Countable modules

by

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Abstract. In this paper I prove that if R is a countable ring and M is an R-module, then either M is $\aleph_0$-categorical or there are infinitely many pairwise non-isomorphic countable R-modules elementarily equivalent to M (in the usual language for R-modules). I also show that for an arbitrary theory T, (not necessarily a theory of modules), if every formula is equivalent modulo T to a Boolean combination of certain 'nice' formulae, and T is not $\aleph_0$-categorical, then T has at least four countable models. This latter elimination property was suggested to us by the example of modules.

Let me sum up the earlier results on the number of models of theories of modules. Let R be a countable ring and T a complete theory of R-modules. T is always stable. The possible spectra (number of models of T in each cardinality) for *uncountable* models of T is known [9]. In the case in which T is ω-stable, the complete spectrum problem is solved ([2], [7] and [9]). In particular the number of countable models of T is 1, $\aleph_0$ or $2^{\aleph_0}$. An important fact in the ω-stable case is that every model of T is uniquely expressible as a direct sum of indecomposable modules. For superstable T, the number of countable models is 1 or infinite, but this is a property of *all* superstable theories. On the other hand there is at present no *general* theorem on the number of countable models of an *arbitrary* stable theory.

In Section 1, I present some preliminary information on modules, forking *and* forking in modules. In Section 2, I show that any theory of R-modules, T, has one or infinitely many countable models. In fact I show that such a theory always has a non-isolated type of U-rank at most ω, if it is not $\aleph_0$-categorical. The treatment is *model*-theoretic, rather than *module*-theoretic. In Section 3, I consider a theory T satisfying more general conditions, and I show that such a theory has either one or at least four countable models. The treatment is in the style of [5], that is, without use of stability machinery (although the theories under consideration will be stable).

I thank G. Cherlin for some very helpful suggestions.

1. T will always be a countable complete first order theory, and L the language of T. A positive primitive (p.p.) formula of L is a formula of the form $(\exists \bar{x})(\bigwedge_{i<n} \Theta_i)$, where the Θ_i are atomic formulae of L.

If R is a ring with a 1 (in our case a countable ring), then the language L_R for (right) R-modules contains "+" for module addition, "0" for the zero element of a module, and also a symbol "r" for each element r of R, to represent multiplication of module elements by r.

The crucial tool in the model theory of modules is the following:

PROPOSITION 1.1 [1]. *Let T be a complete theory of R-modules. Then any formula $\varphi(\bar{x})$ of L_R is equivalent modulo T to a formula $\psi(\bar{x})$, where $\psi(\bar{x})$ is a Boolean combination of p.p. formulae.*

Now let $\varphi(\bar{x}, \bar{y})$ be a p.p. formula of L_R, where $l(\bar{x}) = n$. Then it is not difficult to see that, for any R-module M, $\varphi(\bar{x}, \bar{0})^M$ (which is the set of n-tuples of M satisfying $\varphi(\bar{x}, \bar{0})$) is a subgroup of M^n. Moreover, if $\bar{a} \in M$, then $\varphi(\bar{x}, \bar{a})^M$ is either empty or is a coset of $\varphi(\bar{x}, \bar{0})^M$. An easy consequence of this fact and Proposition 1.1 is that any theory of R-modules is stable.

For general stable theories a notion of forking has been developed by S. Shelah. Namely, let $A \subset B$ be subsets of a model of a stable theory, and p a complete type over B: then a certain meaning is given to the expression "p does not fork over A" (or equivalently "p is a non-forking extension of $q = p\restriction A$"), the intuitive content of which is that if a realises p then a depends on B no more than it depends on A. For the details, see [4]. It is worth mentioning that an important fact about forking is that $tp(\bar{a}/A \cup \bar{b})$ does not fork over A iff $tp(\bar{b}/A \cup \bar{a})$ does not fork over A (symmetry).

A useful rank on types is the U-rank (introduced by Lascar), which tells one how many times a type can fork. To be more precise:

DEFINITION 1.2. Let M be a very big model of a stable theory. Let $n < \omega$. The ordinal valued *rank* U_n on complete n-types over subsets of M is defined as follows:

i) $U_n(p) \geqslant 0$ for any such type p, and

ii) Let p be over A. Then $U_n(p) \geqslant \alpha+1$ if there is $B \supset A$ and n-type q over B which extends p such that q forks over A and $U_n(q) \geqslant \alpha$.

We will normally omit n, its value being clear from the context. If $U(p) > \alpha$ for all α, then we put $U(p) = \infty$. If $U(p) = \alpha < \infty$, then the nonforking extensions of p are precisely the extensions of p with U-rank α. A stable theory is superstable if and only if all types have U-rank less than ∞.

Let $\alpha \oplus \beta$ denote the natural sum of ordinals α and β. Namely, if

$$\alpha = \omega^{\gamma_1} n_1 + \omega^{\gamma_2} n_2 + \ldots + \omega^{\gamma_k} n_k \quad \text{and} \quad \beta = \omega^{\gamma_1} m_1 + \omega^{\gamma_2} m_2 + \ldots + \omega^{\gamma_k} m_k$$

where $n_1, n_2, \ldots, n_k, m_1, m_2, \ldots, m_k$ are natural numbers and $\langle \gamma_1, \gamma_2, \ldots, \gamma_k \rangle$ is a strictly decreasing sequence of ordinals, then

$$\alpha \oplus \beta = \omega^{\gamma_1}(n_1+m_1) + \omega^{\gamma_2}(n_2+m_2) + \ldots + \omega^{\gamma_k}(n_k+m_k).$$

FACT 1.3 [3]. *Let $\bar{a}, \bar{b}$ be finite tuples and A a subset, all in a model of a stable theory. Then $U(tp(\bar{a}^\frown \bar{b}/A)) \leqslant U(tp(\bar{a}/A \cup \bar{b})) \oplus U(tp(\bar{b}/A))$.*

We will now return to modules and see what forking means in that context. Firstly, if $p(\bar{x})$ is a type over A and $\varphi(\bar{x}, \bar{y})$ is a L-formula, we say that $\varphi(\bar{x}, \bar{y})$ *is represented in* p if there is $\bar{a} \in A$ such that $\varphi(\bar{x}, \bar{a}) \in p$.

PROPOSITION 1.4 [6]. *Let M be an R-module, $A \subset B$ be subsets of M, and p and q be complete n-types over A and B respectively, such that $p \subset q$. Then q is a forking extension of p if and only if there is a p.p. formula $\varphi(\bar{x}, \bar{y})$ represented in q such that for every p.p. formula $\psi(\bar{x}, \bar{z})$ represented in p $(\varphi(\bar{x}, \bar{0}) \wedge \psi(\bar{x}, \bar{0}))^M$ has infinite index in $\psi(\bar{x}, \bar{0})^M$.*

Thus with any type over a subset of a module, one can associate a set of p.p. definable subgroups, namely those subgroups cosets of which are in the type. Moreover it is by looking at these associated sets of subgroups that one can determine whether or not one type is a forking extension of another.

2. Let R be a countable ring and T a complete theory of R-modules.

In this section, I will, until otherwise stated, *use φ, ψ to denote p.p. formulae of L_R*. A formula will be assumed to contain no parameters unless those parameters are exhibited. Also I will tend to identify a formula and the subset which it defines (in some big model of T, say). Thus, for example, I will talk about the inclusion of formulae when I really mean the inclusion of the subsets they define.

I will prove first:

PROPOSITION 2.1. *If T is not $\aleph_0$-categorical, then T has a non-isolated type p (over the empty set) such that $U(p) \leqslant \omega$.*

Thus assume T to be non $\aleph_0$-categorical. Thus $S_n(T)$ (the complete n-types over $\emptyset$ of T) *is infinite for some $n < \omega$*. For notational convenience, *I take n to be* 1. (In fact if $S_1(T)$ is finite, then T is ω-stable and we have a classification of all the models.)

DEFINITION 2.2. $\psi(x)$ is *small*, if there are only a finite number of $\varphi(x)$ (up to equivalence modulo T of course) such that $\varphi(x) \subset \psi(x)$. If $\psi(x)$ is small I define $h(\psi)$ to be the largest $n < \omega$ such that there are $\psi_0(x) \subsetneqq \psi_1(x) \subsetneqq \ldots \subsetneqq \psi_n(x) = \psi(x)$. ($\psi_0(x)$ is always $x = 0$.) If $\psi(x)$ is not small, I say $\psi(x)$ is big.

The following is obvious:

FACT 2.3. *Suppose that $\psi_i(x)$ for each $i < m$ (where $m < \omega$) is small, and that $\psi(x) \subset \bigvee_{i<m} \psi_i(x)$. Then $\psi(x)$ is small.*

Now I commence the construction of the required type (with thanks to Cherlin for his help). First let $\Psi = \{\neg\psi(x)\colon \psi \text{ is small}\}$.

LEMMA 2.4. *Ψ is consistent (modulo T of course).*

Proof. If not, then there are small $\psi_i(x)$ for $i < m$ such that $T \vdash (\forall x)(\bigvee_{i<m} \psi_i(x))$. But then there are clearly only a finite number of p.p. 1-

formula up to equivalence modulo T. By Proposition 1.1 (p.p. elimination), this implies that there are only a finite number of 1-formulae modulo T, which contradicts our assumption that $S_1(T)$ is infinite.

Now let Φ be a maximal set of p.p. 1-formulae such that $\Phi \cup \Psi$ is consistent with T. Thus by Proposition 1.1 again, $T \cup \Phi \cup \Psi$ generates a complete consistent 1-type of T, say $p(x)$.

Note that Φ is closed under finite conjunctions (the conjunction of a set of p.p.'s being equivalent to a p.p.), and moreover that if $\varphi \in \Phi$ then φ is big.

FACT 2.5. *$p(x)$ is not isolated.*

Proof. Suppose that $\chi(x)$ isolates $p(x)$. Then as p is generated by $T \cup \Phi \cup \Psi$ it is clear that χ is equivalent modulo T to $\varphi(x) \wedge \bigwedge_{i<n} \neg\psi_i(x)$ where φ is in Φ and thus big, and the ψ_i are small. But then clearly $\chi(x)$ contains infinitely many p.p. formulae (i.e. the subset defined by $\chi(x)$ contains infinitely many p.p. definable subsets), which is a contradiction.

I now show that $U(p) \leqslant \omega$.

LEMMA 2.6. *Let $q \in S_1(A)$ be a forking extension of p. Then there is $\psi(x, \bar{y})$ represented in q such that $\psi(x, \bar{0})$ is small.*

Proof. By Proposition 1.4, there is $\psi(x, \bar{y})$ represented in q such that for all $\varphi(x)$ in p, $\psi(x, \bar{0}) \wedge \varphi(x)$ is a proper subgroup of $\varphi(x)$. In particular we must have that $\neg\psi(x, \bar{0}) \in p$. But then

$$T \cup \{\varphi(x) \wedge \bigwedge_{i<n} \neg\psi_i(x)\} \vdash \neg\psi(x, \bar{0}),$$

where φ is big and the ψ_i are small. Then we have

$$T \vdash \varphi(x) \wedge \psi(x, \bar{0}) \rightarrow \bigvee_{i<n} \psi_i(x).$$

So by Fact 2.3, $\varphi(x) \wedge \psi(x, \bar{0})$ is *small*. Clearly $\varphi(x) \wedge \psi(x, \bar{y})$ is represented in q. So the lemma is proved.

LEMMA 2.7. *Let $q \in S_1(A)$, and $\psi(x, \bar{y})$ be represented in q, where $\psi(x, \bar{0})$ is small and $h(\psi(x, \bar{0})) \leqslant n < \omega$. Then $U(q) \leqslant n$.*

Proof. Easy, by induction, using again Proposition 1.4.

COROLLARY 2.8. *$p(x)$, the type constructed above, has U rank at most ω.*

Proof. Immediate by Lemma 2.6, Lemma 2.7, and Definition 1.2 of U rank.

So Proposition 2.1 is proved, by 2.5 and 2.8.

Now by Theorem 10, Chapter 10 of [3], the fact that T has a nonisolated type with *some* U rank (less than ∞) is enough to prove that T has infinitely many countable models. However I will go on to show that the condition that the type has U rank at most ω, enables one to construct a

particularly nice sequence of models which witnesses T having infinitely many isomorphism types of countable models. This being true in general, I now drop the assumption that T is a theory of R-modules. Thus T is now simply a countable complete theory. Recall first that a set $\{\bar{a}_i\colon i < \alpha\}$ of tuples in a model of a stable theory, is said to be *independent* if for every $i < \alpha$ $tp(\bar{a}_i / \bigcup \{\bar{a}_j\colon j < \alpha, j \neq i\})$ does not fork over $\emptyset$. Moreover for any type p there are always independent sets of realisations of p of any cardinality.

PROPOSITION 2.9. *Let T be stable and p a non-isolated (complete) type of T (over $\emptyset$) such that $U(p) \leqslant \omega$. Suppose that $\{\bar{a}_i\colon i < \omega\}$ is an independent set of realisations of p in some model of T, and that further, for each $n < \omega$ the model M_n is prime over $\bigcup \{\bar{a}_i\colon i < n\}$. Then $m < n$ implies that M_n and M_m are not isomorphic.*

Proof. Note that the models M_n are all countable models of T. Let me remark first that for each $n < \omega$ $tp(\bar{a}_n / \bigcup \{\bar{a}_i\colon i < n\})$ is not isolated. This is a consequence of the "Open map theorem" [4].

There are now two cases to consider.

Case I.) $U(p)$ is finite. It then follows from Fact 1.3 that for all $n < \omega$ $tp(\bar{a}_0 ^\frown \bar{a}_1 ^\frown \ldots ^\frown \bar{a}_{n-1})$ has finite U rank. It is now a consequence of Theorem 13, Chapter 13 [3], that for each n, M_n is *relatively homogeneous*. This means that if $\bar{b} \in M_n$ and $tp(\bar{b}) = tp(\bar{a}_0 ^\frown \bar{a}_1 ^\frown \ldots ^\frown \bar{a}_{n-1})$, then there is an automorphism of M_n taking $\bar{a}_0 ^\frown \bar{a}_1 ^\frown \ldots ^\frown \bar{a}_{n-1}$ to $\bar{b}$. Thus, as $tp(\bar{a}_n / \bar{a}_0 ^\frown \bar{a}_1 ^\frown \ldots ^\frown \bar{a}_{n-1})$ is not realised in $(M_n, \bar{a}_0 ^\frown \bar{a}_1 ^\frown \ldots ^\frown \bar{a}_{n-1})$, it follows that $tp(\bar{a}_0 ^\frown \bar{a}_1 ^\frown \ldots ^\frown \bar{a}_{n-1} ^\frown \bar{a}_n)$ is not realised in M_n. It is then clear that if $n < m$ then M_n and M_m cannot be isomorphic.

Case II). $U(p) = \omega$. It follows from Fact 1.3 that p is *regular*. Namely, if $tp(\bar{a}/A)$ is a nonforking extension of p and $tp(\bar{b}/A)$ is a forking extension of p, then $\bar{a}$ and $\bar{b}$ are independent over A. Then, as in [8], for any model M of T, all maximal independent sets of realisations of p in M have the same cardinality. It now suffices to see that for each $n < \omega$, $\{\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_{n-1}\}$ is a maximal independent set of realisations of p in M_n. But this is a consequence of the Open Map theorem, and the fact that M_n is prime over $\bar{a}_0 ^\frown \bar{a}_1 ^\frown \ldots ^\frown \bar{a}_{n-1}$. Thus again the models $\{M_n\colon n < \omega\}$ are pairwise nonisomorphic.

Thus we have shown that a non $\aleph_0$-categorical theory of R-modules has infinitely many countable models. Note that we can assume that T has prime models over all finite sets, for if not, T has $2^{\aleph_0}$ types, and thus $2^{\aleph_0}$ countable models.

3. T is again simply a countable complete theory in a language L.

DEFINITION 3.1. I will say that T is *uniform* if there is a set Φ of formulae of L, closed under conjunction such that:

i) any formula $\varphi(\bar{x})$ of L is equivalent modulo T to a Boolean combination of formulae in Φ (with free variables among $\bar{x}$), and

ii) if $\varphi(\bar{x}, \bar{y}) \in \Phi$, then

$$T \vdash \forall \bar{y}_1 \bar{y}_2 \big((\exists \bar{x})(\varphi(\bar{x}, \bar{y}_1) \wedge \varphi(\bar{x}, \bar{y}_2)) \to (\forall \bar{x})(\varphi(\bar{x}, \bar{y}_1) \leftrightarrow \varphi(\bar{x}, \bar{y}_2))\big).$$

Note that by Proposition 1.1 and the remarks following it, any theory of R-modules is uniform, where Φ is just the set of p.p. formulae of L_R. However an important property of theories of modules is *not* captured by the definition of uniform. This is the property that if $\varphi(x, \bar{a}_1)$ and $\varphi(x, \bar{a}_2)$ are p.p. definable subsets of a big module, then the Boolean algebras of the definable (with parameters) subsets of $\varphi(x, \bar{a}_1)$ and $\varphi(x, \bar{a}_2)$ are isomorphic. It is this property that accounts for the simple expressions of forking and ranks in modules, that we have used in the previous sections. Thus such reductions will not be a priori available in the case of uniform theories.

PROPOSITION 3.2. *If T is uniform then T is stable.*

Proof. Let M be a model of T and $p(\bar{x})$ a complete type over M. Then it is easy to see that p is definable. It is enough by 3.1 i) to check this for formulae in Φ. So let $\varphi(\bar{x}, \bar{y})$ be such a formula. If say $\varphi(\bar{x}, \bar{a}) \in p$, then by 3.1 ii) for all $\bar{b}$ in M $\varphi(\bar{x}, \bar{b}) \in p$ iff $M \models (\exists \bar{x})(\varphi(\bar{x}, \bar{a}) \wedge \varphi(\bar{x}, \bar{b}))$. If on the other hand $\varphi(\bar{x}, \bar{y})$ is not represented in p, then we have for all $\bar{b}$ in M $\varphi(\bar{x}, \bar{b}) \in p$ iff $M \models \bar{b} \neq \bar{b}$.

However *within* the class of stable theories, there are a variety of uniform theories. The case of modules provides such theories which are ω-stable, superstable and non-ω-stable, or non-superstable. I leave the reader to find examples which are multidimensional, nonmultidimensional, with the f.c.p. or without the f.c.p.

I will prove:

PROPOSITION 3.3. *Let T be a "non-$\aleph_0$-categorical" uniform theory. Then T has at least four countable models up to isomorphism.*

DEFINITION 3.4. A model M is *relatively homogeneous*, if M is prime over a finite set $\bar{a}$, and whenever $\bar{b} \in M$ and $tp(\bar{b}) = tp(\bar{a})$ then there is an automorphism of M taking $\bar{a}$ to $\bar{b}$.

I will use the following result which can be extracted from Theorem 2.1 of [5].

PROPOSITION 3.5. *Let $p(\bar{x})$ be a complete n-type of T (an arbitrary complete countable theory), such that p has Cantor–Bendixon rank 1. Let M be prime over $\bar{b}$, where $tp(\bar{b}) = p$, and suppose that M is not relatively homogeneous. Then there is a $2n$-formula $\alpha(\bar{x}, \bar{y})$ of $L(T)$ and n-formulae $\varphi_i(\bar{x})$ of $L(T)$ for $i < \omega$, such that*

i) *$\alpha(\bar{b}, \bar{y})$ is a complete n-formula of* Th$(M, \bar{b})$.

ii) *$\varphi_i(\bar{x})$ is a complete n-formula of T, for all $i < \omega$.*

iii) *$M \models (\exists \bar{x})(\varphi_i(\bar{x}) \wedge \alpha(\bar{x}, \bar{b}))$, for all $i < \omega$.*

iv) *For all $i, j < \omega$, $M \models (\exists \bar{x}\bar{y})(\varphi_i(\bar{x}) \wedge \varphi_j(\bar{y}) \wedge \alpha(\bar{x}, \bar{y}))$ iff $i < j$.*

Proof of Proposition 3.3. Let T be a uniform theory which is not $\aleph_0$-categorical. We can assume that $S(T)$ is countable, and thus for some $n < \omega$, there is a complete n-type $p(\bar{x})$ of Cantor–Bendixon rank 1. Let M be a model of T which is prime over $\bar{b}$, where $tp(\bar{b}) = p$. *I will show that M is relatively homogeneous. Suppose not.* Then there are formulae α and φ_i, $i < \omega$, as in 3.5 satisfying i)–iv) of 3.5. Let Φ be a set of formulae as in Definition 3.1 which witnesses the uniformity of T. Thus by Definition 3.1 i) and 3.5 i), we can assume that $\alpha(\bar{x}, \bar{y})$ is of the form $\varphi(\bar{x}, \bar{y}) \wedge \bigwedge_{i=1}^{n} \neg\psi_i(\bar{x}, \bar{y})$, where φ and the ψ_i are in Φ.

By 3.5 iii) we can choose for each $i < \omega$, $\bar{a}_i$ in φ_i^M such that $M \models \alpha(\bar{a}_i, \bar{b})$.

OBSERVATION 3.6. *There is, for each $i < j < \omega$, an n-tuple $\bar{c}_{ij}$ in φ_j^M such that $M \models \alpha(\bar{a}_i, \bar{c}_{ij})$.*

Proof. Take $i < j$. By 3.5 iv) we have

$$M \models (\exists \bar{x}\bar{y})(\varphi_i(\bar{x}) \wedge \varphi_j(\bar{y}) \wedge \alpha(\bar{x}, \bar{y})).$$

As $\varphi_i(\bar{x})$ is complete, it follows that

$$M \models (\forall \bar{x})\big(\varphi_i(\bar{x}) \to (\exists \bar{y})(\varphi_j(\bar{y}) \wedge \alpha(\bar{x}, \bar{y}))\big).$$

But $M \models \varphi_i(\bar{a}_i)$. Thus we can find the required $\bar{c}_{ij}$.

OBSERVATION 3.7. *Let $k < \omega$ and $i < j < \omega$. Then $M \models \varphi(\bar{a}_k, \bar{c}_{ij})$.*

Proof. Let k, i, j be as in the line above. Now $M \models \alpha(\bar{a}_k, \bar{b})$ and $M \models \alpha(\bar{a}_i, \bar{b})$. Thus (as α is just $\varphi \wedge \bigwedge_{i=1}^{n} \neg\psi_i$), $M \models \varphi(\bar{a}_k, \bar{b}) \wedge \varphi(\bar{a}_i, \bar{b})$. But $M \models \varphi(\bar{a}_i, \bar{c}_{ij})$, by Observation 3.6. Thus as $\varphi \in \Phi$ and by Definition 3.1ii), it follows that $M \models \varphi(\bar{a}_k, \bar{c}_{ij})$.

OBSERVATION 3.8. i) *If $i < j < \omega$, then $M \models \bigwedge_{r=1}^{n} \neg\psi_r(\bar{a}_i, \bar{c}_{ij})$.*

ii) *If $i < j < k < \omega$, then $M \models \bigvee_{r=1}^{n} \psi_r(\bar{a}_k, \bar{c}_{ij})$.*

Proof. i) is clear because $i < j$ implies $M \models \alpha(\bar{a}_i, \bar{c}_{ij})$.

For ii): Let $i < j < k$. Now $\bar{c}_{ij} \in \varphi_j^M$, and $\bar{a}_k \in \varphi_k^M$. Thus by 3.5 iv) we have $M \models \neg\alpha(\bar{a}_k, \bar{c}_{ij})$. But $\alpha(\bar{x}, \bar{y})$ is $\varphi(\bar{x}, \bar{y}) \wedge \bigwedge_{r=1}^{n} \neg\psi_r(\bar{x}, \bar{y})$, and we know that $M \models \varphi(\bar{a}_k, \bar{c}_{ij})$ (Observation 3.7). Thus $M \models \bigvee_{r=1}^{n} \psi_r(\bar{a}_k, \bar{c}_{ij})$.

Now we try to obtain a contradiction. I define $f\colon [\omega]^3 \to \{1, 2, \ldots, n\}$ as follows: if $i < j < k$, then $f(i, j, k) =$ the least r such that $M \models \psi_r(\bar{a}_k, \bar{c}_{ij})$.

(Such an r must exist by Observation 3.8 ii)). Thus by Ramsey's Theorem there must be an infinite $X \subset \omega$, which is homogeneous for f. That is, there is $s \leqslant n$ such that if i, j and k are in X and $i<j<k$ then

$$(*) \qquad M \models \psi_s(\bar{a}_k, \bar{c}_{ij}).$$

Now choose $i_0 < i_1 < i_2 < i_3 < i_4$ all in X. Thus by (*), we have

$$M \models \psi_s(\bar{a}_{i_4}, \bar{c}_{i_0 i_1}) \wedge \psi_s(\bar{a}_{i_2}, \bar{c}_{i_0 i_1}) \wedge \psi_s(\bar{a}_{i_4}, \bar{c}_{i_2 i_3}).$$

But $\psi_s \in \Phi$. Thus it easily follows (from Definition 3.1 ii)) that $M \models \psi_s(\bar{a}_{i_2}, \bar{c}_{i_2 i_3})$. But this contradicts Observation 3.8 i). This contradiction proves that M is relatively homogeneous. As in the proof of Proposition 2.9 it follows that there is some type q of T which is omitted in M. Let N be prime over a realisation of q. Then M, N and the prime and countable saturated models of T give us our four models. So Proposition 3.3 is proved.

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References

[1] W. Baur, *Elimination of quantifiers for modules*, Israel J. Math. 25 (1976), pp. 64–70.

[2] S. Garavaglia, *Decomposition of totally transcendental modules*, J. Symb. Logic 45 (1980), pp. 155–164.

[3] D. Lascar, *Définissabilité de types en théorie des modèles*, Thèse de Doctorat d'Etat, Université Paris VII, 1975.

[4] — and B. Poizat, *An introduction to forking*, J. Symb. Logic 44 (1979), pp. 77–88.

[5] A. Pillay, *Theories with exactly three countable models and theories with algebraic prime models*, J. Symb. Logic 45 (1980), pp. 302–310.

[6] — and M. Prest, *Forking and pushouts in modules*, preprint 1981.

[7] M. Prest, *Pure-injectives and T-injective hulls of modules*, preprint 1981.

[8] S. Shelah, *Classification Theory and the Number of Non-Isomorphic Models*, North-Holland, Amsterdam 1978.

[9] M. Ziegler, *Model theory of modules*, preprint 1981.

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Fixed point theorems and almost continuity

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Abstract. In 1959, John Stallings asked the following question which he attributed to K. Borsuk: Suppose K is a non-separating planar continuum contained in the interior of a disk D. Is there an almost continuous function $r\colon D \to D$ such that $r(D) = K$ and $r|K = \mathrm{id}$? We answer this question negatively. We also show that if $X_0 \supset X_1 \supset \ldots \supset X_n \supset X_{n+1} \supset \ldots$ is a sequence of ARs, with retractions $f_n\colon X_{n-1} \to X_n$, such that $x \in X_{n-1} \sim X_n$ implies $f_n(x) \in \bigcap X_i$, then $\bigcap X_i$ has the fixed point property.

1. Introduction. Throughout this paper X, Y and Z will denote topological spaces. A *map* is a continuous function. When $f\colon X \to Y$ may not be continuous, we refer to it simply as the function f. An *absolute retract* (AR) is a retract of the Hilbert cube. A space X has the *fixed point property*, if for each map $f\colon X \to X$ there exists $x \in X$ such that $f(x) = x$. The *graph* of a function $f\colon X \to Y$ is the subset of $X \times Y$ consisting of the points $(x, f(x))$; this set will be symbolized $\Gamma(f)$.

J. Stallings [11] defined a class of functions, which he named almost continuous, for the purpose of studying the fixed point property.

DEFINITION 1 [11, p. 252]. A function $f\colon X \to Y$ is *almost continuous* if for each open subset $\mathscr{U}$ of $X \times Y$ such that $\Gamma(f) \subset \mathscr{U}$, there exists a map $g\colon X \to Y$ such that $\Gamma(g) \subset \mathscr{U}$.

THEOREM 1 [11, p. 252]. *A Hausdorff space X has the fixed point property if and only if every almost continuous function $f\colon X \to X$ leaves a point fixed.*

THEOREM 2 [11, p. 260]. *If $f\colon X \to Y$ is almost continuous and $g\colon Y \to Z$ is a map, then $gf\colon X \to Z$ is almost continuous.*

DEFINITION 2. If $Y \subset X$ and $r\colon X \to X$ is an almost continuous function such that $r(X) = Y$ and $r(x) = x$ for all $x \in Y$, then r is called a *quasi retraction* and Y is called a *quasi retract* of X (1).

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(1) In the Literature quasi retractions have been called almost continuous retractions. We have avoided the term "almost continuous retraction" because it has also been used for almost continuous $r\colon X \to Y$ such that $r(x) = x$ for all $x \in Y$.