STUDIA MATHEMATICA, T. LXXVIII. (1984)

On the fixed point index of non-compact mappings

by

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Abstract. We construct the topological index for a special class of mappings, called *admissible* with respect to a filtration. The construction is an extension of the method given by Browder and Petryshyn [1] for approximative proper mappings (A-proper mappings), but the family of linear projections is not considered. Moreover, the class of admissible mappings contains compact mappings and is not contained in the class of A-proper mappings.

We also prove some other results including an analogue of the classical Schauder fixed point theorem (see Theorems 9, 10).

Introduction. In 1968 F. E. Browder and W. V. Petryshyn published the theory of degree for mappings called *A-proper mappings*. Namely, let X, Y be Banach spaces and (X_n), (Y_n) be sequences of linear subspaces of X, Y respectively, such that $\dim X_n = n$, $X_n \subset X_{n+1}$, $\bigcup X_n$ is dense in X, and analogously for (Y_n). Moreover, linear projections $Q_n\colon Y \to Y_n$ are considered. A mapping $f\colon X \supset G \to Y$ is called *A-proper* if $Q_{n_j} f(x_{n_j}) \to a$, $x_{n_j} \in X_{n_j}$ implies that there exists a subsequence of (x_{n_j}) converging to an element of X.

In the present paper linear projections are not considered. The index is defined for "admissible" mappings, i.e. satisfying the condition

$$\lim_{n\to\infty} \sup_{x\in G\cap X_n} d\big(f(x), X_n\big) = 0,$$

where d denotes the distance of the point $f(x)$ from the subspace X_n, and $f\colon X \supset G \to X$. The class of admissible mappings contains compact mappings and is not contained in the class of A-proper mappings.

Also the fixed point theorem for admissible mappings is proved.

1. Definition. Let X be a Banach space and (X_n) be a sequence of its oriented subspaces such that

(a) $\dim X_n = n$ for all natural n,

(b) $X_n \subset X_{n+1}$,

(c) $\overline{\bigcup_{n\in N} X_n} = X$.

The sequence (X_n) is called a *filtration* of X.

Proposition. *A Banach space with filtration (X_n) is a separable space.*

Example. The space l^p of sequences (x_k) such that $\sum |x_k|^p < \infty$ with the norm $\|(x_k)\| = (\sum |x_k|^p)^{1/p}$ has the filtration (X_n), where X_n is the subspace generated by $e_1, e_2, \ldots, e_n$, $e_j = (\delta_{jk})$, δ_{jk} is the Kronecker's symbol.

2. Definition. Let X be a Banach space with a filtration (X_n), $G \subset X$. A continuous mapping $f\colon G \to X$ is called *admissible* with respect to filtration (X_n), or shortly *A-mapping*, if

(1) $$\lim_{n\to\infty} \sup_{x\in G_n} d(f(x), X_n) = 0,$$

where $G_n = G \cap X_n$ and the distance $d(a, B) = \inf_{b\in B} \|a-b\|$.

3. Lemma. *Let X, (X_n) be as above, $G \subset X$, and let $f\colon \bar G \to X$ be continuous. Then f is an A-mapping if and only if*

(2) $$\bigwedge_{\varepsilon>0} \bigvee_{n_0} \bigwedge_{n\geq n_0} \bigvee_{f_n} f_n\colon G_n \to X_n, \quad \sup_{x\in G_n} \|f(x)-f_n(x)\| < \varepsilon,$$

where f_n are continuous functions.

Proof. The implication (2)$\Rightarrow$(1) is easy to prove. Let f be an A-mapping. Set an $\varepsilon > 0$. It follows from (1) that for $n \geq n_0$ we have

(3) $$\sup_{x\in \bar G_n} d(f(x), X_n) < \varepsilon.$$

Let $n \geq n_0$ be fixed and let $V = \bigcup_{p\in X_n} B_p$, where $B_p = B(p, \varepsilon)$ are balls with radius ε and centre p.

By the paracompactness of X for the cover $(B_p)_{p\in X_n}$ of V there exists a locally finite partition of unity $(\varphi_s)_{s\in S}$ inscribed into this cover.

Pick a $p(s) \in X_n$ for every $s \in S$ such that $\varphi_s^{-1}((0, 1]) \subset B_{p(s)}$ and define

(4) $$f_n(x) = \sum_{s\in S} \varphi_s(f(x))\, p(s) \quad \text{for} \quad x \in \bar G_n.$$

The mapping $f_n\colon \bar G_n \to X_n$ is well-defined and continuous, since the sum is locally finite and from (3) we have $f(x) \in V$ for $x \in \bar G_n$ (hence the superposition is correct).

To complete the proof it is enough to show

$$\|f(x)-f_n(x)\| < \varepsilon \quad \text{for} \quad x \in \bar G_n.$$

Let $x \in \bar G_n$. We have

$$\|f(x)-f_n(x)\| \leq \sum_{s\in S} \varphi_s(f(x))\, \|f(x)-p(s)\|$$

and if $\varphi_s(f(x)) > 0$, then $f(x) \in B_{p(s)}$, hence $\|f(x)-p(s)\| < \varepsilon$ and so

$$\|f(x)-f_n(x)\| < \sum_{s\in S} \varphi_s(f(x))\varepsilon = \varepsilon.$$

4. Definition. (a) Let X be a Banach space with a filtration (X_n), and G an open bounded subset of X. Assume that

(5) $$f\colon \bar G \to X \quad \text{is an } A\text{-mapping}, \quad a(f) = \inf_{x\in\partial G} \|x-f(x)\| > 0,$$

where $\partial G = \bar G \setminus G$ denotes the boundary of G.

The set of all mappings satisfying condition (5) we denote by $\Delta(G, X)$ or shortly Δ.

(b) Let $f \in \Delta(G, X)$. Fix an ε, $0 < \varepsilon < \frac{1}{3}a(f)$ and let (f_n) be a sequence of finite-dimensional continuous mappings $f_n\colon \bar G_n \to X_n$ satisfying

$$\bigvee_{n_0} \bigwedge_{n\geq n_0} \sup_{x\in \bar G_n} \|f(x)-f_n(x)\| < \varepsilon.$$

We define the index of f on G to be the set of finite or infinite numbers m such that $m = \lim_{k\to\infty} \operatorname{ind}_{\mathrm{LS}}(f_{n_k}, G_{n_k})$, where (f_{n_k}) is a subsequence of (f_n) and $\operatorname{ind}_{\mathrm{LS}}$ denotes the Leray–Schauder finite-dimensional fixed point index (see [5], [6]).

Index $\operatorname{ind}_{\mathrm{LS}}(f_{n_k}, G_{n_k})$ is well-defined since $f_n(x) \neq x$ for $x \in \partial G_n$ (if $f_n(x) = x$ for $x \in \partial G_n \subset \partial G$, then $\|f(x)-x\| = \|f(x)-f_n(x)\| < \varepsilon < \frac{1}{3}a(f)$).

We denote the index by $\operatorname{Ind}(f, G)$.

5. Lemma. *The index* $\operatorname{Ind}(f, G)$ *is independent of the choice of sequence* (f_n) *and* ε $(0 < \varepsilon < \frac{1}{3}a(f))$.

Proof. Let $0 < \varepsilon < \frac{1}{3}a(f)$, $0 < \bar\varepsilon < \frac{1}{3}a(f)$ and let (f_n), $(\bar f_n)$ satisfy the conditions

$$\bigwedge_{n\geq N} \sup_{x\in \bar G_n} \|f(x)-f_n(x)\| < \varepsilon, \qquad \bigwedge_{n\geq \bar N} \sup_{x\in \bar G_n} \|f(x)-\bar f_n(x)\| < \bar\varepsilon.$$

Let $n_1 = \max(N, \bar N)$. Construct the segment homotopy between f_n and $\bar f_n$ for $n \geq n_1$,

$$H_n(x, t) = t\bar f_n(x) + (1-t) f_n(x), \quad (x, t) \in \bar G_n \times [0, 1].$$

It is easy to see that the homotopy lacks fixed points on ∂G_n and from the homotopy property of the Leray–Schauder index we have

$$\operatorname{ind}_{\mathrm{LS}}(f_n, G_n) = \operatorname{ind}_{\mathrm{LS}}(\bar f_n, G_n).$$

6. Definition. A mapping $H\colon \bar G \times [0, 1] \to X$ is said to be an *admissible homotopy* between f and g, $f, g \in \Delta(G, X)$, if it satisfies the conditions:

(H1) $$\lim_{n\to\infty} \sup_{(x,t)\in \bar G_n \times [0,1]} d(H(x, t), X_n) = 0,$$

(H2) $$a_H = \inf_{(x,t)\in \partial G \times [0,1]} \|H(x, t)-x\| > 0$$

and $H(x, 0) = f(x)$, $H(x, 1) = g(x)$. We use the notation $H\colon f \simeq g$.

7. Lemma. *If $f\colon \bar G \to X$ is compact (G — an open bounded subset of X),*

i.e. the set $f(\bar{G})$ *is relatively compact*, $f(x) \neq x$ *for* $x \in \partial G$, *then* $f \in \Delta(G, X)$ *and*

$$\mathrm{Ind}(f, G) = \{\mathrm{ind}_{\mathrm{L,S}}(f, G)\}.$$

Proof. The mapping f belong to $\Delta(G, X)$ since $a(f) > 0$ (see [6]) and by Lemma 3 is admissible. The assertion about the index follows from the equality

$$\mathrm{ind}_{\mathrm{L,S}}(f_n, G_n) = \mathrm{ind}_{\mathrm{L,S}}(f, G) \quad \text{for } f_n \text{ as above.}$$

8. Properties. Let $f \in \Delta(G, X)$ (G open bounded subset of X).

(a) If $f(x) = x_0 \in G$ for $x \in \bar{G}$, then $\mathrm{Ind}(f, G) = \{1\}$.

(b) If the set $(I-f)(\bar{G})$ is closed ($I(x) = x$), $\mathrm{Ind}(f, G) \neq \{0\}$, then there exists an $x \in G$ such that $f(x) = x$.

(c) If $\bar{G} = \overline{G^1} \cup \overline{G^2}$, $G^1 \cap G^2 = \emptyset$, $f \in \Delta(G^i, X)$, G^i open bounded sets for $i = 1, 2$, then

$$\mathrm{Ind}(f, G) \subseteq \mathrm{Ind}(f, G^1) + \mathrm{Ind}(f, G^2) \quad \text{(the algebraic sum)}$$

with equality holding if either $\mathrm{Ind}(f, G^1)$ or $\mathrm{Ind}(f, G^2)$ is a singleton (finite) integer.

We define here $+\infty - \infty = Z' = Z \cup \{+\infty, -\infty\}$.

(d) If $f, g \in \Delta(G, X)$ and $H\colon f \simeq g$, then $\mathrm{Ind}(f, G) = \mathrm{Ind}(g, G)$.

Proof. (a) follows from Lemma 7 and the compactness of a fixed mapping.

(b) Let $x - f(x) \neq 0$ for every $x \in \bar{G}$. We have $\varepsilon = \inf_{x \in \bar{G}} \|x - f(x)\| > 0$ since $0 \notin (I-f)(\bar{G})$ and $(I-f)(\bar{G})$ is closed. The mapping f_n, satisfying $\|f_n(x) - f(x)\| < \varepsilon/2$, lacks fixed points on $\bar{G}_n$. Thus we have $\mathrm{ind}_{\mathrm{L,S}}(f_n, G_n) = 0$ and so $\mathrm{Ind}(f, G) = \{0\}$.

(c) Let $m \in \mathrm{Ind}(f, G)$, $m = \lim_{k\to\infty} \mathrm{ind}_{\mathrm{L,S}}(f_{n_k}, G_{n_k}) = \lim_{k\to\infty} [\mathrm{ind}(f_{n_k}, G^1_{n_k}) + {} + \mathrm{ind}(f_{n_k}, G^2_{n_k})]$.

We choose appropriate subsequences convergent to a finite or infinite limit. Then we have

$$m \in \mathrm{Ind}(f, G^1) + \mathrm{Ind}(f, G^2).$$

The second part is easy to prove.

(d) From (H1) we may construct, as in the proof of Lemma 3, mappings $H_n\colon \bar{G}_n \times [0, 1] \to X_n$ such that for fixed $\varepsilon > 0$ ($0 < \varepsilon < \frac{1}{3} a_H$) and $n \geqslant n_0$ we have

$$\|H(x, t) - H_n(x, t)\| < \varepsilon \quad \text{for every } (x, t) \in \bar{G}_n \times [0, 1]. \tag{6}$$

By (H2) and (6) we obtain

$$H_n(x, t) \neq x \quad \text{for every } (x, t) \in \partial G_n \times [0, 1], \tag{7}$$

and so, by the homotopy property, $\mathrm{ind}_{\mathrm{L,S}}(H_n(\cdot, t), G_n)$ is independent of t. Hence the theorem follows.

9. Theorem. *Let* $(X, (X_n))$ *be a Banach space with a filtration, and* $G \neq \emptyset$ *an open bounded subset of* X, $0 \in G$. *Moreover, let* $f\colon \bar{G} \to X$ *be an admissible mapping satisfying the conditions*:

(A1) $\bigwedge_{t \in [0,1]} (I - tf)(\partial G)$ *closed*, $(I-f)(\bar{G})$ *closed*, $f(\partial G)$ *bounded*;

(A2) $\bigwedge_{(x,t) \in \partial G \times (0,1)} tf(x) \neq x$.

Then there exists an $x \in \bar{G}$ *such that* $f(x) = x$.

Remark. Condition (A2) holds if G is convex and $f(\partial G) \subset G$.

Proof of the theorem. If there exists a fixed point on ∂G, then the theorem is true. Suppose f lacks fixed points on ∂G. We have $a(f) > 0$ since $(I-f)(\partial G)$ is closed, and so $f \in \Delta(G, X)$.

The homotopy $H(x, t) = tf(x)$, $(x, t) \in \bar{G} \times [0, 1]$, satisfies (H1). We shall prove (H2). Assume that (H2) is false. Then there exists a sequence of points $(x_n, t_n) \in \partial G \times [0, 1]$ and p such that

$$\bigwedge_{n \geqslant p} \|x_n - t_n f(x_n)\| = \|x_n - H(x_n, t_n)\| < 1/n. \tag{8}$$

Hence for $t_{n_k} \to t_0 \in [0, 1]$ we have

$$\|x_{n_k} - t_0 f(x_{n_k})\| \leqslant |t_{n_k} - t_0| \, \|f(x_{n_k})\| + \|x_{n_k} - t_{n_k} f(x_{n_k})\| \xrightarrow[k\to\infty]{} 0$$

since $f(\partial G)$ is bounded. Therefore $0 \in (I - t_0 f)(\partial G)$, because $(I - t_0 f)(\partial G)$ is closed and $x_{n_k} - t_0 f(x_{n_k}) \xrightarrow[k\to\infty]{} 0$.

And this contradicts (A2) for $t_0 \in (0,1)$ or $0 \in G$ for $t_0 = 0$ or $f(x) \neq x$ on ∂G for $t_0 = 1$. The proof of (H2) is completed.

We have the admissible homotopy $H\colon f \simeq g$, where $g(x) = 0$ on $\bar{G}$. Hence by Properties 8 (a), (b), (d) f has a fixed point on G.

10. Corollary. *If an admissible mapping* $f\colon \bar{G} \to X$ *satisfies* (A2) *and*

(A3) $\bigwedge_{t \geqslant 1} f - tI$ *is proper*, $f(\partial G)$ *is bounded, then* f *has a fixed point*.

(The mapping g is called *proper* iff for every compact set K the set $g^{-1}(K)$ is compact.)

Proof. $I - tf$ are proper for $t \in [0, 1]$ since $f - tI$ are proper; hence they are closed. Therefore, by Theorem 9, f has a fixed point.

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Received April 20, 1982
Revised version June 14, 1982 (1754)

STUDIA MATHEMATICA, T. LXXVIII. (1984)

Connected subgroups of nuclear spaces

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Abstract. It is proved that closed and connected subgroups of nuclear spaces are real linear subspaces.

1. It is well known that a closed subgroup of a Lie group is a Lie subgroup. In the simplest linear case this amounts to the elementary fact that every closed and connected subgroup of $\boldsymbol{R}^n$ is its linear subspace. However, considering infinite-dimensional topological linear spaces we encounter essential differences. For example, the subset of all integer-valued functions in $L^2(0, 1)$ is a closed and connected subgroup of $L^2(0, 1)$ but it fails to contain any line.

Our aim in this note is to show that in the case of nuclear spaces the situation is analogous to the finite-dimensional case, namely:

THEOREM 1. *Closed and connected subgroups of nuclear spaces are real linear subspaces.*

This theorem substanties a conjecture of W. Wojtyński and provides one more example that nuclear spaces are closer to finite-dimensional spaces than normed spaces are.

After proving the theorem we have found that it can be derived also from the results of the first named author concerning unitary representations of groups which are quotiens of nuclear spaces by its closed subgroups.

We are indebted to many persons for stimulating discussions, especially to T. Dobrowolski, S. Kwapień, W. Wojtyński and T. Wolniewicz.

2. All linear spaces we shall deal in the sequel are assumed to be real. We shall obtain Theorem 1 as a consequence of the following

THEOREM 2. *Let G be a subgroup of a real nuclear space X such that for each neighbourhood U of zero in X* $\operatorname{span}(G\cap U)$ *is dense in X. Then G is dense in X.*

In fact, assuming that Theorem 2 holds true let G be a closed connected subgroup of a nuclear space X, let $X_1 = \operatorname{cl}(\operatorname{span} G)$ and let U be a neighbourhood of 0 in X.

The set $G\cap U$ generates G, hence $\operatorname{span}(G\cap U)$ is dense in X_1. Then, by Theorem 2, G is dense in X_1, whence $G = \operatorname{cl} G = X_1$.