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On some dense subspaces of topological linear spaces

by

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Abstract. We consider the following completeness-type property of a metrizable topological linear space X introduced by S. Mazur and W. Orlicz: (K) Every sequence (x_n) in X with $x_n \to 0$ contains a subsequence (x_{n_k}) such that the series $\sum_{k=1}^{\infty} x_{n_k}$ is convergent. We give a general example showing that property (K) is not multiplicative. On th other hand, we represent every F-space of dimension $2^{\aleph_0}$ as the direct algebraic sup of two dense subspaces X_1 and X_2 with property (K) such that $X_1 \times X_2$ also has pro- erty (K).

The present paper deals with the existence and some completeness-type properties of dense (linear) subspaces of infinite-dimensional topological linear spaces. It falls into three sections.

Section 1 establishes the existence of a decomposition into a large family of dense subspaces with the same dimension as the whole space (Theorem 1).

Sections 2 and 3, which are independent of Section 1, contain some results related to the following property of a metrizable topological linear space X:

(K) *Every sequence* (x_n) *in* X *with* $x_n \to 0$ *contains a subsequence* (x_{n_k}) *such that the series* $\sum_{k=1}^{\infty} x_{n_k}$ *is convergent.*

This property was first isolated (in the wider context of linear spaces with convergence), already in the forties, by Mazur and Orlicz. They realized that in some basic theorems of functional analysis property (K) can be used as a substitute for completeness (see their paper [10], p. 169; cf. also [1], postulate (a_2'), p. 203). Property (K) was rediscovered (and given the present name) in the seventies at the seminar guided by Professor Jan Mikusiński at Katowice.

The existence of a noncomplete (metrizable) topological linear space with property (K) was first proved, under the continuum hypothesis, by Kliś ([8], Theorem 2) and next, without that hypothesis, by Labuda and the author ([9], Theorem 2). In [2], Theorem 2, it was proved (in the wider context of Abelian groups) that property (K) implies that X is a Baire space. Those results were already applied by Drewnowski [4].

Below we give a general example showing that property (K) is not multiplicative (Theorem 3). On the other hand, we represent every F-space X with $\dim X = 2^{\aleph_0}$ as the direct algebraic sum of two dense subspaces X_1 and X_2 with property (K) such that $X_1 \times X_2$ also has property (K) (Theorem 4). As a consequence, X admits a strictly stronger linear topology with property (K) (Corollary 3).

Most proofs are based on ideas already applied in [8], [9] and [2]. Therefore some arguments are presented fairly concisely.

1. Existence of dense subspaces of large dimension and codimension. It is well known that a subspace of a topological linear space which is of codimension 1 is either closed or dense in the space. Recently the interest in dense subspaces of large dimension and codimension was raised by De Wilde and Tsirulnikov ([3], Example 3.6). (A related problem had already been considered by Klee [7].) Independently of [3], such subspaces, with an additional property, were constructed in [9]. Here we shall establish a result which implies [3], Remark (ii). It also partially generalizes [9], Theorem 2.

THEOREM 1. *Let X be a topological linear space which has a base of cardinality $\leqslant \dim X$. Then there exists a sequence $(X_\alpha)_{\alpha<\varphi}$, where φ is the initial ordinal of cardinality $\dim X$, of dense subspaces of X such that $\dim X_\alpha = \dim X$ for all $\alpha < \varphi$ and X is the direct algebraic sum of $(X_\alpha)_{\alpha<\varphi}$.*

Proof (cf. [9], proof of Theorem 2, and [2], proofs of Lemma 1 and Theorem 1). Let $\mathscr{U}$ be a base for X with $\operatorname{card}\mathscr{U} \leqslant \dim X$. Arrange $\mathscr{U}$ into a transfinite sequence $(U_\alpha)_{\alpha<\varphi}$ with each element of $\mathscr{U}$ repeated $(\dim X)$-times. Then it is easy to construct inductively a double sequence $(x_\alpha^\beta)_{\beta\leqslant\alpha<\varphi}$ of elements of X such that for all $\beta \leqslant \alpha < \varphi$

$$x_\alpha^\beta \notin \operatorname{lin}\{x_{\alpha'}^{\beta'}\colon \beta' \leqslant \alpha' < \alpha \text{ or } \alpha' = \alpha \text{ and } \beta' < \beta\} \quad \text{and} \quad x_\alpha^\beta \in U_\alpha.$$

Let Y be an algebraic complement of $\operatorname{lin}\{x_\alpha^\beta\colon \beta \leqslant \alpha < \varphi\}$ and put

$$X_0 = Y + \operatorname{lin}\{x_\gamma^0\colon \gamma < \varphi\} \quad \text{and} \quad X_\alpha = \operatorname{lin}\{x_{\alpha'}^\alpha\colon \alpha \leqslant \alpha' < \varphi\}$$

for $1 \leqslant \alpha < \varphi$. As easily seen, $(X_\alpha)_{\alpha<\varphi}$ is then as desired.

We note that the assumption of Theorem 1 holds if X is metrizable and infinite-dimensional (cf. [9], p. 94, footnote (*)). This assumption is also satisfied by the nonmetrizable space $\boldsymbol{R}^{\boldsymbol{R}}$. On the other hand, it fails if X is a linear space such that every subspace of X is closed (cf. [9], Example 1).

LEMMA 1. *Every linear space Z can be represented as the union of $\dim Z$ pairwise disjoint convex sets.*

Proof (cf. [7], proof of (3)). Let $(x_\alpha)_{\alpha<\varphi}$, where φ is an ordinal of cardinality $\dim Z$, be a Hamel basis for Z and let $(f_\alpha)_{\alpha<\varphi}$ be the corresponding sequence of coefficient functionals. Put

$$Z_\alpha = \{x \in Z\colon f_\beta(x) \leqslant 0 \text{ for } \beta < \alpha \text{ and } f_\alpha(x) > 0\},$$
$$Z_\varphi = \{x \in Z\colon f_\beta(x) \leqslant 0 \text{ for } \beta < \varphi\}.$$

Then $Z = \bigcup_{\alpha\leqslant\varphi} Z_\alpha$ and the Z_α's are pairwise disjoint convex sets.

The following corollary slightly generalizes an old result of Klee ([7], (4)).

COROLLARY 1. *Let X be a topological linear space which has a base of cardinality $\leqslant \dim X$ and let $\mathfrak{m}$ be a cardinal with $\mathfrak{m} \leqslant \dim X$. Then X can be represented as the union of $\mathfrak{m}$ pairwise disjoint dense convex sets.*

Proof. By assumption and Theorem 1, there exist subspaces Y and Z of X such that $X = Y \oplus Z$, Y is dense in X and $\dim Z = \mathfrak{m}$. Let $(Z_\alpha)_{\alpha\leqslant\varphi}$ be the decomposition of Z given by Lemma 1. Then $(Y+Z_\alpha)_{\alpha\leqslant\varphi}$ is the desired decomposition of X.

2. $\varkappa$-subspaces. Following Drewnowski [4], we call a subspace Y of a topological linear space X a *$\varkappa$-subspace* provided every linearly independent sequence (x_n) in X such that $\sum_{n=1}^{\infty} x_n$ is subseries convergent contains a subsequence (x_{n_k}) with $\sum_{k=1}^{\infty} x_{n_k} \in Y$.

Under the continuum hypothesis, the existence of a proper $\varkappa$-subspace of l_2 is implicit in [8], proof of Theorem 2. A more general result, without the continuum hypothesis, is given in [9], Theorem 2.

Clearly, every $\varkappa$-subspace of an F-space (i.e., a complete metrizable topological linear space) has property (K). However, there exist F-spaces having dense subspaces with property (K) which are not $\varkappa$-subspaces.

EXAMPLE (J. Burzyk). Let X and Z be F-spaces such that $\dim X = 2^{\aleph_0}$ and $\dim Z \geqslant 2^{\aleph_0}$. Let Y be a (proper) dense subspace of X with property (K) (see [9], Theorem 2). Then $Y \times Z$ is a dense subspace of $X \times Z$ with property (K). Nevertheless, $Y \times Z$ is not a $\varkappa$-subspace of $X \times Z$. Indeed, fix $x \in X \setminus Y$ and a linearly independent sequence (u_n) in Z such that $\sum_{n=1}^{\infty} u_n$ is subseries convergent. Then the sequence $\big((2^{-n}x, u_n)\big)$ is also linearly independent and $\sum_{n=1}^{\infty}(2^{-n}x, u_n)$ is subseries convergent in $X \times Z$. However, for every $n_1 < n_2 < \ldots$

$$\sum_{k=1}^{\infty}(2^{-n_k}x, u_{n_k}) = \Big(\Big(\sum_{k=1}^{\infty} 2^{-n_k}\Big)x, \sum_{k=1}^{\infty} u_{n_k}\Big) \notin Y \times Z.$$

Drewnowski showed that every $\varkappa$-subspace of an F-space has dimension $\geqslant 2^{\aleph_0}$ ([4], Lemma). We shall show that $\varkappa$-subspaces are large in two other respects.

THEOREM 2. *Suppose X is a topological linear space which contains a linearly independent sequence (u_n) such that $\sum_{n=1}^{\infty} u_n$ is subseries convergent. If Y is a $\varkappa$-subspace of X, then*

(a) $\operatorname{codim} Y \leqslant 2^{\aleph_0}$;

(b) *Y is dense in X* (1).

Proof. (a): Assume, to get a contradiction, that $\operatorname{codim} Y > 2^{\aleph_0}$. Then X contains a linearly independent set $\{x_t\colon t \in T\}$ such that $\operatorname{card} T > 2^{\aleph_0}$ and

$$\operatorname{lin}\{x_t\colon t \in T\} \cap \operatorname{lin}(Y \cup \{u_n\colon n \in \boldsymbol{N}\}) = \{0\}.$$

Then, for every $t \in T$, the sequence $(2^{-n}x_t + u_n)$ is linearly independent. Moreover, $\sum_{n=1}^{\infty}(2^{-n}x_t + u_n)$ is subseries convergent. Since $\operatorname{card} T > 2^{\aleph_0}$, there exist $t_1, t_2 \in T$ with $t_1 \neq t_2$ and $n_1 < n_2 < \ldots$ such that

$$\Big(\sum_{k=1}^{\infty} 2^{-n_k}\Big) x_{t_i} + \sum_{k=1}^{\infty} u_{n_k} \in Y$$

for $i = 1, 2$. Hence $x_{t_1} - x_{t_2} \in Y$, a contradiction.

(b): Fix $x \in X$ with $x \neq 0$. Then $x \notin \operatorname{lin}\{u_{n+k}\colon k \in \boldsymbol{N}\}$ for n large enough. We assume without loss of generality that $x \notin \operatorname{lin}\{u_n\colon n \in \boldsymbol{N}\}$.

Next fix a neighbourhood U of 0 in X. By passing to a subsequence, we may assume that for every $k \in \boldsymbol{N}$ and all scalars $\gamma_1, \ldots, \gamma_k$ with $|\gamma_n| \leqslant 2^n$ for $n = 1, \ldots, k$, we have $\sum_{n=1}^{k} \gamma_n u_n \in U$.

Since the sequence $(2^{-n}x + u_n)$ is linearly independent and $\sum_{n=1}^{\infty}(2^{-n}x + u_n)$ is subseries convergent, we have, by assumption,

$$y = \Big(\sum_{k=1}^{\infty} 2^{-n_k}\Big) x + \sum_{k=1}^{\infty} u_{n_k} \in Y$$

for some $n_1 < n_2 < \ldots$ On the other hand,

$$\Big(\sum_{k=1}^{\infty} 2^{-n_k}\Big)^{-1} y - x \in \overline{U},$$

as $\big(\sum_{k=1}^{\infty} 2^{-n_k}\big)^{-1} \leqslant 2^{n_i}$ for $i \in \boldsymbol{N}$.

(1) This assertion was found by L. Drewnowski and, independently, by the author. The proof given here is due to the author.

Theorem 2 above sheds some light on Theorem 2 of [9]. Namely, it shows that if X satisfies the assumption of the former theorem, then the condition that $\dim X \leqslant 2^{\aleph_0}$ of the latter theorem is necessary, while the denseness assertion is redundant.

3. Products of spaces with property (K). Obviously, the (Cartesian) product of two complete metric spaces is again complete. Similarly, the product of two Baire spaces one of which is second countable is again a Baire space (see, e.g., [6], Theorem 5.1). Somewhat suprisingly, property (K), which is in between, fails to be multiplicative. A general counterexample is given by the forthcoming Theorem 3.

Before stating it, we quote the following definition ([9], Definition 1). A sequence (x_n) in a topological linear space X is called (*topologically linearly*) *m-independent* if for each bounded sequence (λ_n) of scalars such that $\sum_{n=1}^{\infty} \lambda_n x_n = 0$ we have $(\lambda_n) = 0$. Note that if $\dim X \geqslant \aleph_0$, then X contains an m-independent sequence (see [5], Theorem, for a stronger result).

THEOREM 3 (2). *Let X be a topological linear space with $\operatorname{card} X = 2^{\aleph_0}$ and let Y be a subspace of X with $\dim Y < 2^{\aleph_0}$. Then there exist $\varkappa$-subspaces X_1 and X_2 of X such that $X_1 \cap X_2 = Y$.*

In particular, if $Y = \operatorname{lin}\{u_n\colon n \in \boldsymbol{N}\}$, where (u_n) is an m-independent sequence such that $\sum_{n=1}^{\infty} u_n$ is subseries convergent, then $X_1 \times X_2$ does not have property (K).

Proof. We omit the construction of X_1 and X_2, which follows a known pattern ([8], proof of Theorem 2, and [9], proof of Theorem 2).

In order to prove the last assertion observe that the sequence $((u_n, u_n))$ is linearly independent, $(u_n, u_n) \to 0$ and $(u_n, u_n) \in X_1 \times X_2$. Moreover, since (u_n) is m-independent, for every $n_1 < n_2 < \ldots$ we have $\sum_{k=1}^{\infty} u_{n_k} \notin Y$, and so

$$\Big(\sum_{k=1}^{\infty} u_{n_k}, \sum_{k=1}^{\infty} u_{n_k}\Big) \notin X_1 \times X_2.$$

We note that, in the case where X is an infinite-dimensional separable F-space, Theorem 3 provides an example of a Baire space which does not have property (K). The first example to this effect was given in [2], Theorem 3.

The next two auxiliary results are concerned with an arbitrary topological linear space X. They improve [9], Proposition 3 and Corollary 1.

(2) A similar result was obtained independently by J. Burzyk.

PROPOSITION. *Assume* (x_n) *is a linearly independent sequence in* X *such that* $\sum_{n=1}^{\infty} x_n$ *is subseries convergent. Then there exists a sequence* $n_1 < n_2 < \ldots$ *of odd natural numbers such that the sequence*

$$x_{n_1}, x_{n_1+1}, x_{n_2}, x_{n_2+1}, \ldots$$

is m-independent.

Sketch of proof. Arguing as in [9], proof of Proposition 3, we may assume that $\sum_{n=1}^{\infty} x_n$ is bounded multiplier convergent.

Put

$$C_n = \Big\{\sum_{i=1}^{n+1} \mu_i x_i\colon |\mu_i| \leqslant 1,\ i = 1, \ldots, n+1;\ \max(|\mu_n|, |\mu_{n+1}|) \geqslant \tfrac{1}{2}\Big\}.$$

Clearly, C_n is compact and $0 \notin C_n$. Hence for each n we can find $m > n$ such that

$$\Big\{\sum_{i=m}^{\infty} \lambda_i x_i\colon |\lambda_i| \leqslant 1, i = m, m+1, \ldots\Big\} \subset X \setminus C_n.$$

This allows us to define a sequence $n_1 < n_2 < \ldots$ of odd natural numbers such that $\sum_{i=n_k+1}^{\infty} \lambda_i x_i \in X \setminus C_{n_k}$ whenever $|\lambda_i| \leqslant 1$. The last condition implies the assertion (cf. [9], proof of Proposition 1).

COROLLARY 2. *If* (x_n) *and* (y_n) *are linearly independent sequences in* X *such that* $\sum_{n=1}^{\infty} x_n$ *and* $\sum_{n=1}^{\infty} y_n$ *are subseries convergent and*

$$\operatorname{lin}\{x_n\colon n \in \boldsymbol{N}\} \cap \operatorname{lin}\{y_n\colon n \in \boldsymbol{N}\} = \{0\},$$

then, given a family $\{(\nu_M, \varrho_M)\colon M \subset \boldsymbol{N}\}$ *of pairs of scalars with* $|\nu_M| + |\varrho_M| > 0$ *for all* $M \subset \boldsymbol{N}$, *we have*

$$\dim \operatorname{lin}\Big\{\nu_M \sum_{n=1}^{\infty} 1_M(n) x_n + \varrho_M \sum_{n=1}^{\infty} 1_M(n) y_n\colon M \subset \boldsymbol{N}\Big\} = 2^{\aleph_0}.$$

Proof. The sequence

$$x_1, y_1, x_2, y_2, \ldots$$

is linearly independent and the corresponding series is subseries convergent. Hence, applying the Proposition, we may assume that the sequences

(x_n) and (y_n) have the property that for every pair (λ_n), (μ_n) of bounded sequences of scalars such that

$$\sum_{n=1}^{\infty} \lambda_n x_n + \sum_{n=1}^{\infty} \mu_n y_n = 0$$

we have $(\lambda_n) = (\mu_n) = 0$.

Let $(M_k)_{1 \leqslant k \leqslant l}$ be a sequence of pairwise almost disjoint infinite subsets of $\boldsymbol{N}$. We claim that the corresponding sequence

$$\Big(\nu_{M_k} \sum_{n=1}^{\infty} 1_{M_k}(n) x_n + \varrho_{M_k} \sum_{n=1}^{\infty} 1_{M_k}(n) y_n\Big)_{1 \leqslant k \leqslant l}$$

of vectors is linearly independent. Indeed, if

$$\sum_{k=1}^{l} \gamma_k \Big(\nu_{M_k} \sum_{n=1}^{\infty} 1_{M_k}(n) x_n + \varrho_{M_k} \sum_{n=1}^{\infty} 1_{M_k}(n) y_n\Big) = 0,$$

where $(\gamma_k)_{1 \leqslant k \leqslant l}$ is a sequence of scalars, then, by the assumption introduced at the beginning of the proof, we have

$$\sum_{k=1}^{l} \gamma_k \nu_{M_k} 1_{M_k}(n) = \sum_{k=1}^{l} \gamma_k \varrho_{M_k} 1_{M_k}(n) = 0$$

for all $n \in \boldsymbol{N}$. Since

$$M_k \setminus \bigcup_{\substack{1 \leqslant j \leqslant l \\ j \neq k}} M_j \neq \emptyset,$$

we have $\gamma_k \nu_{M_k} = \gamma_k \varrho_{M_k} = 0$, and so $\gamma_k = 0$ for $k = 1, \ldots, l$.

The assertion now follows from Sierpiński's theorem on the existence of $2^{\aleph_0}$ pairwise almost disjoint infinite subsets of $\boldsymbol{N}$.

Corollary 2 allows us to strengthen partially Theorem 2 of [9] as follows.

THEOREM 4. *Let* X *be an F-space with* $\dim X = 2^{\aleph_0}$. *Then there exist subspaces* X_1 *and* X_2 *of* X *such that*

(i) $X = X_1 \oplus X_2$;

(ii) *For every pair* $(x_n), (y_n)$ *of linearly independent sequences in* X *such that* $\sum_{n=1}^{\infty} x_n$ *and* $\sum_{n=1}^{\infty} y_n$ *are subseries convergent and*

$$\operatorname{lin}\{x_n\colon n \in \boldsymbol{N}\} \cap \operatorname{lin}\{y_n\colon n \in \boldsymbol{N}\} = \{0\}$$

there exists an infinite set $M \subset \boldsymbol{N}$ *with*

$$\sum_{n=1}^{\infty} 1_M(n) x_n \in X_1 \quad \text{and} \quad \sum_{n=1}^{\infty} 1_M(n) y_n \in X_2.$$

In particular, X_1 and X_2 are $\varkappa$-subspaces of X and $X_1 \times X_2$ has property (K).

Proof. Arrange the family of all pairs of sequences satisfying the conditions of (ii) into a transfinite sequence $\big((x_n^a), (y_n^a)\big)_{a<\varphi}$, where φ is the initial ordinal of cardinality $2^{\aleph_0}$. We shall construct inductively two increasing sequences $(X_1^a)_{a<\varphi}$ and $(X_2^a)_{a<\varphi}$ of subspaces of X such that for all $a < \varphi$

(1) $X_1^a \cap X_2^a = \{0\}$;

(2) There exists an infinite set $M_a \subset N$ with $\sum_{n=1}^{\infty} 1_{M_a}(n) x_n^a \in X_1^a$ and $\sum_{n=1}^{\infty} 1_{M_a}(n) y_n^a \in X_2^a$;

(3) $\dim X_i^a < 2^{\aleph_0}$ for $i = 1, 2$.

Suppose the construction has been carried out for all $\beta < a$. Put $Y_i^a = \bigcup_{\beta<a} X_i^\beta$. Since $\dim(Y_1^a + Y_2^a) < 2^{\aleph_0}$, it follows from Corollary 2 that etere exists an infinite set $M_a \subset N$ such that

$$(Y_1^a + Y_2^a) \cap \operatorname{lin}\Big\{\sum_{n=1}^{\infty} 1_{M_a}(n) x_n^a, \sum_{n=1}^{\infty} 1_{M_a}(n) y_n^a\Big\} = \{0\}.$$

Put $X_1^a = \operatorname{lin}\big(Y_1^a \cup \{\sum_{n=1}^{\infty} 1_{M_a}(n) x_n^a\}\big)$ and $X_2^a = \operatorname{lin}\big(Y_2^a \cup \{\sum_{n=1}^{\infty} 1_{M_a}(n) y_n^a\}\big)$. Clearly, (1)–(3) hold.

Let Y be an algebraic complement of the subspace $\big(\bigcup_{a<\varphi} X_1^a\big) + \big(\bigcup_{a<\varphi} X_2^a\big)$ and put $X_1 = Y + \bigcup_{a<\varphi} X_1^a$ and $X_2 = \bigcup_{a<\varphi} X_2^a$. Then (1) and (2) imply (i) and (ii), respectively.

In connection with Theorems 3 and 4, we note that nothing is known about squares of noncomplete spaces with property (K).

COROLLARY 3. *Every F-space X with $\dim X = 2^{\aleph_0}$ admits a strictly stronger (metrizable) linear topology with property* (K).

Proof [3]. Let X_1 and X_2 be given by Theorem 4. Then the addition in X when restricted to $X_1 \times X_2$ is continuous and one-to-one and maps $X_1 \times X_2$ onto X. Since, in view of Theorem 2 (b), $X_1 \times X_2$ is noncomplete, the inverse mapping is not continuous. This yields the assertion.

Corollary 3 contrasts with a classical theorem stating that the complete metrizable linear topologies on a given linear space are mutually incomparable.

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(3) The idea of this proof was shown to me by L. Drewnowski.

Added in proof. 1. Theorem 1 is related to another result due to V. L. Klee (see R. R. Phelps, *Subreflexive normed linear spaces*, Arch. Math. (Basel) 8 (1957), 444–450, Theorem 3.1).

2. In the literature there exist analogues of Theorem 1 for metric spaces (see, e.g., W. Sierpiński, *Sur la décomposition des espaces métriques en ensembles disjoints*, Fund. Math. 36 (1949), 68–71).

3 (Drewnowski). Example on p. 415 can be generalized as follows: Every F-space X for which there exists an infinite-dimensional closed subspace Z with $\dim X/Z = 2^{\aleph_0}$ contains a dense subspace with property (K) which is not a $\varkappa$-subspace of X_j(cf. [4], p. 63).

4 (Drewnowski and the author). The argument given in Example on p. 415 shows that if X is a topological linear space, Z is an infinite-dimensional F-space and X_1 and Z_1 are subspaces of X and Z, respectively, such that $X_1 \times Z_1$ is a $\varkappa$-subspace of $X \times Z$, then $X_1 = X$. In particular, in the situation of Theorem 4, $X_1 \times X_2$ is not a $\varkappa$-subspace of $X \times X$.

5 (Drewnowski). Corollary 3 can be deduced from the decomposition $X = X_1 \oplus X_2$, where X_1 is a dense subspace of X with property (K) and $\dim X_2 = 1$, which follows from [9], Theorem 2. Moreover, Corollary 3 holds if X contains a closed subspace Z with $\dim X/Z = 2^{\aleph_0}$.

The author is much indebted to L. Drewnowski for these and other related remarks.

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