(iii) *If* $\xi \in C''$, *then* $\xi \circ s \in C''$ *for each* s.

(iv) *If* $\xi \in C''$, *then* ξ *is either convergent or totally divergent.*

The Example 3 of the C^+-group $(X, \mathfrak{L}, \lambda, +)$ of equivalence classes of B-measurable functions with convergence almost everywhere shows that in general for C^+-groups we have $C' = C'' \subsetneqq C$. Indeed, if $f_n \to f$ in measure but not almost everywhere, then $\langle [f_n] \rangle \in C - C''$ in view of (iv) of Corollary 3. In fact, it is easy to see that $(X, \mathfrak{L}, \lambda, +)$ is "C''-complete" in the sense that each $\langle [f_n] \rangle \in C''$ converges in $(X, \mathfrak{L}, \lambda, +)$.

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Received November 12, 1982 (1830)

STUDIA MATHEMATICA, T. LXXVII. (1984)

On the structure of L_φ-solution sets of integral equations in Banach spaces

by

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Abstract. In this paper we consider the integral equation

$$(1)\qquad x(t) = p(t) + \int_0^t f(t, s, x(s))\,ds$$

in a Banach space X. We prove that under suitable assumptions the set of all solutions of (1), belonging to a certain Orlicz space $L_\varphi(J, X)$, is a compact R_δ.

Let X be a separable Banach space. For any compact interval J and for any N-function φ (cf. [4], [6]) we shall denote by $L_\varphi(J, X)$ the Orlicz space of all strongly measurable functions $u\colon J \to X$ for which the number

$$\|u\|_\varphi = \inf\Big\{r > 0\colon \int_J \varphi\left(\|u(s)\|/r\right) ds \leqslant 1\Big\}$$

is finite. It is well known that $\langle L_\varphi(J, X), \|\cdot\|_\varphi\rangle$ is a Banach space. Moreover, we shall denote by $E_\varphi(J, X)$ the closure in $L_\varphi(J, X)$ of the set of all bounded functions. For properties of the spaces $L_\varphi(J, X)$ and $E_\varphi(J, X)$ see [4], pp. 76–106.

In [7] we gave some conditions which guarantee that the integral equation

$$(1)\qquad x(t) = p(t) + \int_0^t f\big(t, s, x(s)\big)\,ds$$

has at least one solution x belonging to a certain space $L_\varphi(J, X)$. In this paper we shall show that under the same assumptions as in [7] the set S of all solutions $x \in L_\varphi(J, X)$ of (1) is a compact R_δ in the sense of Aronszajn, i.e., S is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts.

Let $L^1(J, X)$ denote the Lebesgue space of Bochner integrable functions $u\colon J \to X$ provided with the norm $\|u\|_1 = \int_J \|u(s)\|\,ds$, and let β and β_1 be the ball measures of noncompactness in X and $L^1(J, X)$, respectively.

Without loss of generality we shall always assume that all functions from $L^1(J, X)$ are extended to R by putting $u(t) = 0$ outside J. For any set V of functions belonging to $L^1(J, X)$ denote by v the function defined by $v(t) = \beta(V(t))$ for $t \in J$ (under the convention that $\beta(A) = \infty$ if A is unbounded), where $V(t) = \{u(t): u \in V\}$. The following result has been proved in [7]:

LEMMA 1. *Suppose that V is a countable subset of $L^1(J, X)$ and there exists $\mu \in L^1(J, R)$ such that $\|u(t)\| \leqslant \mu(t)$ for all $u \in V$ and $t \in J$. Then the function v is integrable on J and for any measurable subset T of J*

(2) $$\beta\Big(\Big\{\int_T u(t)\,dt:\ u \in V\Big\}\Big) \leqslant \int_T v(t)\,dt.$$

If in addition $\limsup_{h\to 0}\sup_{u\in V} \int_J \|u(t+h) - u(t)\|\,dt = 0$, then

(3) $$\beta_1(V) \leqslant \int_J v(t)\,dt.$$

Assume that $D = [0, d]$ and M, N are complementary N-functions. We introduce the following conditions concerning f.

C1. $(t, s, x) \to f(t, s, x)$ is a function from $D^2 \times X$ into X which is continuous in x for almost every $t, s \in D$, and strongly measurable in (t, s) for every $x \in X$.

C2. $\|f(t, s, x)\| \leqslant K(t, s)\big(b(s) + H(\|x\|)\big)$ for $t, s \in D$ and $x \in X$, where $b \in L_N(D, R)$, H is a nonnegative nondecreasing continuous function defined on $[0, \infty)$ and $K(t, s) \geqslant 0$ for $t, s \in D$.

C3. 1° N satisfies the condition Δ', i.e., there exist $\lambda, u_0 \geqslant 0$ such that $N(uv) \leqslant \lambda N(u) N(v)$ for $u, v \geqslant u_0$.

2° $K \in E_M(D^2, R)$.

3° φ is an N-function and there exist $\alpha, \gamma, u_0 \geqslant 0$ such that

$$N\big(\alpha H(u)\big) \leqslant \gamma\varphi(u) \leqslant \gamma M(u) \quad \text{for} \quad u \geqslant u_0.$$

C4. 1° N satisfies the condition Δ_3, i.e., there exist $\lambda, u_0 \geqslant 0$ such that

$$N(\lambda u) \geqslant u N(u) \quad \text{for} \quad u \geqslant u_0.$$

2° $K \in L_M(D^2, R)$.

3° There exist $\lambda, u_0 > 0$ such that

$$H(u) \leqslant \lambda M(u)/u \quad \text{for} \quad u \geqslant u_0.$$

4° φ is an N-function satisfying the condition Δ' and such that

$$\iint_{D^2} \varphi\big(M(K(t, s))\big)\,ds\,dt < \infty$$

(the existence of φ follows from 1° and 2°).

C5. 1° φ is an N-function and the function N satisfies the condition Δ_2, i.e., there exist $\lambda, u_0 \geqslant 0$ such that $N(2u) \leqslant \lambda N(u)$ for $u \geqslant u_0$.

2° There exists $\gamma > 0$ such that $H(u) \leqslant \gamma N^{-1}(\varphi(u))$ for $u \geqslant 0$.

3° $K(t, \cdot) \in E_M(D, R)$ for almost every $t \in D$ and the function $t \to \|K(t, \cdot)\|_M$ belongs to $E_\varphi(D, R)$.

It is well known (cf. [4], Th. 19.1, Th. 19.2, L. 16.3, Th. 17.6) that conditions C1–C3 imply that the operator F defined by

$$F(x)(t) = \int_0^t f\big(t, s, x(s)\big)\,ds$$

maps the unit ball in $L_\varphi(D, X)$ into $E_\varphi(D, X)$, and conditions C1, C2, plus C4 or C5, imply that F is a mapping of $E_\varphi(D, X)$ into itself.

Now we shall present some inequalities concerning the operator F. We distinguish three cases:

(1) If C3 holds, then by Lemma 19.1 of [4] there exists a constant C such that for any measurable subset T of D and $x \in L_\varphi(D, X)$, $\|x\|_\varphi \leqslant 1$, we have

(4) $$\|F(x)\chi_T\|_\varphi \leqslant C\|K\chi_{T\times D}\|_M.$$

Moreover, there exist $\alpha, \gamma, u_0 > 0$ such that

(5) $$\big\|H(\|x\|)\big\|_N \leqslant \frac{1}{\alpha}\Big(1 + \int_0^d N\big(\alpha H(\|x(s)\|)\big)\,ds\Big)$$
$$\leqslant \frac{1}{\alpha}\Big(1 + N\big(\alpha H(u_0)\big) + \gamma\int_0^d \varphi(\|x(s)\|)\,ds\Big)$$

for any $x \in L_\varphi(D, X)$ with $\|x\|_\varphi \leqslant 1$.

(2) If C4 holds, then by Th. 19.2 of [4] there exists a constant C such that for any measurable subset T of D and $x \in L_\varphi(D, X)$ we have

(6) $$\|F(x)\chi_T\|_\varphi \leqslant C\|K\chi_{T\times D}\|_{\varphi\circ M}\big(\|b\|_N + \big\|H(\|x\|)\big\|_N\big).$$

Moreover, as C4, 3° implies that there exist $\alpha, \eta, u_0 > 0$ such that $N\big(\alpha H(u)\big) \leqslant \eta u$ for $u \geqslant u_0$, we see that for any $x \in L_\varphi(D, X)$ and $t \in D$

(7) $$\big\|H(\|x\chi_{[0,t]}\|)\big\|_N \leqslant \frac{1}{\alpha}\Big(1 + \int_0^t N\big(\alpha H(\|x(s)\|)\big)\,ds\Big)$$
$$\leqslant \frac{1}{\alpha}\Big(1 + N\big(\alpha H(u_0)\big) + \eta\int_0^t \|x(s)\|\,ds\Big).$$

(3) Under condition C5 for any $x \in E_\varphi(D, X)$ we have

(8) $$\big\|H(\|x\chi_{[0,t]}\|)\big\|_N \leqslant \gamma\big\|N^{-1}\big(\varphi(\|x\chi_{[0,t]}\|)\big)\big\|_N \leqslant \gamma + \gamma\int_0^t \varphi(\|x(s)\|)\,ds.$$

Moreover, using C2 and applying the Hölder inequality, we obtain

(9) $\|F(x)(t)\| \leqslant k(t)\big(\|b\|_N + \big\|H(\|x\chi_{[0,t]}\|)\big\|_N\big)$ for almost every $t \in D$,

and consequently

(10) $$\|F(x)\chi_T\|_\varphi \leqslant \|k\chi_T\|_\varphi\big(\|b\|_N + \big\|H(\|x\|)\big\|_N\big)$$

for any measurable subset T of D and $x \in E_\varphi(D,X)$, where $k(t) = 2\|K(t,\cdot)\chi_{[0,t]}\|_M$. Furthermore, by the Hölder inequality, from C2 it follows that for any $t \in D$ such that $K(t,\cdot) \in E_M(D,R)$

(11) $$\int_P \|f(t,s,x(s))\|\,ds \leqslant 2\|K(t,\cdot)\chi_P\|_M\big(\|b\|_N + \big\|H(\|x\|)\big\|_N\big)$$

for any measurable subset P of $[0,t]$ and $x \in E_\varphi(D,X)$.

Denote by $B^r_\varphi(D,X)$ the closed ball in $E_\varphi(D,X)$ with center 0 and radius r. Our fundamental results are given by the following theorems:

THEOREM 1. *Assume that conditions* C1, C2 *and* C3 *hold. Assume in addition that*

(12) $$\lim_{h\to 0}\sup_{x\in B^1_\varphi(D,X)}\int_0^d \|F(x)(t+h) - F(x)(t)\|\,dt = 0$$

and

(13) $$\beta\big(f(t,s,Z)\big) \leqslant h\big(t,s,\beta(Z)\big)$$

for almost every $t, s \in D$ *and for every bounded subset* Z *of* X, *where* $(t,s,u) \to h(t,s,u)$ *is a nonnegative function defined for* $0 \leqslant s \leqslant t \leqslant d$, $u \geqslant 0$, *satisfying the following conditions*:

(i) *for any nonnegative* $u \in L^1(D,R)$ *there exists the integral* $\int_0^t h(t,s,u(s))\,ds$ *for almost every* $t \in D$;

(ii) *for any* a, $0 < a \leqslant d$, $u = 0$ *a.e. is the only nonnegative integrable function on* $[0,a]$ *which satisfies* $u(t) \leqslant \int_0^t h(t,s,u(s))\,ds$ *almost everywhere on* $[0,a]$.

Then for any $p \in E_\varphi(D,X)$ *there exists an interval* $J = [0,a]$ *such that the set* S *of all solutions of* (1) *belonging to* $B^1_\varphi(J,X)$ *is a compact* R_δ.

THEOREM 2. *Assume that conditions* C1, C2, C4 and (13) *hold. If for any* $r > 0$

(14) $$\lim_{h\to 0}\sup_{x\in B^r_\varphi(D,X)}\int_0^d \|F(x)(t+h) - F(x)(t)\|\,dt = 0,$$

then for any $p \in E_\varphi(D,X)$ *the set* S *of all solutions of* (1) *belonging to* $E_\varphi(D,X)$ *is a compact* R_δ.

THEOREM 3. *Assume that conditions* C1, C2, C5 and (14) *hold. If the function* f *satisfies* (13) *with a function* h *satisfying the conditions*

(i)′ *for any nonnegative* $u \in E_\varphi(D,R)$ *there exists the integral*

$$\int_0^t h(t,s,u(s))\,ds \quad \textit{for almost every } t \in D;$$

(ii)′ *for any* a, $0 < a \leqslant d$, $u = 0$ *a.e. is the only nonnegative function on* $[0,a]$ *which belongs to* $E_\varphi([0,a],R)$ *and satisfies*

$$u(t) \leqslant \int_0^t h(t,s,u(s))\,ds \quad \textit{almost everywhere on } [0,a],$$

then for any $p \in E_\varphi(D,X)$ *there exists an interval* $J = [0,a]$ *such that the set* S *of all solutions of* (1) *belonging to* $E_\varphi(J,X)$ *is a compact* R_δ.

Proof. Fix a function $p \in E_\varphi(D,X)$. We choose a number a, $0 < a \leqslant d$, in such a way that

$C\|K\chi_{[0,a]\times D}\|_M + \|p\chi_{[0,a]}\|_\varphi < 1$ under condition C3;

$a = d$ under condition C4;

$a < \min(d, \omega_+)$, where $[0,\omega_+)$ is the maximal interval of existence of the maximal continuous solution z of the integral equation

(15) $$z(t) = \frac{1}{2}\int_0^t \varphi\Big(2\|p(s)\| + 2k(s)\big(\|b\|_N + \gamma + \gamma z(s)\big)\Big)\,ds,$$

under condition C5.

Let $J = [0,a]$. For simplicity we introduce the following denotations:

$$L^1 = L^1(J,X), \quad L_\varphi = L_\varphi(J,X), \quad E_\varphi = E_\varphi(J,X), \quad B^r_\varphi = B^r_\varphi(J,X)$$

and

$$Q_\varphi = \begin{cases} B^1_\varphi & \text{if C3 holds,} \\ E_\varphi & \text{if C4 or C5 holds.}\end{cases}$$

First we shall show that F is a continuous mapping of Q_φ into E_φ. Let $x_n, x_0 \in Q_\varphi$ and $\lim_{n\to\infty}\|x_n - x_0\|_\varphi = 0$. Suppose that $\|F(x_n) - F(x_0)\|_\varphi$ does not converge to 0 as $n \to \infty$. Then there are $\varepsilon > 0$ and a subsequence (x_{n_j}) such that

(16) $$\|F(x_{n_j}) - F(x_0)\|_\varphi > \varepsilon \quad \text{for } j = 1, 2, \ldots$$

and $\lim_{j\to\infty} x_{n_j}(t) = x_0(t)$ for almost every $t \in J$. As $x_n \to x_0$ in Q_φ, the sequences $\big(\int_0^a \varphi(\|x_n(s)\|)\,ds\big)$ and $\big(\int_0^a \|x_n(s)\|\,ds\big)$ are bounded. By (11), (5), (7) and (8), from this we deduce that for almost every $t \in J$ the sequence $\big(\|f(t,s,x_n(s))\|\big)$ is equi-integrable on $[0,t]$. Since for almost every $t \in J$

$$\lim_{j\to\infty} f(t,s,x_{n_j}(s)) = f(t,s,x_0(s)) \quad \text{for almost every } s \in [0,t],$$

the Vitali convergence theorem proves that

$$\lim_{j\to\infty} F(x_{n_j})(t) = F(x_0)(t) \quad \text{for almost every } t \in J.$$

On the other hand, from inequalities (4), (6) and (7) or (10) and (8) (according as C3, C4 or C5 holds) it follows that the sequence $(F(x_{n_j}))$ has equi-absolutely continuous norms in L_φ. This implies that $\lim_{j\to\infty} \|F(x_{n_j}) - F(x_0)\|_\varphi = 0$, in contradiction with (16).

Choose a positive number r in such a way that

(1) $r = 1$ if C3 holds;

(2) under condition C4,

$$r = 1 + \|p\|_\varphi + C\|K\|_{\varphi O M}\Bigg(\|b\|_N + \frac{1}{\alpha}\bigg(1 + N\big(\alpha H(u_0)\big) + \eta\Big(m_\varphi + \int_0^a q(s)\,ds\Big)\exp\Big(\frac{\eta}{\alpha}\int_0^a k(s)\,ds\Big)\bigg)\Bigg),$$

where

$$m_\varphi = \sup\{\|u\|_1 \colon u \in B^1_\varphi\}, \quad k(t) = 2\|K(t,\cdot)\chi_{[0,t]}\|_M$$

and

$$q(t) = \|p(t)\| + \Big(\|b\|_N + \frac{1}{\alpha} + \frac{1}{\alpha}N\big(\alpha H(u_0)\big)\Big)k(t);$$

(3) if C5 holds, then $r = 2 + \sup_{t\in J} z(t)$, where z is the maximal solution of (15).

Let

$$U = \begin{cases} B^r_\varphi & \text{if C3 or C4 holds,} \\ \Big\{x \in E_\varphi \colon \int_0^a \varphi\big(\|x(s)\|\big)\,ds \leqslant r-1\Big\} & \text{if C5 holds.}\end{cases}$$

It is clear that $U \subset B^r_\varphi$.

Let us remark that

(17) $$\sup\big\{\|H(\|x\|)\|_N \colon x \in U\big\} < \infty.$$

It follows immediately from (5), (7) or (8), according as C3, C4 or C5 holds. Hence, by (4), (6) or (10), the set $\{F(x) \colon x \in U\}$ has equi-absolutely continuous norms in L_φ.

Note also that

(18) $$\limsup_{h\to 0}\sup_{x\in U} \|F(x)(\cdot + h) - F(x)\|_\varphi = 0.$$

Indeed, if we suppose the contrary, then there exist $\varepsilon > 0$ and sequences (x_n), (h_n) such that $\lim_{n\to\infty} h_n = 0$, $x_n \in U$ and

(19) $$\|F(x_n)(\cdot + h_n) - F(x_n)\|_\varphi > \varepsilon \quad \text{for} \quad n = 1, 2, \ldots$$

By (12) or (14) we have

$$\lim_{n\to\infty} \|F(x_n)(\cdot + h_n) - F(x_n)\|_1 = 0.$$

As the functions $t \to \|F(x_n)(t + h_n) - F(x_n)(t)\|$, $n = 1, 2, \ldots$, have equi-absolutely continuous norms in L_φ, this implies that

$$\lim_{n\to\infty} \|F(x_n)(\cdot + h_n) - F(x_n)\|_\varphi = 0,$$

which contradicts (19).

For any positive integer n we define a mapping G_n by

$$G_n(x)(t) = p(t) + F(x)\big(r_n(t)\big) \quad \text{for} \quad x \in U \text{ and } t \in J,$$

where

$$r_n(t) = \begin{cases} 0 & \text{for} \quad 0 \leqslant t \leqslant a_n, \\ t - a_n & \text{for} \quad a_n \leqslant t \leqslant a \end{cases}$$

and $a_n = a/n$.

From the continuity of F it follows that G_n is a continuous mapping of U into E_φ.

Fix n. It can easily be verified that for any $x, y \in U$ the following implication holds

(20) $$x - G_n(x) = y - G_n(y) \Rightarrow x = y.$$

Suppose that $x_j, x_0 \in U$ and

(21) $$\lim_{j\to\infty} \|x_j - G_n(x_j) - x_0 + G_n(x_0)\|_\varphi = 0.$$

Since $G_n(x_j)(t) = G_n(x_0)(t) = p(t)$ for $0 \leqslant t \leqslant a_n$, (21) implies that $\lim_{j\to\infty} \|(x_j - x_0)\chi_{[0,a_n]}\|_\varphi = 0$. Further,

$$\begin{aligned} x_j(t) &= x_j(t) - G_n(x_j)(t) + p(t) + F(x_j)(t - a_n) \\ &= x_j(t) - G_n(x_j)(t) + p(t) + F(x_j\chi_{[0,a_n]})(t - a_n) \end{aligned}$$

for $a_n \leqslant t \leqslant 2a_n$ and $j = 0, 1, 2, \ldots$ By (21) and the continuity of F, this proves that $\lim_{j\to\infty} \|(x_j - x_0)\chi_{[a_n, 2a_n]}\|_\varphi = 0$. By repeating this argument we obtain $\lim_{j\to\infty} \|(x_j - x_0)\chi_{[0, ia_n]}\|_\varphi = 0$ for $i = 1, \ldots, n$, so that $\lim_{j\to\infty} \|x_j - x_0\|_\varphi = 0$.

From this and (20) we deduce that the mapping $I - G_n \colon U \to E_\varphi$ is a homeomorphism into (I — the identity mapping).

Now we shall show that there exist a positive number ϱ and a positive integer n_0 such that

$$B_\varphi^\varrho \subset (I-G_n)(U) \quad \text{for all } n \geqslant n_0. \tag{22}$$

(1) Assume C3 and put $\varrho = 1 - C\|K\chi_{[0,a]\times D}\|_M - \|p\chi_{[0,a]}\|\varphi$. For given n and $y \in B_\varphi^\varrho$ we define a sequence of functions x_i, $i = 1, \ldots, n$, by the formulas

$$x_1(t) = y(t) + p(t) \quad \text{for} \quad 0 \leqslant t \leqslant a_n,$$

$$\tilde{x}_i(t) = \begin{cases} x_i(t) & \text{for} \quad 0 \leqslant t \leqslant ia_n, \\ 0 & \text{for} \quad ia_n < t \leqslant a, \end{cases} \tag{23}$$

$$x_{i+1}(t) = y(t) + p(t) + F(\tilde{x}_i)\big(r_n(t)\big) \quad \text{for} \quad ia_n \leqslant t \leqslant (i+1)a_n.$$

We observe that

$$x_i(t) = y(t) + p(t) + F(\tilde{x}_i)\big(r_n(t)\big) \quad \text{for} \quad 0 \leqslant t \leqslant ia_n,$$

$$x_{i+1}|[0, ia_n] = x_i \quad \text{and} \quad x_i \in E_\varphi([0, ia_n], X).$$

Obviously, $\|\tilde{x}_1\|_\varphi \leqslant \|y\|_\varphi + \|p\|_\varphi \leqslant 1$. Moreover, if $\|\tilde{x}_i\|_\varphi \leqslant 1$ for some $i \leqslant n-1$, then from (23) and (4) it follows that

$$\|\tilde{x}_{i+1}\|_\varphi \leqslant \|y\|_\varphi + \|p\|_\varphi + C\|K\chi_{[0,a]\times D}\|_M \leqslant 1.$$

This shows that $x_n \in B_\varphi^1$. Since $x_n - G_n(x_n) = y$, we conclude that $B_\varphi^\varrho \subset (I-G_n)(B_\varphi^1)$.

(2) Assume C4. For given n and $y \in B_\varphi^1$ let x_n be the function defined by (23). Then

$$x_n(t) = y(t) + p(t) + F(x_n)\big(r_n(t)\big) \quad \text{for} \quad t \in J. \tag{24}$$

By the Hölder inequality, from C2 and (24) it follows that

$$\|x_n(t)\| \leqslant \|y(t)\| + \|p(t)\| + k_n(t)\big(\|b\|_N + \|H(\|x_n\chi_{[0,t]}\|)\|_N\big),$$

where $k_n(t) = k\big(r_n(t)\big)$ and $k(t) = 2\|K(t,\cdot)\chi_{[0,t]}\|_M$. Hence, by (7),

$$\|x_n(t)\| \leqslant \|y(t)\| + q_n(t) + \frac{\eta}{\alpha} k_n(t) \int_0^t \|x_n(s)\|\,ds, \tag{25}$$

where $q_n(t) = \|p(t)\| + \left(\|b\|_N + \frac{1}{\alpha} + \frac{1}{\alpha} N\big(\alpha H(u_0)\big)\right) k_n(t)$. As $p \in E_\varphi(D, X)$ and $K \in E_M(D^2, R)$, the functions p and k are integrable on J. Putting $w(t) = \int_0^t \|x_n(s)\|\,ds$ and integrating (25) between 0 and t we get

$$w(t) \leqslant \int_0^a \|y(s)\|\,ds + \int_0^a q_n(s)\,ds + \frac{\eta}{\alpha} \int_0^t k_n(s) w(s)\,ds \quad \text{for} \quad t \in J,$$

which implies

$$w(t) \leqslant \left(\int_0^a \|y(s)\|\,ds + \int_0^a q_n(s)\,ds\right) \exp\left(\frac{\eta}{\alpha}\int_0^t k_n(s)\,ds\right) \quad \text{for} \quad t \in J.$$

Since $\int_0^a k_n(s)\,ds \leqslant \int_0^a k(s)\,ds$ and $\int_0^a \|y(s)\|\,ds \leqslant m_\varphi \|y\|_\varphi \leqslant m_\varphi$, this shows that

$$\int_0^a \|x_n(s)\|\,ds \leqslant \left(m_\varphi + \int_0^a q(s)\,ds\right) \exp\left(\frac{\eta}{\alpha}\int_0^a k(s)\,ds\right).$$

Consequently, by (6), (7) and (24), we obtain

$$\begin{aligned} \|x_n\|_\varphi &\leqslant \|y\|_\varphi + \|p\|_\varphi + \|F(x_n)\|_\varphi \leqslant 1 + \|p\|_\varphi + \\ &\quad + C\|K\|_{\varphi\circ M}\left(\|b\|_N + \frac{1}{\alpha}\left(1 + N\big(\alpha H(u_0)\big) + \eta\int_0^a \|x_n(s)\|\,ds\right)\right) \\ &\leqslant 1 + \|p\|_\varphi + C\|K\|_{\varphi\circ M}\left(\|b\|_N + \frac{1}{\alpha}\left(1 + N\big(\alpha H(u_0)\big) + \right.\right. \\ &\quad \left.\left. + \eta\left(m_\varphi + \int_0^a q(s)\,ds\right)\exp\left(\frac{\eta}{\alpha}\int_0^a k(s)\,ds\right)\right)\right) = r. \end{aligned}$$

Thus $B_\varphi^1 \subset (I-G_n)(B_\varphi^r)$, i.e., (22) holds with $\varrho = 1$.

(3) Assume C5. We choose a positive number c such that $c \leqslant 1/2$ and the maximal continuous solution z_c of the equation

$$z(t) = c + \tfrac{1}{2}\int_0^t \varphi\Big(2\|p(s)\| + 2k(s)\big(\|b\|_N + \gamma + \gamma z(s)\big)\Big)\,ds$$

is defined on J and $z_c(t) \leqslant 1 + z(t)$ for $t \in J$, where z is the maximal solution of (15). As $p \in E_\varphi$, we may choose a positive integer n_0 such that $l = \sup_{n \geqslant n_0} \|p - p(\cdot - a_n)\|_\varphi < c$. Let $\varrho = c - l$. For given $n \geqslant n_0$ and $y \in B_\varphi^\varrho$ let x_n be the function defined by (23). Then, by (9), (8) and (24), we have

$$\|x_n(t)\| \leqslant \|y(t)\| + \|p(t)\| \quad \text{for} \quad 0 \leqslant t \leqslant a_n$$

and

$$\|x_n(t)\| \leqslant \|y(t)\| + \|p(t)\| + k(t-a_n)\left(\|b\|_N + \gamma + \gamma\int_0^{t-a_n} \varphi\big(\|x_n(s)\|\big)\,ds\right)$$

for almost every $t \in [a_n, a]$, which implies

$$\begin{aligned} \varphi\big(\|x_n(t)\|\big) &\leqslant \tfrac{1}{2}\varphi\big(2\|y(t)\| + 2\|p(t) - p(t-a_n)\|\big) + \\ &\quad + \tfrac{1}{2}\varphi\left(2\|p(t-a_n)\| + 2k(t-a_n)\left(\|b\|_N + \gamma + \gamma\int_0^{t-a_n}\varphi\big(\|x_n(s)\|\big)\,ds\right)\right). \end{aligned}$$

Put $w(t) = \int_0^t \varphi(\|x_n(s)\|)\,ds$. As $k \in E_\varphi(D, R)$ and $p \in E_\varphi$, we may integrate the last inequality between 0 and $t \in [a_n, a]$ which yields

$$w(t) \leqslant \tfrac{1}{2}\int_0^t \varphi\big(2\|y(s)\|+2\|p(s)-p(s-a_n)\|\big)\,ds + \\ +\tfrac{1}{2}\int_{a_n}^t \varphi\Big(2\|p(s-a_n)\|+2k(s-a_n)\big(\|b\|_N+\gamma+\gamma w(s-a_n)\big)\Big)\,ds$$

(under the convention that $p(s) = 0$ if $s < 0$).

Hence

$$w(t) \leqslant \tfrac{1}{2}\int_0^t \varphi\big(2\|y(s)\|+2\|p(s)-p(s-a_n)\|\big)\,ds + \\ +\tfrac{1}{2}\int_0^t \varphi\Big(2\|p(s)\|+2k(s)\big(\|b\|_N+\gamma+\gamma w(s)\big)\Big)\,ds \quad \text{for} \quad t \in J.$$

Since $\|y\|_\varphi+\|p-p(\cdot-a_n)\|_\varphi \leqslant c \leqslant \frac{1}{2}$, we have

$$\tfrac{1}{2}\int_0^t \varphi\big(2\|y(s)\|+2\|p(s)-p(s-a_n)\|\big)\,ds \leqslant \|y\|_\varphi+\|p-p(\cdot-a_n)\|_\varphi \leqslant c,$$

and consequently

$$w(t) \leqslant c+\tfrac{1}{2}\int_0^t \varphi\Big(2\|p(s)\|+2k(s)\big(\|b\|_N+\gamma+\gamma w(s)\big)\Big)\,ds \quad \text{for} \quad t \in J.$$

Applying now Theorem 2 of [1] we get $w(t) \leqslant z_c(t)$ for $t \in J$, so that

$$\int_0^a \varphi(\|x_n(s)\|)\,ds \leqslant 1+\sup_{t\in J} z(t) = r-1.$$

This ends the proof of (22).

Consider now the mapping G defined by

$$G(x) = p+F(x) \quad \text{for} \quad x \in U.$$

As $G_n(x)(t)-G(x)(t) = F(x)\big(r_n(t)\big)-F(x)(t)$, from (18) it follows that

(26) $$\lim_{n\to\infty} \|G_n(x)-G(x)\|_\varphi = 0 \quad \text{uniformly in } x \in U.$$

We shall show that $I - G$ is a proper mapping, that is,

(27) $(I-G)^{-1}(Y)$ is compact for any compact subset Y of E_φ.

Let Y be a given compact subset of E_φ, and let (u_n) be an infinite sequence in $(I-G)^{-1}(Y)$. Since $u_n-p-F(u_n) \in Y$ for $n = 1, 2, \ldots$, we can find a subsequence (u_{n_j}) and $y \in Y$ such that

(28) $$\lim_{j\to\infty} \|u_{n_j}-p-F(u_{n_j})-y\|_\varphi = 0$$

and

(29) $$\lim_{j\to\infty} \big(u_{n_j}(t)-p(t)-F(u_{n_j})(t)\big) = y(t) \quad \text{for almost every } t \in J.$$

Let $V = \{u_{n_j}: j = 1, 2, \ldots\}$ and $W = F(V)$. It is clear from (28) and (29) that

(30) $$\beta_1(V) = \beta_1(W) \quad \text{and} \quad \beta\big(V(t)\big) = \beta\big(W(t)\big) \quad \text{for almost every } t \in J,$$

where β and β_1 are the measures of noncompactness in X and L^1, respectively. Moreover, from (11) and (17) it follows that

(31) $$\|F(x)(t)\| \leqslant Ak(t) \quad \text{for} \quad x \in U \text{ and } t \in J,$$

where $k(t) = 2\|K(t,\cdot)\chi_{[0,t]}\|_M$ and $A = \|b\|_N+\sup\{\|H(\|x\|)\|_N: x \in U\}$. As the function k is integrable and $V \subset U$, by Lemma 1 from (18) and (31) we deduce that the function $t \to v(t) = \beta\big(W(t)\big)$ is integrable on J. Obviously, $v(t) \leqslant Ak(t)$ for $t \in J$. Note that under condition C5 the function $k \in E_\varphi(J, R)$ and therefore $v \in E_\varphi(J, R)$.

Fix $t \in J$ for which (13) holds and $K(t,\cdot) \in E_M(J, R)$. Then, by (11) and (17), we have

(32) $$\int_P \big\|f(t, s, x(s))\big\|\,ds \leqslant 2A\|K(t,\cdot)\chi_P\|_M$$

for any measurable subset P of $[0, t]$ and $x \in U$.

Furthermore, by the Iegorov theorem and (29), for every $\varepsilon > 0$ there exists a closed subset J_ε of J such that $\operatorname{mes}(J \setminus J_\varepsilon) < \varepsilon$ and

$$\lim_{j\to\infty} \big(u_{n_j}(s)-p(s)-F(u_{n_j})(s)\big) = y(s) \quad \text{uniformly on } J_\varepsilon.$$

Hence, in view of the Luzin theorem, from (31) and (32) we infer that for a given $\varepsilon > 0$ there exist a closed subset T of $[0, t]$ and a positive number δ such that

(33) $$\|u_{n_j}(s)\| \leqslant \delta \quad \text{for} \quad s \in T \text{ and } j = 1, 2, \ldots$$

and

(34) $$\int_P \big\|f(t, s, u_{n_j}(s))\big\|\,ds \leqslant \varepsilon \quad \text{for} \quad j = 1, 2, \ldots,$$

where $P = [0, t] \setminus T$. As $\|f(t, s, u_{n_j}(s))\| \leqslant K(t, s)\big(b(s) + H(\|u_{n_j}(s)\|)\big)$, from (33) it follows that

$$\|f(t, s, u_{n_j}(s))\| \leqslant \eta(s) \quad \text{for} \quad s \in T \text{ and } j = 1, 2, \ldots,$$

where $\eta(s) = K(t, s)\big(b(s) + H(\delta)\big)$. It is clear that under the assumptions of Theorem 1, 2 or 3 the function η is integrable on T.

Let $Z = \{f(t, \cdot, u_{n_j}(\cdot)) : j = 1, 2, \ldots\}$ and

$$\int_T Z(s)\, ds = \Big\{\int_T f(t, s, u_{n_j}(s))\, ds : j = 1, 2, \ldots\Big\}.$$

By (2) we have

$$\beta\Big(\int_T Z(s)\, ds\Big) \leqslant \int_T \beta(Z(s))\, ds. \tag{35}$$

Moreover, (34) implies that

$$\beta\Big(\int_P Z(s)\, ds\Big) \leqslant \varepsilon. \tag{36}$$

Since $F(V)(t) \subset \int_T Z(s)\, ds + \int_P Z(s)\, ds$, from (35) and (36) we obtain

$$\beta\big(F(V)(t)\big) \leqslant \int_T \beta\big(Z(s)\big)\, ds + \varepsilon.$$

On the other hand, by (13),

$$\beta\big(Z(s)\big) \leqslant h\big(t, s, \beta(V(s))\big) \quad \text{for almost every } s \in [0, t].$$

Hence, by (30),

$$v(t) \leqslant \int_T h\big(t, s, v(s)\big)\, ds + \varepsilon \leqslant \int_0^t h\big(t, s, v(s)\big)\, ds + \varepsilon.$$

As ε is arbitrary, this shows that

$$v(t) \leqslant \int_0^t h\big(t, s, v(s)\big)\, ds.$$

Since the last inequality holds for almost every $t \in J$ and $v \in L^1(J, R)$ or $v \in E_\varphi(J, R)$ (according as C3, C4 or C5 holds), we deduce that $v(t) = 0$ for almost every $t \in J$. Applying now Lemma 1 and using (30), we get

$$\beta_1(V) = \beta_1(W) \leqslant \int_0^a v(t)\, dt = 0.$$

Thus the set V is relatively compact in L^1, so that we can find a subsequence $(u_{n_{j_i}})$ of (u_{n_j}) which is convergent in L^1. On the other hand, the sequence $(F(u_{n_j}))$ has equi-absolutely continuous norms in L_φ, and consequently, by (28), the sequence (u_{n_j}) has equi-absolutely continuous norms in L_φ. Hence the sequence $(u_{n_{j_i}})$ is convergent in L_φ. As U is a complete metric subspace of L_φ, this proves (27).

Owing to (22), (26) and (27) it is clear that the mapping $G: U \to E_\varphi$ satisfies all assumptions of Theorem 7 of [2], which proves that the set $(I - G)^{-1}(0)$ is a compact R_δ. Moreover, similarly as for x_n in the proof of (22), it can be shown that if $x \in S$, then $x \in U$. Hence $S = (I - G)^{-1}(0)$ and consequently S is a compact R_δ.

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Received November 12, 1982 (1831)