δ and r_0 as in Lemma 2.2, we have, by Theorem 4, two sequences of operators, (V_n) and (F_n), on H such that for each n, V_n restricted to $B'(0, r_0)$ is a contraction with coefficient $\delta_0 < 1$, F_n is completely continuous with $F_n(H)$ contained in a line and

$$V_n + F_n \to U \quad \text{uniformly on } B'(0, r_0).$$

Since $Ux+Fx$ is in $B'(0, \delta r_0)$, $0 < \delta < 1$, for each x in $B'(0, r_0)$, there exists n_0 such that for all $n \geqslant n_0$

$$U_n x + Fx \text{ is in } B'(0, r_0) \quad \text{for each } x \text{ in } B'(0, r_0).$$

Here $U_n = V_n + F_n$. We have

$$V_n + F_n + F\colon B'(0, r_0) \to B'(0, r_0).$$

Since V_n is a contraction and $F_n + F$ is completely continuous, the operator $V_n + F_n + F$ has, by Theorem 1, a fixed point x_n, i.e.,

$$x_n = U_n x_n + Fx_n.$$

Then

$$x_n - Ux_n - Fx_n = U_n x_n - Ux_n.$$

Since $U_n \to U$ uniformly on $B'(0, r_0)$, we have

$$\lim_{n\to\infty} \|x_n - Ux_n - Fx_n\| = \lim_{n\to\infty} \|U_n x_n - Ux_n\| = 0.$$

By the corollary of Lemma 2.1, $U+F$ has a fixed point. This completes the proof. ∎

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