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Added in proof. Any locally slack semialgebraic set T has in fact — as conjectured — a locally slack decomposition $T = \bigcup T_i$, $T_i = \{g_{i_1} \geqslant 0, \ldots, g_{i_m} \geqslant 0\}$ (cf. Dubois–Recio, unpublished). Also, if V is an algebraic surface then $V_c = \{g_1 \geqslant 0, g_2 \geqslant 0\} \cap V$ (cf. Ruiz "Geometric and arithmetic aspects of the 17th Hilbert Problem for real analytic germs", unpublished dissertation, U. Madrid).

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NEW MEXICO
Albuquerque, NM 87131
DEPARTAMENTO DE ALGEBRA
UNIVERSIDAD DE SANTANDER
Santander, Spain

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Applications of certain $\mathfrak{K}$-families

by

Wolfgang Sander (Braunschweig)

Dedicated to Professor J. Aczél on his 60th birthday

Abstract. We give some applications of Morgan's abstract Baire category theory to Cauchy's functional equation and to Hamel bases. Especially we can give a unified representation for all results of Kuczma's report on topologically saturated non measurable sets in [7].

1. Introduction. In the real analysis and in the theory of functional equations there are many results, which remain true, replacing measure theoretical conditions by appropriate topological conditions, though the proofs are sometimes quite different (see [3] and [13]). The aim of this paper is to show, that Morgan's theory of $\mathfrak{K}$-families (cf. [9]–[11]) is an appropriate theory for a unified representation of many well-known results, concerning measure and category especially in the theory of real-valued additive functions and in the herewith closely related theory of Hamel bases. We remark that all theorems are formulated in $\boldsymbol{R}$, though they are sometimes valid in $\boldsymbol{R}^n$ or in more general spaces.

2. Preliminaries. We shall use the terminology of [9]–[11] and assume, that the reader is familiar especially with the theory of $\mathfrak{P}$- and $\mathfrak{S}$-families on $\boldsymbol{R}$. If $\mathscr{C}$ is a $\mathfrak{K}$-family on $\boldsymbol{R}$, we say, that $\mathscr{C}$ is *multiplication invariant* respectively *inversion invariant*, if for all $x \in \boldsymbol{R} \setminus \{0\}$ we have $A \cdot x \in \mathscr{C}$ respectively $-A \in \mathscr{C}$. Now if $\mathscr{C}$ is multiplication invariant respectively inversion invariant, it is clear, that the families of $\mathscr{C}$-singular sets, $\mathscr{C}_1$-sets and sets with the Baire property are also multiplication invariant respectively inversion invariant. Moreover a $\mathfrak{S}^*$-family $\mathscr{C}$ on $\boldsymbol{R}$ is said to be an *inversion invariant* $\mathfrak{S}$-family $\mathscr{C}$ on $\boldsymbol{R}$ such that $\mathscr{C}$ contains

$$\{\{x \in \boldsymbol{R}\colon |x-y| < 1/n\}\colon y \in \boldsymbol{Q}, n \in \boldsymbol{N}\}.$$

We define a class of sets, which will play a key role in our considerations:

$$\mathscr{C}_{\mathrm{III}} := \{A \subset \boldsymbol{R}\colon \exists B \subset A\colon B \in \mathfrak{B}(\mathscr{C}) \cap \mathscr{C}_{\mathrm{II}}\}.$$

The next theorem will be used rather often in this note ([16], Theorem 3.4 and [17], Theorem 2):

THEOREM 2.1. (1) *If $\mathscr{C}$ is a $\mathfrak{S}$-family on $\boldsymbol{R}$ and if $A \in \mathscr{C}_{\mathrm{III}}$, then $A-A$ contains an interval.*

(2) *If $\mathscr{C}$ and $\mathscr{D}$ are nonequivalent $\mathfrak{P}$- and $\mathfrak{S}$-families on $\boldsymbol{R}$ (simultaneously), then $\boldsymbol{R}$ is the disjoint union of a $\mathscr{C}_{\mathrm{I}}$-set and a $\mathscr{D}_{\mathrm{I}}$-set.*

We here consider two examples. Let

$$\mathscr{C} := \{\{x \in \boldsymbol{R}\colon\ |x-y| \leqslant 1/n\}\colon\ y \in \boldsymbol{R},\, n \in \boldsymbol{N}\}$$

and let

$$\mathscr{D} := \{A \subset \boldsymbol{R}\colon A \text{ is closed, } \mu(U \cap A) > 0 \text{ for all } x \in A \text{ and for all open sets } U \text{ of } \boldsymbol{R}, \text{ containing } x\}$$

(μ denotes the Borel measure in $\boldsymbol{R}$). Then $\mathscr{C}$ and $\mathscr{D}$ are nonequivalent, multiplication invariant $\mathfrak{P}$- and $\mathfrak{S}^*$-families on $\boldsymbol{R}$ (see [17]). Since the members of $\mathscr{C}$ and $\mathscr{D}$ have positive Borel measure, $\mathscr{C}$ and $\mathscr{D}$ also satisfy c.c.c., that is, each family of disjoint sets of $\mathscr{C}$ or $\mathscr{D}$ is at most countable ([15], p. 123). Moreover $\mathscr{C}_{\mathrm{I}}$ consists of all first category sets and $\mathscr{D}_{\mathrm{I}}$ consists of all sets of Lebesgue measure zero. Thus $\mathscr{C}_{\mathrm{III}}$ consists of all subsets of $\boldsymbol{R}$, containing a Baire set of second category and $\mathscr{D}_{\mathrm{III}}$ consists of all subsets of positive inner L-measure (or equivalently: $\mathscr{D}_{\mathrm{III}}$ consists of all subsets of $\boldsymbol{R}$, containing a L-measurable set of positive L-measure). Theorem 2.1, (1) is an extension of two well-known results of Piccard [14] and Steinhaus [20]:

If $A \subset \boldsymbol{R}$ contains a Baire set of second category or if A is of positive inner L-measure, then $A-A$ contains an interval.

The second statement in Theorem 2.1 yields that $\boldsymbol{R}$ is the disjoint union of a set of first category and a set of L-measure zero. We remark that by the above two examples all theorems in § 3 have topological and measure theoretical versions. We regard $\boldsymbol{R}$ as a vector space over $\boldsymbol{Q}$ and call each basis of $\boldsymbol{R}$ over $\boldsymbol{Q}$ a Hamel basis. If $T \subset \boldsymbol{R}$ then $E(T)$ is the linear subspace of $\boldsymbol{R}$ over $\boldsymbol{Q}$, spanned by T. The complement of a set $A \subset \boldsymbol{R}$ will be denoted by cA.

A set $A \subset \boldsymbol{R}$ is called *Q-convex* iff $a\cdot x+(1-a)\cdot y \in A$ for all $x, y \in A$ and for all $a \in [0,1]$; $A \subset \boldsymbol{R}$ is J-convex iff $\frac{1}{2}x+\frac{1}{2}y \in A$ for all $x, y \in A$. By $J(B)$ respectively $Q(B)$ we denote the J-convex hull respectively the Q-convex hull of a set $B \subset \boldsymbol{R}$ (i.e. the smallest J-convex respectively Q-convex set containing B). Thus

$$J(B) = \bigcup_{n=0}^{\infty}\left(\frac{1}{2^n}\sum_{1}^{2^n} B\right)$$

and $Q(B)$ is the set of all $\alpha_1 x_1+\ldots+\alpha_n x_n$, where $\alpha_i \in \boldsymbol{Q} \cap [0,1]$, $\alpha_1+\ldots+\alpha_n = 1$, $x_1, \ldots, x_n \in B$ and $n \in \boldsymbol{N}$.

Finally we need the following definition, which occurs in the theory of additive, real-valued functions (see [6], p. 385): If $x_0 \in A \subset \boldsymbol{R}$, then A is called *Q-radial at the point x_0* iff for every $x \in \boldsymbol{R}$ there is a real $c_x > 0$ such that $x_0+\alpha\cdot x \in A$ for all $\alpha \in \boldsymbol{Q}$, $0 \leqslant \alpha < c_x$.

3. Main results. The first results in this section were motivated by Kuczma's report on topologically saturated non-measurable sets during the 17th international Symposium about functional equations in Oberwolfach 1979 [7].

Definition 3.1. Let $\mathscr{C}$ be a $\mathfrak{K}$-family on $\boldsymbol{R}$. Then $A \subset \boldsymbol{R}$ is called *non $\mathfrak{B}(\mathscr{C})$-saturated* iff $A \notin \mathscr{C}_{\mathrm{III}}$ and $cA \notin \mathscr{C}_{\mathrm{III}}$. By $\mathfrak{S}(\mathscr{C})$ we denote all subsets of $\boldsymbol{R}$, which are non $\mathfrak{B}(\mathscr{C})$-saturated.

If $\mathscr{C}$ and $\mathscr{D}$ are the topological respectively the measure theoretical example of § 2, then $\mathfrak{S}(\mathscr{C})$ and $\mathfrak{S}(\mathscr{D})$ are the sets of all topologically saturated non-measurable sets respectively saturated non-measurable sets (cf. [7], [8], p. 422).

The following characterization of $\mathfrak{S}(\mathscr{C})$-sets comes immediately from Lemma 3.2 in [16].

Theorem 3.2. *Let $\mathscr{C}$ be a $\mathfrak{K}$-family on $\boldsymbol{R}$ with $\mathscr{C} \subset \mathscr{C}_{\mathrm{II}}$.*

$$A \in \mathfrak{S}(\mathscr{C}) \Leftrightarrow (\forall B \in \mathscr{C}\colon\ B \cap cA \in \mathscr{C}_{\mathrm{II}}) \wedge (\forall C \in \mathscr{C}\colon\ C \cap A \in \mathscr{C}_{\mathrm{II}})\,.$$

We give some examples for $\mathfrak{S}(\mathscr{C})$-sets.

Theorem 3.3. (1) *If $\mathscr{C}$ is a $\mathfrak{P}$-family on $\boldsymbol{R}$, then each Bernstein set of $\boldsymbol{R}$ is non $\mathfrak{B}(\mathscr{C})$-saturated.*

(2) *If $\mathscr{C}$ is a $\mathfrak{S}$-family on $\boldsymbol{R}$, then each set of real numbers, whose representation with respect to a Hamel basis H does not contain a fixed element of H, is non $\mathfrak{B}(\mathscr{C})$-saturated.*

(3) *If $\mathscr{C}$ is a $\mathfrak{S}$-family on $\boldsymbol{R}$, then each linear subspace of $\boldsymbol{R}$ over $\boldsymbol{Q}$, which does not have the Baire property, is non $\mathfrak{B}(\mathscr{C})$-saturated.*

Proof. (1) Let A be a Bernstein set, that is, neither A nor cA contains a perfect set ([13], p. 23). Thus by Theorem 12 in [10] $A \in \mathfrak{S}(\mathscr{C})$.

(2) This follows from Theorem 8 in [9] and from Theorem 3.2.

(3) Let V be a linear subspace of $\boldsymbol{R}$ over $\boldsymbol{Q}$ such that $V \notin \mathfrak{B}(\mathscr{C})$. Assume that $V \in \mathscr{C}_{\mathrm{III}}$. By Theorem 2.1 $V = V-V$ contains an interval. So V is an open and thus closed subgroup of $\boldsymbol{R}$. This yields $V = \boldsymbol{R}$. Since we always can assume that $\boldsymbol{R} \in \mathscr{C}$ (cf. Lemma 3 in [10]), we get the contradiction $V \in \mathscr{C} \subset \mathfrak{B}(\mathscr{C})$ ([11], Theorem 4). Since $V \notin \mathfrak{B}(\mathscr{C})$ we get $V \in \mathscr{C}_{\mathrm{II}}$. On the other hand we have $V = V+\boldsymbol{Q}\cdot v$. But $\boldsymbol{Q}\cdot v$ is dense in $\boldsymbol{R}$ for $v \in V$ and so Theorem 2 in [9] yields that $A \cap V \in \mathscr{C}_{\mathrm{II}}$ for all $A \in \mathscr{C}$, that is $cV \notin \mathscr{C}_{\mathrm{III}}$.

The measure theoretical version of Theorem 3.3 (3) was proved by Kuczma and Smital ([8], p. 423).

The next result, whose measure theoretical version is due to Ostrowski [12] and whose category version is due to Kuczma [7], shows that non $\mathfrak{B}(\mathscr{C})$-saturated sets occur in the theory of additive functions.

Theorem 3.4. *Let $\mathscr{C}$ be a multiplication invariant $\mathfrak{S}$- and $\mathfrak{P}$-family on $\boldsymbol{R}$. If $f\colon \boldsymbol{R} \to \boldsymbol{R}$ is discontinuous and additive, then $f^{-1}(U)$ is non $\mathfrak{B}(\mathscr{C})$-saturated, whenever $U \subset \boldsymbol{R}$ is a closed interval.*

Proof. Let $U := [a, b]$, $a, b \in \boldsymbol{R}$, $a<b$, and suppose that $f^{-1}(U) \in \mathscr{C}_{\mathrm{III}}$. Then Theorem 4.1 in [16] implies the contradiction that f is continuous. Thus $f^{-1}(U) \notin \mathscr{C}_{\mathrm{III}}$. Now suppose that $cf^{-1}(U) \in \mathscr{C}_{\mathrm{III}}$. Then there exists a $\mathfrak{B}(\mathscr{C}) \cap \cap\, \mathscr{C}_{\mathrm{II}}$-set $A \subset \boldsymbol{R}$, such that $f(x) \notin [a, b]$ for all $x \in A$. We first prove that without loss of generality we can assume, that $f(1) = 0$. If $f(1) \neq 0$, we consider the additive, discontinuous function $g\colon \boldsymbol{R} \to \boldsymbol{R}$, defined by $g(x) := f(x)-f(1)x$, which satisfies

$g(1) = 0$. Since $A \in \mathscr{C}_{\mathrm{II}}$ we infer from Theorem 2 in [11], that there is a $\mathscr{C}$-set C, such that $B \cap A \in \mathscr{C}_{\mathrm{II}}$ for all $B \in \mathscr{C}$ with $B \subset C$. Now take an $x \in C \cap A$. From the definition of a $\mathfrak{P}$-family we get a descending sequence (A_n) of $\mathscr{C}$-sets, such that $x \in A_n$, $A_n \subset C$, and $\operatorname{diam} A_n \leqslant 1/n$ for all $n \in N$. Choose $m \in N$ such that $\frac{1}{m} < \frac{1}{2} \cdot \frac{b-a}{|f(1)|}$. By our construction we have $A' := A_m \cap A \in \mathscr{C}_{\mathrm{II}}$ and it follows $A' \in \mathfrak{B}(\mathscr{C}) \cap \mathscr{C}_{\mathrm{II}}$. But now $g(x)$ does not take on values of an interval of positive length for all $x \in A'$: indeed, if $x \in A'$, then by our assumption $f(x)$ does not take on values of an interval of length $b-a$ and $f(1)x$ takes on values of an interval of length at most $\frac{b-a}{2}$. Since f is $\boldsymbol{Q}$-homogeneous, $f(1) = 0$ implies $f(r) = rf(1) = 0$ for all $r \in \boldsymbol{Q}$. It follows that $f(x) \notin [a, b]$ for all $x \in M := A + \boldsymbol{Q}$, where $cM \in \mathscr{C}_{\mathrm{I}}$ (see the corollary after Theorem 2 in [9]). So we have $M_0 := \{x \in \boldsymbol{R}\colon f(x) \in [a, b]\} \subset cM \in \mathscr{C}_{\mathrm{I}}$. Without loss of generality we may suppose that $b = ra > a > 0$, where $r \in \boldsymbol{Q} \setminus \{0\}$ (Observe that $-f$ is also an additive function). Using that f is $\boldsymbol{Q}$-homogeneous we get for all $n \in N$, that $f(x) \in [r^n a, r^{n+1} a]$ only if x is an element of the $\mathscr{C}_{\mathrm{I}}$-set $r^n M_0$. Defining $B := \bigcup_{n \in N} r^n M_0$ it follows $B \in \mathscr{C}_{\mathrm{I}}, f(B) \subset [a, \infty)$ and $cB \in \mathfrak{B}(\mathscr{C}) \cap \mathscr{C}_{\mathrm{II}}$. But f is bounded above on cB by a. Again Theorem 4.1 in [16] implies the continuity of f, which is impossible. (We remark that in the proof only "rational multiplication invariance" is needed.)

DEFINITION 3.5. [7]. Let $A \subset \boldsymbol{R}$ be an uncountable Borel set. A Hamel basis H is called a *Burstin basis relative to* A iff H intersects each uncountable Borel subset of A.

The existence of a Burstin basis relative to $\boldsymbol{R}$ was proved by Burstin [2] and by Abian [1]. In a similar manner the following result can be proven.

THEOREM 3.6. [7] *Every Borel set $A \subset \boldsymbol{R}$, containing a Hamel basis, contains a Burstin basis relative to A.*

Now we can prove immediately the following two results.

THEOREM 3.7. *Let $\mathscr{C}$ be a $\mathfrak{P}$- and $\mathfrak{S}$-family on $\boldsymbol{R}$. If $A \subset \boldsymbol{R}$ is a $\mathscr{C}$-residual Borel set, then each Burstin basis relative to A is non $\mathfrak{B}(\mathscr{C})$-saturated.*

Proof. (1) Let H be a Burstin basis relative to A. If $H \in \mathscr{C}_{\mathrm{III}}$, then by Theorem 2.1 $H-H$ contains an interval U. Now take any $a \in H$. Then there is an $r \in \boldsymbol{Q} \setminus \{0\}$ such that $ra \in U$. Thus there exist $b, d \in H$ satisfying $ra = b-d$, contradicting the linear independence of a, b, d.

(2) If we assume that $A \cap cH \in \mathscr{C}_{\mathrm{III}}$, then by Theorem 12 in [10] $A \cap cH$ contains a non empty perfect set P; but P is an uncountable Borel subset of A such that $P \cap H = \emptyset$, which is impossible. Since A is $\mathscr{C}$-residual, that is $cA \in \mathscr{C}_{\mathrm{I}}$, we get $cH \notin \mathscr{C}_{\mathrm{III}}$.

THEOREM 3.8. *Let $\mathscr{C}$ and $\mathscr{D}$ be nonequivalent $\mathfrak{P}$- and $\mathfrak{S}$-families on $\boldsymbol{R}$. Then there exist Hamel bases H and B such that $H \in \mathfrak{S}(\mathscr{D}) \cap \mathscr{C}_{\mathrm{I}}$ and $B \in \mathfrak{S}(\mathscr{C}) \cap \mathscr{D}_{\mathrm{I}}$.*

Proof. By Theorem 2.1 $\boldsymbol{R}$ can be decomposed into a $\mathscr{C}_{\mathrm{I}}$-set C and a $\mathscr{D}_{\mathrm{I}}$-set D. Observing that $\mathscr{C}$ and $\mathscr{D}$ consist of perfect sets, the proof of Theorem 2.1 in [17] yields that C and D are uncountable Borel sets (indeed, C and D are either F_σ- or G_δ-sets). Moreover we have $C = cD \in \mathfrak{B}(\mathscr{D}) \cap \mathscr{D}_{\mathrm{II}}$ and $D = cC \in \mathfrak{B}(\mathscr{C}) \cap \mathscr{C}_{\mathrm{II}}$. From Theorem 2.1 we infer again that $E(C) = \boldsymbol{R}$ and $E(D) = \boldsymbol{R}$. But it is known, that this is a necessary and sufficient condition for C and D to contain a Hamel basis (cf. [3], p. 4.29). By Theorem 3.6 C contains a Burstin basis H relative to C and D contains a Burstin basis B relative to D. Now Theorem 3.7 yields $H \in \mathfrak{S}(\mathscr{D}) \cap \mathscr{C}_{\mathrm{I}}$ and $B \in \mathfrak{S}(\mathscr{C}) \cap \mathscr{D}_{\mathrm{I}}$.

The next two results are extensions of theorems in [4]. We introduce some notations. If H is any Hamel basis, then we denote by H^* the set of all real numbers of the form $\sum z_i h_i$ (finite sum) and by H^+ the set of all real numbers of the form $\sum \alpha_i h_i$ (finite sum); here $h_i \in H$, $z_i \in \boldsymbol{Z}$ and $\alpha_i \in \boldsymbol{Q} \cap [0, \infty)$. Moreover we define for any $\mathfrak{K}$-family on $\boldsymbol{R}$:

$$\mathfrak{L}(\mathscr{C}) := \{A \subset \boldsymbol{R}\colon |A| > \aleph_0, \forall C \in \mathscr{C}_{\mathrm{I}}\colon |C \cap A| \leqslant \aleph_0\}.$$

If for example $\mathscr{C}$ is the topological example of § 2, then $\mathfrak{L}(\mathscr{C})$ consists of all Lusin sets (see [11], Definition 9).

THEOREM 3.9. *If H is any Hamel basis and if $\mathscr{C}$ is a multiplication invariant $\mathfrak{S}$-family on $\boldsymbol{R}$, then $H^* \in \mathfrak{S}(\mathscr{C})$.*

Proof. It is obvious, that for all $x \in \boldsymbol{R}$ there is an $z_x \in \boldsymbol{Z}$ such that $z_x \cdot x \in H^*$. Thus

$$\boldsymbol{R} = \bigcup \left\{\frac{1}{z} \cdot H^*\colon z \in \boldsymbol{Z} \setminus \{0\}\right\}.$$

Since $\boldsymbol{R} \notin \mathscr{C}_{\mathrm{I}}$, we have $\frac{1}{n} \cdot H^* \in \mathscr{C}_{\mathrm{II}}$ for some $n \in \boldsymbol{Z} \setminus \{0\}$ and thus $H^* \in \mathscr{C}_{\mathrm{II}}$. Moreover it is known that H^* is dense in $\boldsymbol{R}$. Since $H^* + H^* = H^*$ (H^* is an additive group), Theorem 2 in [9] yields that $cH \notin \mathscr{C}_{\mathrm{III}}$.

Now suppose that $H^* \in \mathscr{C}_{\mathrm{III}}$. By Theorem 2.1 the interior of H^* is non empty. Thus H^* is open and also closed and we get $H^* = \boldsymbol{R}$, which is a contradiction.

THEOREM 3.10. *Let $\mathscr{C}$ and $\mathscr{D}$ be nonequivalent, multiplication invariant $\mathfrak{P}$- and $\mathfrak{S}$-families on $\boldsymbol{R}$, such that $|\mathscr{C}| \leqslant c$, $|\mathscr{D}| \leqslant c$ and such that $\mathscr{C}$ and $\mathscr{D}$ satisfy c.c.c. If $c = \omega_1$, then there exist Hamel bases H and B satisfying $H^+ \in \mathfrak{L}(\mathscr{C}) \cap \mathscr{D}_{\mathrm{I}}$ and $B^+ \in \mathfrak{L}(\mathscr{D}) \cap \mathscr{C}_{\mathrm{I}}$.*

Proof. (1) By $\mathfrak{E}$ we denote the family of all sets, which are complements of members of $\mathscr{C}$. Because of $|\mathscr{C}| \leqslant c$ we have $|\mathfrak{E}_{\delta\sigma} \cap \mathscr{C}_{\mathrm{I}}| \leqslant c$. Now let $\mathfrak{E}_{\delta\sigma} \cap \mathscr{C}_{\mathrm{I}} = \{F_\alpha\colon \alpha < \omega_1\}$. Like in [4] we can construct a Hamel basis H such that $|H^+ \cap F_\alpha| \leqslant \aleph_0$ for all $\alpha < \omega_1$ (In this step of the proof we need the multiplication invariance of $\mathscr{C}$). Now let $A \in \mathscr{C}_{\mathrm{I}}$. By Theorem 3 in [11] A is contained in a certain set F_β, $\beta < \omega_1$, which proves that $H^+ \in \mathfrak{L}(\mathscr{C})$. Theorem 2.1 yields that H^+ is the disjoint union of a set $A \in \mathscr{C}_{\mathrm{I}}$ and a set $B \in \mathscr{D}_{\mathrm{I}}$. Since $H^+ \in \mathfrak{L}(\mathscr{C})$ we get $|A| \leqslant \aleph_0$. Thus $A \in \mathscr{D}_{\mathrm{I}}$ and also $H^+ \in \mathscr{D}_{\mathrm{I}}$.

(2) The second statement can be proved in exactly the same manner.

We now prove a result, which can be compared with Theorem 5 in [9].

THEOREM 3.11. *Let $\mathscr{C}$ be a $\mathfrak{P}$-family on $\boldsymbol{R}$ such that $|\mathscr{C}| \leqslant c$ and $\mathscr{C}$ satisfies c.c.c. If $c = \omega_1$, then each $\mathscr{C}_{\mathrm{II}}$-set can be decomposed into c disjoint sets, none of which has the Baire property.*

Proof. Let $A \in \mathscr{C}_{\mathrm{II}}$. By Theorem 17 in [11] and by Theorem 6 in [10] A contains a set $L \in \mathfrak{L}(\mathscr{C})$. If $f\colon \boldsymbol{R}\times\boldsymbol{R} \to L$ is a bijective function (Observe that $c = \omega_1$ implies $|\boldsymbol{R}\times\boldsymbol{R}| = |L|$), then

$$\{f(\{x\}\times\boldsymbol{R})\colon x \in \boldsymbol{R}\}$$

are c disjoint $\mathscr{C}_{\mathrm{II}}$-sets, contained in A. Now Theorem 19 in [10] yields, that A contains c disjoint sets B_α, $\alpha<\omega_1$, such that $B_\alpha \notin \mathfrak{B}(\mathscr{C})$ for all $\alpha<\omega_1$. Consider

$$D := A \cap c \bigcup \{B_\alpha\colon \alpha<\omega_1\}.$$

If $D \notin \mathfrak{B}(\mathscr{C})$, then $\{B_\alpha\colon \alpha<\omega_1\} \cup \{D\}$ is the desired decomposition of A. If $D \in \mathfrak{B}(\mathscr{C})$, then $\{B_0 \cup D\} \cup \{B_\alpha\colon 0<\alpha<\omega_1\}$ is a decomposition of A with $B_0 \cup D \notin \mathfrak{B}(\mathscr{C})$. Indeed, if $B_0 \cup D \in \mathfrak{B}(\mathscr{C})$, then $(B_0 \cup D) \cap cD = B_0 \in \mathfrak{B}(\mathscr{C})$, which is impossible.

We close our considerations with two results concerning real-valued additive functions. Smital ([18], [19]) could give necessary and sufficient conditions for sets $T \subset \boldsymbol{R}$ such that every additive function, bounded (respectively bounded above) on T, is continuous in $\boldsymbol{R}$. We here replace these conditions by equivalent conditions using $\mathfrak{S}^*$-families on $\boldsymbol{R}$.

THEOREM 3.12. *Let $\mathscr{C}$ be a $\mathfrak{S}^*$-family on $\boldsymbol{R}$ and let $T \subset \boldsymbol{R}$. Then every additive function $f\colon \boldsymbol{R} \to \boldsymbol{R}$ bounded on T is continuous in $\boldsymbol{R}$ iff $Q(T-T) \in \mathscr{C}_{\mathrm{III}}$.*

Proof. (I) Let $Q(T-T) \in \mathscr{C}_{\mathrm{III}}$ and let $|f(x)| \leqslant M$ for all $x \in T$ and for some $M \in \boldsymbol{R}$. Using that f is $\boldsymbol{Q}$-homogeneous we get that $|f(x)| \leqslant 2M$ for all $x \in Q(T-T)$. Now Theorem 4.1 in [16] implies that f is continuous in $\boldsymbol{R}$.

(II) Assume that $Q(T-T) \notin \mathscr{C}_{\mathrm{III}}$. If $Q(T-T)$ would contain an interval, it would also contain a member A of the family $\{\{x \in \boldsymbol{R}\colon |x-y| < 1/n\}\colon y \in \boldsymbol{Q}, n \in \boldsymbol{N}\}$. Since each $\mathfrak{S}$-family satisfies $\mathscr{C} \subset \mathscr{C}_{\mathrm{II}}$, we get $A \in \mathscr{C} \subset \mathfrak{B}(\mathscr{C}) \cap \mathscr{C}_{\mathrm{II}}$, which is impossible. Thus $Q(T-T)$ contains no interval and by Theorem 4 in [18] there is a discontinuous additive function bounded on T.

Using again Theorem 4.1 in [16] and the main result in [19], we can prove immediately the following theorem.

THEOREM 3.13. *Let $\mathscr{C}$ be a $\mathfrak{S}^*$-family on $\boldsymbol{R}$ and let $T \subset \boldsymbol{R}$. Then every additive function $f\colon \boldsymbol{R} \to \boldsymbol{R}$ bounded above on T is continuous in $\boldsymbol{R}$ iff $Q(T-A) \in \mathscr{C}_{\mathrm{III}}$ for all subsets A of $\boldsymbol{R}$, which are Q-radial at a point.*

Let $\mathscr{C}$ be a $\mathfrak{S}$-family on $\boldsymbol{R}$ with $\boldsymbol{R} \in \mathscr{C}$. We here remark, that the condition "Every additive function $f\colon \boldsymbol{R} \to \boldsymbol{R}$ upper-bounded on $T \subset \boldsymbol{R}$ is continuous in $\boldsymbol{R}$" does not

imply that $J(T)-J(T) \in \mathscr{C}_{\mathrm{III}}$. Let H be a Hamel basis such that $1 \in H$ and let T be the set of all real numbers, which can be written in the form $\sum \alpha_i h_i$ (finite sum), where $h_i \in H$ and α_i are dyadic rational numbers (compare [19]). Now if f is any additive function bounded above on $T = T-T$, then f is bounded above on $Q(T) = \boldsymbol{R} \in \mathfrak{B}(\mathscr{C}) \cap \mathscr{C}_{\mathrm{II}}$. By Theorem 4.1 in [16] f is continuous in $\boldsymbol{R}$. If $J(T)-J(T) = T-T = T$ would be a $\mathscr{C}_{\mathrm{III}}$-set, then by Theorem 2.1 T would contain an interval. So there exists a non dyadic number $a = a\cdot 1 \in T$, which is a contradiction. Thus $J(T)-J(T) \notin \mathscr{C}_{\mathrm{III}}$.

Contrary to the last statement the following positive result is true: Let $\mathscr{C}$ be a translation and multiplication invariant $\mathfrak{K}$-family on $\boldsymbol{R}$. Then the condition "Every additive function $f\colon \boldsymbol{R} \to \boldsymbol{R}$ bounded on a set $T \subset \boldsymbol{R}$ is continuous" implies $J(T)-J(T) \in \mathscr{C}_{\mathrm{II}}$. The proof is analogous to the proof of Theorem 6 in [5], replacing the measure theoretical notions by the corresponding notions of a $\mathfrak{K}$-family.

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TECHNICAL UNIVERSITY OF BRAUNSCHWEIG
Braunschweig

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Fixed point sets of continuum-valued mappings

by

Jack T. Goodykoontz, Jr. and **Sam B. Nadler, Jr.** (Morgantown, WV)

Abstract. Let X be a metric continuum and let $C(X)$ denote the hyperspace of subcontinua of X. The following question is investigated: When does X have the property that for each nonempty closed subset A of X there exists a continuous function $F\colon X \to C(X)$ such that $x \in F(x)$ if and only if $x \in A$?

1. Introduction. By a *continuum* we mean a nonempty compact connected metric space. If X is a continuum, then $2^X(C(X))$ denotes the hyperspace of closed subsets (subcontinua) of X, each with the Hausdorff metric.

A *Peano continuum* is a locally connected continuum. By a *mapping* we mean a continuous function. If X is a space and $f\colon X \to X$ is a mapping, then the *fixed point set* of f is $\{x \in X\colon f(x) = x\}$. In [16] L. E. Ward, Jr. defines a space X to have the *complete invariance property* (CIP) provided that for each nonempty closed subset A of X there exists a mapping $f\colon X \to X$ such that A is the fixed point set of f. Some spaces known to have CIP are one-dimensional Peano continua [9], convex subsets of Banach spaces [16], compact n-manifolds [14], locally compact metrizable groups [8], and polyhedra [3]. In [16] Ward asked if every Peano continuum has CIP. This question was answered negatively in [7]. A rather complete bibliography of the literature on fixed point sets and CIP may be found in the survey article by H. Schirmer [14].

Part of the literature on the fixed point property has been concerned with multi-valued (set-valued) mappings. However, the question of which sets can be fixed point sets of multi-valued mappings has not been investigated before. If X is a continuum, $F\colon X \to 2^X$ is a mapping, and $x \in X$, then x is said to be a *fixed point* of F provided $x \in F(x)$. The *fixed point set* of F is $\{x \in X\colon x \in F(x)\}$. By a *continuum-valued mapping* we mean a mapping $F\colon X \to C(X)$.

In this paper we introduce and study the following generalization of CIP to the setting of multi-valued mappings. A continuum X is said to have the *complete invariance property for continuum-valued mappings* (MCIP) provided that for each nonempty closed subset A of X there exists a mapping $F\colon X \to C(X)$ such that A is the fixed point set of F.