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Strong differentiability with respect to product measures

by

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Abstract. We study the strong differentiability of integrals with respect to measures which are products of Radon measures on the real line, thus generalizing a classical result on the subject.

1. Introduction and statement of results. All functions considered in this note are assumed to have compact support: differentiability being a local problem, this assumption represents no serious restriction here.

Our point of departure is the strong maximal operator S defined for each measurable function f on $\boldsymbol{R}^n$ by means of the formula

$$Sf(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y)|\, dy, \tag{1}$$

where vertical bars outside the integral stand for Lebesgue measure and the supremum is taken over the set of all intervals (cells with faces parallel to the axes) containing the point x.

For several years after the work of Jessen, Marcinkiewicz and Zygmund [7] and its generalization by Zygmund [9], the subject of strong differentiability of integrals remained almost forgotten. This may be due to the fact that the solution attained by those authors looks fairly complete in the context of Lebesgue measure, in view of the negative results concerning the basis formed by all rectangles (rotated intervals) in the plane ([6], p. 226). However, in the beginning of the seventies several proofs appeared ([1], [2] and [5]) of the so-called *strong maximal theorem*, which we state in the following form: there exists a finite constant C such that for all functions f and all positive numbers λ we have the inequality

$$|\{Sf > 4\lambda\}| \leqslant C \int \frac{|f|}{\lambda} \left(\log^+ \frac{|f|}{\lambda}\right)^{n-1} dx. \tag{2}$$

This theorem allowed us to give a concise proof of the now classical theorem of Jessen, Marcinkiewicz and Zygmund.

The subject seems to have attracted much attention since that time as attested by the work of de Guzmán [6], and what we have presently in mind is a generalization of the above mentioned theorem.

The expression *Radon measure* is used in the sequel to mean a positive Borel measure μ on $\boldsymbol{R}^n$, such that $\mu(K) < \infty$ if K is compact; by $\delta(E)$ we denote the diameter of the set E. Provisionally we assume that all intervals I are open intervals and that $\mu(I) > 0$ for every non-void I.

Letting I be an interval containing the point x, our purpose in this note is to prove that the limit

$$\lim_{\delta(I)\to 0} \frac{1}{\mu(I)} \int_I f\,d\mu \tag{3}$$

equals $f(x)$ almost everywhere with respect to μ, provided that

$$\mu = \mu_1 \otimes \ldots \otimes \mu_n$$

is the product of n Radon measures μ_i on the real line $\boldsymbol{R}^1$ and f is a function satisfying

$$\int |f| (\log^+ |f|)^{n-1}\,d\mu < \infty. \tag{4}$$

In view of the fact that f has compact support, the last condition amounts to saying that the integral

$$\int \frac{|f|}{\lambda}\left(\log^+ \frac{|f|}{\lambda}\right)^{n-1} d\mu$$

is finite for every positive number λ.

The existence of the limit (3) has been studied by R. Fefferman [3] in the case of an absolutely continuous measure satisfying an additional requirement, so that our present theorem represents a generalization of the classical result in a different direction.

2. The maximal inequality in $\boldsymbol{R}^1$. In this section we consider a Radon measure μ on the real line and for each measurable function f on $\boldsymbol{R}^1$, we consider the maximal operator M defined by

$$Mf(t) = \sup_{I \ni t} \frac{1}{\mu(I)} \int_I |f|\,d\mu, \tag{5}$$

the supremum being taken over the set of all linear (one dimensional) intervals I containing the real number t. Next we prove that for every positive number λ, we have the inequality

$$\mu\{Mf > \lambda\} \leqslant \frac{2}{\lambda} \int |f|\,d\mu. \tag{6}$$

For this purpose we need the following covering lemma:

LEMMA 1. *From every finite family $\mathscr{A}$ of open intervals in the real line, we can select a disjoint family $(I_j,\ j = 1, \ldots, N)$ such that*

$$\mu(\bigcup \mathscr{A}) \leqslant 2 \sum_{j=1}^{N} \mu(I_j).$$

The proof of the lemma is (save for some obvious modification) the same as the one given in [4], p. 106.

In order to prove (6), take an arbitrary compact set $K \subset \{Mf > \lambda\}$. For each point $t \in K$ there exists an interval I containing t such that

$$\frac{1}{\mu(I)} \int_I |f|\,d\mu > \lambda. \tag{7}$$

Hence, there is a finite family $\mathscr{A}$ of open intervals covering K, each of whose members satisfies (7) and, according to Lemma 1, we can select a disjoint family $(I_j,\ j = 1, \ldots, N)$ of members of $\mathscr{A}$ such that

$$\mu(K) \leqslant 2 \sum_{j=1}^{N} \mu(I_j) < 2 \sum_{j=1}^{N} \frac{1}{\lambda} \int_{I_j} |f|\,d\mu \leqslant \frac{2}{\lambda} \int |f|\,d\mu,$$

and the proof of (6) is complete, since μ is a Radon measure.

3. The strong maximal inequality. Let now $\mu = \mu_1 \otimes \ldots \otimes \mu_n$ be the product of n Radon measures μ_i on the real line. For each measurable function f on $\boldsymbol{R}^n$, we define the strong maximal operator S by the equation

$$Sf(x) = \sup_{I \ni x} \frac{1}{\mu(I)} \int_I |f|\,d\mu,$$

the supremum being taken over the set of all intervals I containing the point $x = (x_1, \ldots, x_n)$. Then Sf is lower semicontinuous and the following theorem holds:

THEOREM 1 (strong maximal inequality). *There exists a finite constant C depending only on the dimension n such that for every positive number λ we have the inequality*

$$\mu\{Sf > 4\lambda\} \leqslant C \int \frac{|f|}{\lambda}\left(\log^+ \frac{|f|}{\lambda}\right)^{n-1} d\mu.$$

Proof. We consider the "partial" one dimensional operators M_i ($i = 1, \ldots, n$) defined for each measurable function f on $\boldsymbol{R}^n$ by the formulae

$$M_i f(x) = \sup_{I \ni x_i} \frac{1}{\mu_i(I)} \int_I |f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)|\, d\mu_i(t),$$

the supremum being taken over the set of all linear intervals I containing the i-th coordinate of the point x. The proof that the functions $M_i f$ are measurable will be given in an appendix at the end of the note, in order not to interrupt the main line of reasoning here.

Now, from Fubini's theorem it follows that each of the M_i's is a maximal operator in the sense of the definition given in [2] and also that

$$Sf \leqslant M_n \ldots M_1 f \quad \text{almost everywhere.} \tag{8}$$

On the other hand we know [2], Theorem 1, p. 276, that there exists a finite constant C depending only on n such that

$$\mu\{M_n \ldots M_1 f > 4\lambda\} \leqslant C \int \frac{|f|}{\lambda} \left(\log^+ \frac{|f|}{\lambda}\right)^{n-1} d\mu, \tag{9}$$

and our present theorem follows from (8) and (9).

As a consequence of the preceding, we can state the following theorem.

THEOREM 2. *Assuming that f satisfies* (4) *and letting I be any interval containing the point x, we have*

$$\lim_{\delta(I) \to 0} \frac{1}{\mu(I)} \int_I |f(y) - f(x)|\, d\mu(y) = 0 \tag{10}$$

almost everywhere with respect to μ.

Proof. For each function f, we write

$$Lf(x) = \limsup_{\delta(I) \to 0} \frac{1}{\mu(I)} \int_I |f(y) - f(x)|\, d\mu(y).$$

Then L is a sublinear operator which satisfies

(a) $Lf(x) \leqslant Sf(x) + |f(x)|$ and

(b) $Lf = 0$ for continuous f.

From (a) we get the inequality

(c) $$\mu\{Lf > 8\lambda\} \leqslant C \int \frac{|f|}{\lambda} \left(\log^+ \frac{|f|}{\lambda}\right)^{n-1} d\mu,$$

where C is another constant depending only on the dimension.

Assuming first that f is a simple function, by the regularity of μ we can assert the existence of a uniformly bounded sequence (f_k) of continuous functions with uniformly bounded supports, converging to f almost everywhere. Since $Lf \leqslant L(f - f_k) + Lf_k = L(f - f_k)$, we will have

$$\mu\{Lf > 8\lambda\} \leqslant \mu\{L(f - f_k) > 8\lambda\} \leqslant C \int \frac{|f - f_k|}{\lambda} \left(\log^+ \frac{|f - f_k|}{\lambda}\right)^{n-1} d\mu,$$

and the last integral tends to zero as $k \to \infty$, so that $Lf = 0$ almost everywhere when f is a simple function.

Assuming that f satisfies (4), we consider a sequence of simple functions (f_k) converging pointwise to f and such that $|f - f_k| \leqslant |f|$ ($k = 1, 2, \ldots$). Now from Lebesgue's dominated convergence theorem, it follows as before that $Lf = 0$ almost everywhere, and the proof is complete.

4. Final remarks. 1° Our assumption that the support of μ is all of $\boldsymbol{R}^n$ can be dispensed with if we agree to leave the limits (3) and (10) undefined outside the support of μ, that is, within a set of measure zero.

2° Letting J be any (not necessarily open) interval of positive measure, we consider a decreasing sequence of open intervals (I_k) whose intersection equals J. Then for any integrable f, we will have

$$\frac{1}{\mu(J)} \int_J |f|\, d\mu = \lim_{k \to \infty} \frac{1}{\mu(I_k)} \int_{I_k} |f|\, d\mu.$$

This shows that the values of the maximal functions we have defined are not affected by considering the set of all (not just the open) intervals.

3° Theorem 2 holds true for any function f whose support is not bounded, provided only that $|f|(\log^+ |f|)^{n-1}$ is locally integrable with respect to μ.

Appendix

Assuming for simplicity that $n = 2$, let μ be a Radon measure on $\boldsymbol{R}^1$. We prove that for every non-negative Borel function f on $\boldsymbol{R}^2$, the function

$$M_1 f(x, y) = \sup_{a, b > 0} \frac{1}{\mu(x - a, x + b)} \int_{x-a}^{x+b} f(t, y)\, d\mu(t),$$

where the integral extends over the open interval $(x - a, x + b)$, is also a Borel measurable function. The argument here represents a substantial modification of the one given by S. Saks [8], Chapter IV, § 13.

First we note that $\mu(x - a, x + b)$ is a lower semicontinuous function of x. Secondly, if we denote by J the interval $(-b, a)$, then the integral of the preceding formula may be written in the form

$$g_{a,b}(x, y) = \int \chi_J(x-t) f(t, y)\, d\mu(t).$$

As a function of x, t and y, the integrand in this formula is a non-negative Borel function on $\boldsymbol{R}^3$. Hence, by Fubini's theorem it follows that $g_{a,b}$ is a Borel function. Finally we note that

$$M_1 f(x, y) = \sup_{r,s>0} \frac{g_{r,s}(x, y)}{\mu(x-r, x+s)},$$

the supremum being taken over the set of all pairs of positive rational numbers r and s.

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Radial convolutors on free groups

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Abstract. Let G be a free group on finitely many generators and let $1 \leqslant p < 2$. We show that any radial function in the Lorentz space $l^{p,1}(G)$ defines a bounded convolution operator on $l^p(G)$.

Let G be a free group on k generators. Every element x in G is a word whose letters are generators or their inverses. We denote by $|x|$ the *length* of the word x, i.e. the number of letters of the word x in its reduced form.

A complex valued function f on the group G is called *radial* if it depends only on the length of a word, that is, if $f(x) = f(y)$ whenever $|x| = |y|$. The subspace of all radial functions in the Lorentz space $l^{p,q}(G)$, $1 \leqslant p, q \leqslant \infty$, will be denoted by $l_r^{p,q}(G)$. Also $l_r^p(G) = l_r^{p,p}(G)$.

A bounded operator T on $l^p(G)$, $1 \leqslant p \leqslant \infty$, is called a *convolutor* if it commutes with all right translations. Since the characteristic function χ_0 of the identity element in G belongs to $l^p(G)$, one may consider T as convolution by the function $f = T(\chi_0)$, so that $T = \lambda(f)$, where λ is the left regular representation of G on $l^p(G)$. We call T a *radial convolutor* if $T(\chi_0)$ is a radial function. Let $C^p(G)$ denote the Banach algebra of all convolutors on $l^p(G)$ and $C_r^p(G)$ the subset of radial convolutors. It was shown in [2] that $C_r^p(G)$ is a maximal commutative subalgebra in $C^p(G)$ and that $C_r^p(G) = C_r^q(G)$ if $1/p + 1/q = 1$.

Here we want to show that

$$l_r^{p,1}(G) \subset C_r^p(G) \subset l_r^p(G) \quad \text{for} \quad 1 \leqslant p < 2,$$

i.e. that $C_r^p(G)$ "almost" coincide with $l_r^p(G)$ (no result of this type is possible for $p = 2$). We also prove that the necessary and sufficient condition for a non-negative radial function to be in $C_r^p(G)$ is to be in $l_r^{p,1}(G)$. This implies that $l_r^{p,1}(G)$ is a convolution algebra for $p < 2$ and that the inclusion $C_r^p(G) \subset l_r^p(G)$ is proper for all $p > 1$.

Let G_m, $m = 0, 1, 2, \ldots$, be the set of all words in G of length m and χ_m the characteristic function of G_m. Then any radial function f on G has the form

$$f = \sum_{m=0}^{\infty} \alpha_m \chi_m.$$