STUDIA MATHEMATICA, LXXVII. (1984)

Integral extension procedures in weakly σ-complete lattice-ordered groups, I

by

M. WILHELM (Wrocław)

Abstract. An expansion $a \sim \sum_n a_n$ and the classes of weakly σ-complete and conditionally weakly σ-complete commutative lattice-ordered groups (defined by means of this expansion) are studied. Next, given a σ-subadditive l-seminorm ν on L, where L is an l-subgroup of an l-group G, an expansion $a \overset{\nu}{\sim} \sum_n a_n$ and the resulting extension (L_M, ν_M) of (L, ν) are studied. The extension procedure is patterned upon an integral construction due to MacNeille and Mikusiński.

1. Introduction. This is, in principle, a continuation of the author's paper [7]. The notion of weak σ-completeness was introduced there. (We restrict ourselves to commutative l-groups; l-group = lattice-ordered group.) The most significant feature of a weakly σ-complete l-group G is that each σ-subadditive l-seminorm on G is complete ([7], Theorem 5), which need not hold even for a conditionally complete Riesz space ([7], Example 3). For this and other reasons, which will be seen later, such a group seems to be a natural object for modelling some general integration procedures and examining their interrelations. Here we recall the definition of weak σ-completeness (in an equivalent form) and prove some related facts. In parallel, we define and investigate conditional weak σ-completeness; an l-group G is conditionally σ-complete if and only if G is conditionally weakly σ-complete and Archimedean (Theorem 3).

Given an l-subgroup L of an l-group G, we define its extension $L_\sim \subset G$ and prove that $L_\sim$ is closed under the operation of taking all elements $a \in G$ having expansion $a \sim \sum_n a_n$ with $a_n \in L_\sim$ (Theorem 1). The expansion, fundamental here, is defined and studied previously in Section 2.

Next, given a σ-subadditive l-seminorm ν on $L \subset G$, we consider the expansion $a \overset{\nu}{\sim} \sum_n a_n$ meaning that $a \sim \sum_n a_n$, $\{a_n\} \subset L$ and $\sum_n \nu(a_n) < \infty$, define the corresponding extension (L_M, ν_M) of (L, ν) and prove its basic properties (Theorems 4–6). The extension procedure is patterned upon the integral construction due to MacNeille [4] and Mikusiński [5], [6].

In the forthcoming part II of this paper we study other kinds of extensions of (L, ν), their interrelations, integrals and their extensions.

2. The expansion $a \sim \sum_n a_n$ and the extension $L_\sim$. Throughout the paper $(G, +, \leqslant)$ is a commutative l-group (lattice-ordered group; see [1], [3] for fundamentals). We write $a \leqslant' \sup_n a_n$ if $\inf_n (a-a_n)^+ = 0$ (or, equivalently, $a = \sup_n a \wedge a_n$), and $b \leqslant' \sum_n b_n$ if $b \leqslant' \sup_k (\sum_{n\leqslant k} a_n)$; the accents beside "$\leqslant$" mark that $\sup_n a_n$ or $\sum_n b_n$ need not exist in G.

Next, we write $a \sim \sum_n a_n$ (or $a \sim a_1 + a_2 + \ldots$) if

$$\Big|a - \sum_{n\leqslant k} a_n\Big| \leqslant' \sum_{n>k} |a_n| \quad \text{for each } k \in N.$$

EXAMPLE 1. Consider $G = R^S$, where S is an arbitrary set and R denotes the additive group of real numbers with the natural ordering. The relation $a \sim \sum_n a_n$ holds if and only if

for every $s \in S$, $\sum_n |a_n(s)| < \infty$ implies $a(s) = \sum_n a_n(s)$.

PROPOSITION 1. *If $a \sim \sum_n a_n$, then $|a| \leqslant' \sum_n |a_n|$. If, additionally, the series $\sum_n |a_n|$ is bounded by $|a|$ (i.e., $\sum_{n\leqslant k} |a_n| \leqslant |a|$ for $k \in N$), then $|a| = \sum_n |a_n|$.*

Proof. $|a| \leqslant |a_1| + |a - a_1| \leqslant' |a_1| + \sum_{n>1} |a_n|$.

PROPOSITION 2. *If $a \sim \sum_n a_n$ and π is a permutation of N, then $a \sim \sum_n a_{\pi(n)}$.*

Proof. Given $i \in N$, choose $j \in N$ so that

$$I := \{\pi(1), \ldots, \pi(i)\} \subset J := \{1, \ldots, j\}.$$

Then

$$b := \Big|a - \sum_{n\leqslant i} a_{\pi(n)}\Big| \leqslant \Big|a - \sum_{n\leqslant j} a_n\Big| + \sum_{n\in J\setminus I} |a_n| \leqslant' \sum_{n>j} |a_n| + \sum_{n\in J\setminus I} |a_n|,$$

and so

$$b \leqslant' \sum_{n>i} |a_{\pi(n)}|.$$

From now on we can write, e.g., $a \sim \sum_t b_t$ (where t runs through a countable set T) or $a \sim \sum_{n,m} a_m^{(n)}$ (where $n, m \in N$), and this is unambiguous.

LEMMA 1. *If $d = \sup_t d_t$ and $d_t \leqslant' \sup_n e_n$ for all $t \in T$, where T is arbitrary, then $d \leqslant' \sup_n e_n$.*

Proof. $d = \sup_t d_t = \sup_{t,n} d_t \wedge e_n = \sup_n d \wedge e_n$.

PROPOSITION 3. *Let $\{a_n\}$ be a sequence in G. The set A of all elements $a \in G$ satisfying $a \sim \sum_n a_n$ is a convex sublattice of G (possibly empty). If $b_t \in A$ for $t \in T$ and $\inf_t |b_t - b| = 0$, then $b \in A$.*

Proof. Let $a, b \in A$ and $a \wedge b \leqslant c \leqslant a \vee b$. Then

$$\Big|c - \sum_{n\leqslant k} a_n\Big| \leqslant \Big|a - \sum_{n\leqslant k} a_n\Big| \vee \Big|b - \sum_{n\leqslant k} a_n\Big|,$$

which yields $c \in A$ (Lemma 1 for a two-element set T is used here). To prove the second assertion, notice that

$$d := \Big|b - \sum_{n\leqslant k} a_n\Big| = \sup_t \Big(\Big|b - \sum_{n\leqslant k} a_n\Big| \wedge \Big|b_t - \sum_{n\leqslant k} a_n\Big|\Big);$$

Lemma 1 shows that $d \leqslant' \sum_{n>k} |a_n|$.

PROPOSITION 4. *If $\inf_m |a - \sum_{n\leqslant k+m} a_n| = 0$ for all $k \in N$, then $a \sim \sum_n a_n$.*

Proof. The assertion follows from the inequality

$$\Big(\Big|a - \sum_{n\leqslant k} a_n\Big| - \sum_{k<n\leqslant k+m} |a_n|\Big)^+ \leqslant \Big|a - \sum_{n\leqslant k+m} a_n\Big|.$$

PROPOSITION 5. (i) *If $a = o - \sum_n a_n$ (i.e., $|a - \sum_n a_n| \leqslant b_k \searrow_k 0$ for some $\{b_k\} \subset G$), then $a \sim \sum_n a_n$.*

(ii) *If $a \sim \sum_n a_n$ and the sum $\sum_n |a_n|$ exists in G, then $a = o - \sum_n a_n$.*

Proof. Part (i) is a consequence of Proposition 4, part (ii) — of the inequality:

$$\Big|a - \sum_{n\leqslant k} a_n\Big| \leqslant \sum_{n>k} |a_n| \searrow_k 0.$$

PROPOSITION 6. *Let $\{a_n\} \subset G$, and let A be as in Proposition 3. The set A has at most one element provided that*

(i) *the sum $\sum_n |a_n|$ exists in G, or*

(ii) *the series $\sum_n |a_n|$ is bounded and G is Archimedean.*

Proof. The first assertion is a corollary to Proposition 5. If G is Archimedean, it possesses the Dedekind completion G_D (cf. [3], V. 10), and the bounded series $\sum_n |a_n|$ has a sum in G_D. If now $a \sim \sum_n a_n$ holds in G, it also holds in G_D and so, by Proposition 5, a is the order sum $o - \sum_n a_n$ in G_D.

PROPOSITION 7. *Let* $a_n \geqslant 0$. *We have* $a \sim \sum_n a_n$ *if and only if* $a \leqslant' \sum_n a_n$ *and* $2\sum_{n\leqslant k} a_n - a \leqslant' \sum_n a_n$ *for* $k \in N$.

Proof. Both conditions are equivalent to the following one:

$$\Big|a-\sum_{n\leqslant k} a_n\Big| + \sum_{n\leqslant k} a_n \leqslant' \sum_n a_n \quad \text{for} \quad k \in N.$$

PROPOSITION 8. *Let* $a_n \geqslant 0$. *We have* $a = \sum_n a_n$ *if and only if* $A = \{a\}$, *where* A *is as in Proposition* 3.

Proof. *Necessity*. By Proposition 7, $b \in A$ if and only if $b \leqslant a$ and $2\sum_{n\leqslant k} a_n - b \leqslant a$ for $k \in N$ if and only if $b \leqslant a$ and $2a - b \leqslant a$.

Sufficiency. We may deduce from Proposition 7 that each element of the form $a \vee \sum_{n\leqslant k} a_n$ ($k \in N$) in also in A. By the assumption, $a \vee \sum_{n\leqslant k} a_n = a$. Hence $0 \leqslant \sum_{n\leqslant k} a_n \leqslant a$ and Proposition 1 yields $a = \sum_n a_n$.

When we do not assume that $a_n \geqslant 0$, neither $a = o-\sum_n a_n$ implies $A = \{a\}$ (put $a_n = (-1)^{n+1} n^{-1}$; $\ln 2 = \sum_n a_n$ and $A = R$), nor $A = \{a\}$ implies $a = o-\sum_n a_n$ (see Example 2 and also Proposition 15).

PROPOSITION 9. *Let* $a_n \geqslant 0$, *and let* A *be as in Proposition* 3. *If* $a, b \in A$ *and* $a \leqslant b$, *then* $b + p(b-a) \in A$ *for all* $p \in N$.

Proof. It is sufficient to consider $p = 1$. By Proposition 7, $b \leqslant' \sum_n a_n$ and $\sum_{n\leqslant k} 2a_n \leqslant' a + \sum_n a_n$ for $k \in N$. It follows that $2b \leqslant' \sum_n 2a_n$ and, in consequence, $2b \leqslant' a + \sum_n a_n$. Thus the element $2b - a$ satisfies the first condition of Proposition 7, while the second is obvious, because $2b - a \geqslant a$.

A conditionally σ-complete Riesz space X is said to be *weakly σ-distributive* if for every bounded double sequence $\{v_{kj}\} \subset X$ such that $v_{kj} \searrow_j 0$ for each k, $\inf_\varphi \sup_k v_{k\varphi(k)} = 0$, where φ runs through all functions of N into N (see, e.g., Fremlin [2] and references therefrom). We extend this definition to any commutative l-group G (not necessarily conditionally σ-complete) in the following way:

G is weakly σ-distributive if for every double sequence $\{u_{kj}\} \subset G$ such that $u_{kj} \searrow_j 0$ ($k \in N$) and for every element $e > 0$ there exists a function $\varphi\colon N \to N$ such that the inequality $e \leqslant' \sup_k u_{k\varphi(k)}$ does not hold.
(given can readily be verified that the two definitions are consistent
It $e > 0$, consider $v_{kj} = u_{kj} \wedge e$).

PROPOSITION 10. *Let* G *be weakly σ-distributive, and let* $\{a_n\} \subset G$. *If*

(*) $\inf_k 2\sum_{n\in I_k} |a_n| = 0$ *for every disjoint sequence* $\{I_k\}$ *of finite subsets of* N,

then the set A (*corresponding to* $\{a_n\}$ *as in Proposition* 3) *has at most one element.*

Proof. Assume that $a^{(i)} \in A$ for $i = 1, 2$. Put

$$u_{kj} = \sum_{i=1,2} \Big(\Big|a^{(i)} - \sum_{n\leqslant k} a_n\Big| - \sum_{k<n\leqslant k+j} |a_n|\Big)^+.$$

For every $k \in N$ we have $u_{kj} \searrow_j 0$ and

$$|a^{(1)} - a^{(2)}| \leqslant \sum_{i=1,2} \Big|a^{(i)} - \sum_{n\leqslant k} a_n\Big| \leqslant u_{kj} + 2\sum_{k<n\leqslant k+j} |a_n|.$$

For every function $\varphi \in N^N$ we get

$$|a^{(1)} - a^{(2)}| - 2\sum_{k<n\leqslant k+\varphi(k)} |a_n| \leqslant u_{k\varphi(k)} \leqslant' \sup_k u_{k\varphi(k)}, \qquad k \in N.$$

Condition (*) is equivalent to the following one:

(**) $$\inf_k 2 \sum_{k<n\leqslant k+\varphi(k)} |a_n| = 0 \qquad \text{for each } \varphi \in N^N.$$

Taking into account Lemma 1, we infer that

$$|a^{(1)} - a^{(2)}| \leqslant' \sup_k u_{k\varphi(k)}, \qquad \varphi \in N^N,$$

which yields $a^{(1)} = a^{(2)}$.

Remarks. (a) In case G is a vector space, the number "2" in (*) and (**) can be omitted.

(b) Condition (**) (and hence (*)) is weaker than (i) or (ii) of Proposition 6.

The following "transitivity" property of the expansion $a \sim \sum_n a_n$ is essential for further investigations.

LEMMA 2. *If* $a \sim \sum_n a_n$ *and* $a_n \sim \sum_m a_m^{(n)}$ *for each* $n \in N$, *then* $a \sim \sum_{n,m} a_m^{(n)}$.

Proof. Arrange $a_m^{(n)}$ ($n, m \in N$) in the standard diagonal way into a sequence, $\{b_p\}$ say. Given q, we may choose k and $m(n)$ ($n = 1, \ldots, k$) so that

$$c := \Big|a - \sum_{p\leqslant q} b_p\Big| = \Big|a - \sum_{n\leqslant k}\sum_{m\leqslant m(n)} a_m^{(n)}\Big| \leqslant \Big|a - \sum_{n\leqslant k} a_n\Big| + \sum_{n\leqslant k}\Big|a_n - \sum_{m\leqslant m(n)} a_m^{(n)}\Big|$$

$$\leqslant' \sum_{n>k} |a_n| + \sum_{n\leqslant k}\sum_{m>m(n)} |a_m^{(n)}|.$$

By Proposition 1, $|a_n| \leqslant' \sum_m |a_m^{(n)}|$, and so we get

$$c \leqslant' \sum_{n>k} \sum_m |a_m^{(n)}| + \sum_{n \leqslant k} \sum_{m>m(n)} |a_m^{(n)}|.$$

Hence $c \leqslant' \sum_{p>q} |b_p|$.

Now let us consider the following situation. There is given an l-subgroup L of (a commutative l-group) G, and we take all elements $a \in G$ satisfying $a \sim \sum_n a_n$ for some sequence $\{a_n\} \subset L$. These elements form a set, which will be denoted by $L_\sim$.

THEOREM 1. *$L_\sim$ is an l-subgroup of G and contains L. If $\{a_n\} \subset L_\sim$ and $G \ni a \sim \sum_n a_n$, then $a \in L_\sim$. In particular, $L_\sim$ is closed under taking limits of order convergent sequences.*

Proof. Clearly, $L \subset L_\sim$, because $a \sim a+0+0+\ldots$ Suppose $a_n, b_n \in L$, $a \sim \sum_n a_n$ and $b \sim \sum_n b_n$. Then

$$a-b \sim a_1-b_1+a_2-b_2+\ldots,$$

because

$$\Big|(a-b) - \sum_{n\leqslant k}(a_n-b_n)\Big| \leqslant \Big|a - \sum_{n\leqslant k} a_n\Big| + \Big|b - \sum_{n\leqslant k} b_n\Big| \leqslant' \sum_{n>k}|a_n| + \sum_{n>k}|b_n|.$$

To prove that $a^+ \in L_\sim$, define $a_1' = a_1^+$, $a_2' = (a_1+a_2)^+ - a_1^+$, $a_3' = (a_1+a_2+a_3)^+ - (a_1+a_2)^+$, and so on. We have $a_n' \in L$, $\sum_{n\leqslant k} a_n' = \big(\sum_{n\leqslant k} a_n\big)^+$ and

$$\Big|a^+ - \sum_{n\leqslant k} a_n'\Big| \leqslant \Big|a - \sum_{n\leqslant k} a_n\Big| \leqslant' \sum_{n>k}|a_n|.$$

It follows that

$$a^+ \sim a_1' + a_1 - a_1 + a_2' + a_2 - a_2 + \ldots$$

Thus we have shown that $L_\sim$ is an l-subgroup of G. The remaining assertions are consequences of Lemma 2 and Proposition 5.

3. Weak σ-completeness and conditional weak σ-completeness. The considerations of the previous section lead to the following definition, which—in an equivalent form—was introduced in [7]. (As in the whole paper, G stands for a commutative l-group.)

DEFINITION 1. G is *weakly σ-complete* if for every sequence $\{a_n\}$ in G^+ there exists an element $a \in G$ having the expansion $a \sim \sum_n a_n$.

From Example 1 we infer that

PROPOSITION 11. *The function space R^S is weakly σ-complete (S—arbitrary). In particular, R itself is weakly σ-complete.*

A complementary notion is that of conditional weak σ-completeness:

DEFINITION 2. G is *conditionally weakly σ-complete* if for every sequence $\{a_n\} \subset G^+$ such that the series $\sum_n a_n$ is bounded there exists an element $a \in G$ satisfying $a \sim \sum_n a_n$.

PROPOSITION 12. *For a fully ordered G, weak σ-completeness is equivalent to conditional weak σ-completeness.*

Proof. If $\sum_n a_n$ is not bounded, the inequality $c \leqslant' \sum_n a_n$ holds for all elements $c \in G$, and so $a \sim \sum_n a_n$ for each $a \in G$.

PROPOSITION 13. *Let G $(\neq \emptyset)$ be the Cartesian product (= complete direct product) of a family $\{G_t\colon t \in T\}$ of commutative l-groups; G is weakly σ-complete (conditionally weakly σ-complete) if and only if every group G_t is such.*

The direct product of an infinite family of weakly σ-complete groups need not have the same property (Example 2), but Proposition 13 remains true for direct products and conditional weak σ-completeness. The lexicographic product G of two weakly σ-complete groups G_1 and G_2 need not be even conditionally weakly σ-complete (Example 5), but if G is weakly σ-complete (conditionally weakly σ-complete) and non-empty, then both G_1 and G_2 have the same property.

PROPOSITION 14. (i) *Suppose G is weakly σ-complete. Then for every sequence $\{a_n\} \subset G$ there is some $a \in G$ having the expansion $a \sim \sum_n a_n$.*

(ii) *Suppose G is conditionally weakly σ-complete. Then the same holds for each sequence $\{a_n\} \subset G$ such that the series $\sum_n |a_n|$ is bounded.*

Proof. We have $a \sim \sum_n a_n$ for $a = b-c$, where $b \sim \sum_n a_n^+$ and $c \sim \sum_n a_n^-$.

PROPOSITION 15. *Let $\{a_n\}$ be a sequence in G, and let A be as in Proposition 3. Suppose that $A = \{a\}$. Suppose also that* (i) *G is weakly σ-complete, or* (ii) *G is conditionally weakly σ-complete and the series $\sum_n |a_n|$ is bounded. Then the series $\sum_n a_n^+$ and $\sum_n a_n^-$ are both convergent, and so $\sum_n a_n$ order-converges.*

Proof. By (i) or (ii), there are $b, c \in G^+$ satisfying $b \sim \sum_n a_n^+$ and $c \sim \sum_n a_n^-$. Hence $b-c \sim \sum_n a_n$, $a = b-c$ and the elements b, c (fulfiling the expansions) are unique (otherwise a would not be unique). Proposition 8 shows that $b = \sum_n a_n^+$ and $c = \sum_n a_n^-$.

THEOREM 2. *Let G be a weakly σ-distributive commutative l-group. Suppose that $\{a_n\} \subset G$ satisfies condition*

(*) $\inf_k 2 \sum_{n\in I_k} |a_n| = 0$ *for every disjoint sequence* $\{I_k\}$ *of finite subsets of* N.

Suppose also that (i) G *is weakly* σ*-complete, or* (ii) G *is conditionally weakly* σ*-complete and the series* $\sum_n |a_n|$ *is bounded. Then both series* $\sum_n a_n^+$ *and* $\sum_n a_n^-$ *converge, and so the series* $\sum_n a_n$ *is order-convergent.*

Proof. Proposition 14 shows that there is some $a \in A$. Proposition 10 proves that $A = \{a\}$, and we may apply Proposition 15.

THEOREM 3. *A commutative* l*-group* G *is conditionally* σ*-complete (as a lattice) if and only if* G *is Archimedean and conditionally weakly* σ*-complete.*

Proof. *Necessity* follows from the definitions and Proposition 5 (i).

Sufficiency. Let $a_n \in G^+$ and $\sum_n a_n$ be bounded. There exists $a \in G$ satisfying $a \sim \sum_n a_n$. Proposition 6 shows that such an element a is unique. By Proposition 8, $a = \sum_n a_n$. This implies, of course, that G is conditionally σ-complete.

For the rest of this section, let us fix some abbreviations: A is the class of all commutative l-groups G which are Archimedean, NA—non-Archimedean, CC—conditionally σ-complete, WC—weakly σ-complete, CWC—conditionally weakly σ-complete. As well known, no non-trivial G is σ-complete (as a lattice), and CC $\subsetneqq$ A. Obviously WC $\subset$ CWC. By Proposition 11, A$\cap$WC $\neq \emptyset$. By Theorem 3, CC $=$ A$\cap$CWC. Now we are going to show that CC $\setminus$ WC $\neq \emptyset$, NA$\cap$WC $\neq \emptyset$, NA$\cap$CWC $\setminus$ WC $\neq \emptyset$ and NA $\setminus$ CWC $\neq \emptyset$. Thus one can draw a diagram illustrating the situation, consisting of two concentric circles (CWC, WC) and the vertical line going through the centre (NA, A); the right half of the larger circle is CC.

EXAMPLE 2. (CC $\setminus$ WC $\neq \emptyset$) Let $G \subset R^N$ consist of all functions having finite support; G is conditionally complete. There is no $a \in G$ satisfying $a \sim \sum_n 1_{\{n\}}$. Notice also that $0 \sim 1_{\{1\}} - 1_{\{1\}} + 1_{\{2\}} - 1_{\{2\}} + \ldots$, no other element $a \in G$ has this expansion and the series is not order-convergent in G (cf. the passage after Proposition 8).

EXAMPLE 3. (NA$\cap$WC $\neq \emptyset$) Let G be the lexicographic product of Z (the integers) and R. Let $a_n = (z_n, r_n) \in G^+$, and let $\sum_n a_n$ be bounded in G. We have $z_n \in Z^+$; $z_n = 0$ for $n > n_0$, $r_n \in R^+$ for $n > n_0$. Put $z = \sum_n z_n$, $r = \sum_n r_n$ if $\sum_n r_n < \infty$ or $r = 0$ if $\sum_n r_n = \infty$; the element $a = (z, r)$ has the expansion $a \sim \sum_n a_n$. By Proposition 12, $G \in$ WC.

EXAMPLE 4. (NA$\cap$CWC $\setminus$ WC $\neq \emptyset$) Let $G_1 \subset Z^N$ consist of all functions having finite support, let $G_2 \in$ WC be non-trivial, and let G be their lexicographic product. Let $\sum_n a_n$ be bounded in G, where $a_n = (b_n, c_n) \in G^+$. As before, we have $b_n = 0$ for $n > n_0$, and so $c_n \geqslant 0$ for $n > n_0$. Since G_2 is weakly σ-complete, there is $c \in G_2$ satisfying $c \sim \sum_n c_n$. It follows that $a = \left(\sum_n b_n, c\right)$ satisfies $a \sim \sum_n a_n$. Thus $G \in$ CWC. Since $G_1 \notin$ WC (see Example 2), $G \notin$ WC (cf. the passage before Proposition 14).

EXAMPLE 5. (NA $\setminus$ CWC $\neq \emptyset$) Let G be the lexicographic product of R and R. Consider the series $\sum_n a_n$, where $a_n = (x_n, 0)$, $x_1 = -1$, $x_2 = 1/2$, $x_3 = 1/4$, $x_4 = 1/8$, $\ldots$, and an element $a = (x, y) \in G$. Choose k so that $\left|x - \sum_{n\leqslant k} x_n\right| - \sum_{n>k} |x_n| \geqslant 0$. Then

$$\left|a - \sum_{n\leqslant k} a_n\right| - \sum_{k<n\leqslant k+m} |a_n| > (0, 1) > 0 \quad \text{for all } m \in N,$$

and so $a \sim \sum_n a_n$ cannot hold. Thus $G \notin$ CWC.

4. The expansion $a \overset{\nu}{\sim} \sum_n a_n$ and the extension (L_M, ν_M). Let L be an l-subgroup of (a commutative l-group) G, and let ν be an l-seminorm on L, that is, a function of L into $[0, \infty]$ such that $\nu(0) = 0$, $\nu(a+b) \leqslant \nu(a) + \nu(b)$, and $\nu(a) \leqslant \nu(b)$ whenever $|a| \leqslant |b|$ $(a, b \in L)$. We say that ν is σ-subadditive if $\nu(a) \leqslant \sum_n \nu(a_n)$ whenever $|a| = \sum_n |a_n|$ (equivalently: whenever $|a| \leqslant' \sum_n |a_n|$; cf. [7], Theorem 2). It is worth noting that σ-subadditivity is a kind of closed graph property. Indeed, let $\varrho(a, b) = \nu(a-b)$, let ϱ be the relation $a\varrho b \equiv \varrho(a, b) = 0$, let X denote the completion of the semimetric space L/ϱ, and let q be the quotient mapping of L into X; ν is σ-subadditive if and only if

$$(a_n, x_n) \in \operatorname{Gr} q, \quad a = o\text{-}\lim_n a_n \quad \text{and} \quad x = \lim_n x_n \quad \text{imply} \quad (a, x) \in \operatorname{Gr} q.$$

(This follows from Theorem 2, (i) $\Leftrightarrow$ (iv), of [7].)

Given $a, a_n \in G$, we write $a \overset{\nu}{\sim} \sum_n a_n$ if $a \sim \sum_n a_n$, $\{a_n\} \subset L$, and $\sum_n \nu(a_n) < \infty$. (For $\nu = 0$, $a \overset{\nu}{\sim} \sum_n a_n$ reduces to $a \sim \sum_n a_n$ whenever $\{a_n\} \subset L$.)

EXAMPLE 6. Let $(S, \mathscr{A}, \mu)$ be a positive measure space, $G = R^S$, L — the collection of all simple integrable functions in G, $\nu(a) = \int_S |a| \, d\mu$ for $a \in L$. A function $a \in G$ is (Lebesgue) integrable if and only if there exists a sequence $\{a_n\} \subset L$ satisfying $a \overset{\nu}{\sim} \sum_n a_n$. This theorem, due to

MacNeille [4] and Mikusiński [5], was used by the latter author to build an axiomatic theory of integration in his book [6].

Define L_M as the set of all elements $a \in G$ such that $a \overset{\nu}{\sim} \sum_n a_n$ holds for some sequence $\{a_n\} \subset L$. ($L_M \subset L_\sim$, and $L_M = L_\sim$ for $\nu = 0$.)

Theorem 4. *L_M is an l-subgroup of G; L_M contains L provided ν is finite (i.e., takes values in $[0, \infty)$).*

Proof. Suppose $a \overset{\nu}{\sim} \sum_n a_n$ and $b \overset{\nu}{\sim} \sum_n b_n$. Arguing as in the proof of Theorem 1, we get $a-b \overset{\nu}{\sim} a_1-b_1+a_2-b_2+\ldots$, and so L_M is a subgroup of G. Proof that $a^+ \in L_M$ is similar as that $a^+ \in L_\sim$ in Theorem 1; it is sufficient to notice that $|a_n'| \leqslant |a_n|$, $\nu(a_n') \leqslant \nu(a_n)$.

Theorem 5. *Let ν be finite and σ-subadditive. The equality*

$$\nu_M(a) = \lim_k \nu\Big(\sum_{n\leqslant k} a_n\Big),$$

written whenever $a \overset{\nu}{\sim} \sum_n a_n$, defines (correctly) a finite σ-subadditive l-seminorm ν_M on L_M, which extends ν. Furthermore,

$$\nu_M(a) = \inf\Big\{\sum_n \nu(a_n):\ a \overset{\nu}{\sim} \sum_n a_n\Big\} \quad \textit{for} \quad a \in L_M.$$

Proof. Let $a \overset{\nu}{\sim} \sum_n a_n$. The number sequence $\{\nu(\sum_{n\leqslant k} a_n)\}$ satisfies the Cauchy condition, because

$$\Big|\nu\Big(\sum_{n\leqslant k} a_n\Big)-\nu\Big(\sum_{n\leqslant k+m} a_n\Big)\Big| \leqslant \nu\Big(\sum_{k<n\leqslant k+m} a_n\Big) \leqslant \sum_{n>k} \nu(a_n).$$

If also $a \overset{\nu}{\sim} \sum_n a_n'$, then

$$\Big|\nu\Big(\sum_{n\leqslant k} a_n\Big)-\nu\Big(\sum_{n\leqslant k} a_n'\Big)\Big| \leqslant \nu\Big(\sum_{n\leqslant k} a_n - \sum_{n\leqslant k} a_n'\Big) \leqslant \sum_{n>k} \nu(a_n)+\sum_{n>k}\nu(a_n') \searrow_k 0,$$

because

$$\Big|\sum_{n\leqslant k} a_n - \sum_{n\leqslant k} a_n'\Big| \leqslant \Big|a-\sum_{n\leqslant k} a_n\Big| + \Big|a-\sum_{n\leqslant k} a_n'\Big| \leqslant' \sum_{n>k}|a_n| + \sum_{n>k}|a_n'|$$

and ν is σ-subadditive. Thus, a function $\nu_M\colon L_M \to [0, \infty)$ is well defined. Evidently, $\nu_M(a) = \nu(a)$ for $a \in L$. The (finite) subadditivity of ν_M follows easily from the inequality

$$\nu(a_1+b_1+\ldots+a_k+b_k) \leqslant \nu\Big(\sum_{n\leqslant k} a_n\Big)+\nu\Big(\sum_{n\leqslant k} b_n\Big).$$

To prove that $\nu_M(|a|) = \nu_M(a)$, put $a_1' = |a_1|$, $a_2' = |a_1+a_2|-|a_1|$, $a_3' = |a_1+a_2+a_3|-|a_1+a_2|$, $\ldots$, and observe that $|a_n'| \leqslant |a_n|$ and

$$\Big||a|-\sum_{n\leqslant k} a_n'\Big| = \Big||a|-\Big|\sum_{n\leqslant k} a_n\Big|\Big| \leqslant \Big|a-\sum_{n\leqslant k} a_n\Big| \leqslant' \sum_{n>k}|a_n|.$$

It follows that

$$|a| \overset{\nu}{\sim} a_1'+a_1-a_1+a_2'+a_2-a_2+\ldots$$

Hence

$$\nu_M(|a|) = \lim_k \nu\Big(\sum_{n\leqslant k} a_n'\Big) = \lim_k \nu\Big(\Big|\sum_{n\leqslant k} a_n\Big|\Big) = \nu_M(a).$$

Let $b \in L_M$, $0 \leqslant a \leqslant b$, $b \overset{\nu}{\sim} \sum_n b_n$. We have

$$\begin{aligned}\Big(\Big|\sum_{n\leqslant k} a_n\Big|-\Big|\sum_{n\leqslant k} b_n\Big|\Big)^+ &\leqslant \Big(\Big|\sum_{n\leqslant k} a_n\Big|-a\Big)^+ + \Big(b-\Big|\sum_{n\leqslant k} b_n\Big|\Big)^+ \\ &\leqslant \Big|\Big|\sum_{n\leqslant k} a_n\Big|-a\Big| + \Big|b-\Big|\sum_{n\leqslant k} b_n\Big|\Big| \leqslant \Big|a-\sum_{n\leqslant k} a_n\Big| + \Big|b-\sum_{n\leqslant k} b_n\Big| \\ &\leqslant' \sum_{n>k}|a_n| + \sum_{n>k}|b_n|.\end{aligned}$$

Hence

$$\nu\Big(\sum_{n\leqslant k} a_n\Big)-\nu\Big(\sum_{n\leqslant k} b_n\Big) \leqslant \nu\Big(\Big(\Big|\sum_{n\leqslant k} a_n\Big|-\Big|\sum_{n\leqslant k} b_n\Big|\Big)^+\Big) \leqslant \sum_{n>k}\nu(a_n)+\sum_{n>k}\nu(b_n) \xrightarrow[k]{} 0,$$

which yields $\nu_M(a) \leqslant \nu_M(b)$. Thus ν_M is an l-seminorm on L_M. Since $\nu(a_1+ +a_2+\ldots+a_k) \leqslant \sum_n \nu(a_n)$, $\nu_M(a) \leqslant \sum_n \nu(a_n)$. Given $\varepsilon > 0$, we may choose k so large that

$$\Big|\nu_M(a)-\nu\Big(\sum_{n\leqslant k} a_n\Big)\Big| < \varepsilon/2 \quad \text{and} \quad \sum_{n>k}\nu(a_n) < \varepsilon/2;$$

then

$$a \overset{\nu}{\sim} (a_1+\ldots+a_k)+a_{k+1}+a_{k+2}+\ldots$$

and

$$\nu(a_1+\ldots+a_k)+\sum_{n>k}\nu(a_n) < \nu_M(a)+\varepsilon.$$

This yields the asserted equality expressing $\nu_M(a)$.

It remains to prove that ν_M is σ-subadditive. Let $a, a_n \in L_M^+$, $a_m^{(n)} \in L$, $a = \sum_n a_n$ and $a_n \overset{\nu}{\sim} \sum_m a_m^{(n)}$ for $n \in N$. We may assume that $\sum_n \nu_M(a_n) < \infty$ and that $\sum_m \nu(a_m^{(n)}) < \nu_M(a_n)+\varepsilon 2^{-n}$ (by the already proven assertion expressing ν_M). Now we have

$$\sum_{n,m}\nu(a_m^{(n)}) < \sum_n \nu_M(a_n)+\varepsilon,$$

and so, in view of Lemma 2, $a \overset{\nu}{\sim} \sum_{n,m} a_m^{(n)}$. Hence

$$\nu_M(a) \leqslant \sum_{n,m} \nu(a_m^{(n)}).$$

Since ε was arbitrary, the desired inequality follows, and the proof is complete.

Due to Theorem 5, we may consider the expansion $a \overset{\nu_M}{\sim} \sum_n a_n$, meaning, of course, that $a \sim \sum_n a_n$, $\{a_n\} \subset L_M$ and $\sum_n \nu_M(a_n) < \infty$.

THEOREM 6. *Let ν be finite and σ-subadditive.*

(i) *If $a \overset{\nu_M}{\sim} \sum_n a_n$, then $a \in L_M$ and $\lim_k \nu_M(a - \sum_{n\leqslant k} a_n) = 0$. Therefore L is metrically dense in L_M (endowed with the semimetric $\varrho(a, b) = \nu_M(a-b)$).*

(ii) *If G is weakly σ-complete, then the space L_M is metrically complete.*

Proof. (i) The argument used to prove that $a \in L_M$ is nearly the same as in the final part of the proof of Theorem 5; the difference is that now $G \ni a \overset{\nu_M}{\sim} \sum_n a_n$ instead of $L_M^+ \ni a = \sum_n a_n$, but Lemma 2 may be applied as before. Having observed that $a \in L_M$, we may use the σ-subadditivity of ν_M:

$$\nu_M\Big(a - \sum_{n\leqslant k} a_n\Big) \leqslant \sum_{n>k} \nu_M(a_n) \underset{k}{\rightarrow} 0.$$

(ii) It is sufficient to prove that each absolutely convergent series converges. Let $\{a_n\} \subset L_M$ and $\sum_n \nu_M(a_n) < \infty$. Since G is weakly σ-complete, there exists an element $a \in G$ satisfying $a \sim \sum_n a_n$ (Proposition 14), and so $a \overset{\nu_M}{\sim} \sum_n a_n$. By part (i), $a \in L_M$ and $\sum_n a_n$ converges metrically to a.

In case L_M is weakly σ-complete (with respect to G), assertion (ii) follows directly from Theorem 5 of [7]; but we do not make this assumption on L_M here, and the metrical completeness of L_M is a consequence of the construction of L_M and the weak σ-completeness of G.

References

[1] G. Birkhoff, *Lattice Theory*, Providence 1967.
[2] D. H. Fremlin, *A direct proof of the Matthes–Wright integral extention theorem*, J. London Math. Soc. 11 (1975), 276–284.
[3] L. Fuchs, *Partially Ordered Algebraic Systems*, Budapest 1963.
[4] H. M. MacNeille, *A unified theory of integration*, Proc. Nat. Acad. Sci. U.S.A. 27 (1941), 71–76.

[5] J. Mikusiński, *Sur une définition de l'intégrale de Lebesgue*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 12 (1964), 203–204.
[6] J. Mikusiński, *The Bochner Integral*, Basel und Stuttgart 1978.
[7] M. Wilhelm, *Completeness of l-groups and of l-seminorms*, Comment. Math. 21 (1979), 271–281.

INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY, WROCLAW

Received October 21, 1982 (1826)