preceding formula may be written in the form

$$g_{a,b}(x, y) = \int \chi_J(x-t) f(t, y)\, d\mu(t).$$

As a function of x, t and y, the integrand in this formula is a non-negative Borel function on $\boldsymbol{R}^3$. Hence, by Fubini's theorem it follows that $g_{a,b}$ is a Borel function. Finally we note that

$$M_1 f(x, y) = \sup_{r,s>0} \frac{g_{r,s}(x, y)}{\mu(x-r, x+s)},$$

the supremum being taken over the set of all pairs of positive rational numbers r and s.

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UNIVERSITY OF BUENOS AIRES, ARGENTINA

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Radial convolutors on free groups

by

TADEUSZ PYTLIK (Wrocław)

Abstract. Let G be a free group on finitely many generators and let $1 \leqslant p < 2$. We show that any radial function in the Lorentz space $l^{p,1}(G)$ defines a bounded convolution operator on $l^p(G)$.

Let G be a free group on k generators. Every element x in G is a word whose letters are generators or their inverses. We denote by $|x|$ the *length* of the word x, i.e. the number of letters of the word x in its reduced form.

A complex valued function f on the group G is called *radial* if it depends only on the length of a word, that is, if $f(x) = f(y)$ whenever $|x| = |y|$. The subspace of all radial functions in the Lorentz space $l^{p,q}(G)$, $1 \leqslant p, q \leqslant \infty$, will be denoted by $l_r^{p,q}(G)$. Also $l_r^p(G) = l_r^{p,p}(G)$.

A bounded operator T on $l^p(G)$, $1 \leqslant p \leqslant \infty$, is called a *convolutor* if it commutes with all right translations. Since the characteristic function χ_0 of the identity element in G belongs to $l^p(G)$, one may consider T as convolution by the function $f = T(\chi_0)$, so that $T = \lambda(f)$, where λ is the left regular representation of G on $l^p(G)$. We call T a *radial convolutor* if $T(\chi_0)$ is a radial function. Let $C^p(G)$ denote the Banach algebra of all convolutors on $l^p(G)$ and $C_r^p(G)$ the subset of radial convolutors. It was shown in [2] that $C_r^p(G)$ is a maximal commutative subalgebra in $C^p(G)$ and that $C_r^p(G) = C_r^q(G)$ if $1/p + 1/q = 1$.

Here we want to show that

$$l_r^{p,1}(G) \subset C_r^p(G) \subset l_r^p(G) \quad \text{for} \quad 1 \leqslant p < 2,$$

i.e. that $C_r^p(G)$ "almost" coincide with $l_r^p(G)$ (no result of this type is possible for $p = 2$). We also prove that the necessary and sufficient condition for a non-negative radial function to be in $C_r^p(G)$ is to be in $l_r^{p,1}(G)$. This implies that $l_r^{p,1}(G)$ is a convolution algebra for $p < 2$ and that the inclusion $C_r^p(G) \subset l_r^p(G)$ is proper for all $p > 1$.

Let G_m, $m = 0, 1, 2, \ldots$, be the set of all words in G of length m and χ_m the characteristic function of G_m. Then any radial function f on G has the form

$$f = \sum_{m=0}^{\infty} \alpha_m \chi_m.$$

We have (see [2])

(1) $\chi_1 * \chi_1 = \chi_2 + 2k\chi_0, \quad \chi_1 * \chi_m = \chi_{m+1} + (2k-1)\chi_{m-1}, \quad m = 2, 3, \ldots$

It follows that

$$\operatorname{card} G_m = \|\chi_m\|_1 = 2k(2k-1)^{m-1} \quad \text{for} \quad m = 1, 2, \ldots$$

LEMMA 1. *A radial function* $f = \sum_{m=0}^{\infty} \alpha_m \chi_m$ *belongs to the Lorentz space* $l^{p,q}(G)$, $1 \leqslant p, q < \infty$, *if and only if the series*

$$\sum_{m=0}^{\infty} |\alpha_m|^q (\operatorname{card} G_m)^{q/p}$$

is convergent. In this case

(2) $$C\|f\|_{p,q} \leqslant \Big(\sum_{m=0}^{\infty} |\alpha_m|^q (\operatorname{card} G_m)^{q/p}\Big)^{1/q} \leqslant C_p \|f\|_{p,q}$$

with $C = 1-(2k-1)^{-1}$ *and* $C_p = [1-(2k-1)^{-1/p}]^{-1}$.

Proof. Suppose that a measure $\omega = (\omega_1, \omega_2, \ldots)$ on the set N of natural numbers is such that

$$\sup_{n\in N} \frac{\omega_n}{\omega_{n+1}} = d < 1,$$

and for a $t > 0$ let $\omega^t = (\omega_1^t, \omega_2^t, \ldots)$. Then

$$(1-d^t)\,\omega^t(E) \leqslant [\omega(E)]^t \leqslant (1-d)^{-t}\,\omega^t(E)$$

for any finite set $E \subset N$. It follows immediately that $l^{p,q}(N, \omega) = l^q(N, \omega^{q/p})$, $1 \leqslant p, q < \infty$, and that

(3) $$(1-d)^{1/p}\|f\|_{l^{p,q}(N,\omega)} \leqslant \|f\|_{l^q(N,\omega^{q/p})} \leqslant (1-d^{1/p})^{-1}\|f\|_{l^{p,q}(N,\omega)}$$

for any function f in $l^{p,q}(N, \omega)$. Now the lemma follows by (3) with $\omega_1 = 1$, $\omega_{m+1} = 2k(2k-1)^{m-1}$ for $m = 1, 2, \ldots$

Let P_m, $m = 0, 1, 2, \ldots$, denote the polynomials defined by $P_0(z) = 1$, $P_1(z) = z$, $P_2(z) = z^2 - 2k$ and $P_m(z) = zP_{m-1}(z) - (2k-1)P_{m-2}(z)$ for $m = 3, 4, \ldots$

It has been shown in [2], Theorem 3.1, that the maximal ideal space of the algebra $C_r^p(G)$, $1 \leqslant p \leqslant \infty$, coincides with the ellipse

$$E_p = \{z \in C\colon\ |z - 2\omega| + |z + 2\omega| \leqslant 2\omega^{2/p} + 2\omega^{2(1-1/p)}\},$$

where $\omega = (2k-1)^{1/2}$. The Gelfand transform $\hat{T}$ of an operator $T \in C_r^p(G)$ for which $T(\chi_0) = \sum_{m=0}^{\infty} \alpha_m \chi_m$ is given by

$$\hat{T}(z) = \sum_{m=0}^{\infty} \alpha_m P_m(z).$$

Let Ω denote the set of all infinite reduced words in the generators of G and their inverses and consider the natural action of G on Ω by left multiplication. If P denotes the Poisson kernel (see [1]) defined on $G \times \Omega$ by

$$P(x, \omega) = (2k-1)^{\delta(x,\omega)},$$

where $\delta(x, \omega) = n - |x^{-1}\omega_n|$ when $|x| = n$ and ω_n is the word in G_n consisting of the first n letters of ω, then the following cocycle identities hold.

(4) $$P(xy, \omega) = P(y, x^{-1}\omega)\,P(x, \omega), \qquad P(e, \omega) = 1.$$

A simple calculation shows that for any real number t

(5) $$\sum_{x\in G_1} P^t(x, \omega) = (2k-1)^t + (2k-1)^{1-t}$$

which is independent of ω. Put

$$Q_m(t) = P_m\big((2k-1)^t + (2k-1)^{1-t}\big), \qquad m = 0, 1, 2, \ldots$$

Then

(6) $$\sum_{x\in G_m} P^t(x, \omega) = Q_m(t), \qquad m = 0, 1, 2, \ldots, \ \omega \in \Omega.$$

Indeed, the cocycle identity and (5) imply

$$P^t(\cdot, \omega) * \chi_1 = Q_1(t)\,P^t(\cdot, \omega).$$

Thus by the recursive formula for P_m and (1) one has

$$P^t(\cdot, \omega) * \chi_m = Q_m(t)\,P^t(\cdot, \omega), \qquad m = 0, 1, 2, \ldots$$

Evaluating this at the identity e one gets (6).

One can easy verify that an explicit formula for $Q_m(t)$, $t \neq 1/2$, $m = 1, 2, \ldots$ is

(7) $$Q_m(t) = A(t)(2k-1)^{mt} + A(1-t)(2k-1)^{m(1-t)},$$

where

$$A(t) = \frac{(2k-1)^t - (2k-1)^{-t}}{(2k-1)^t - (2k-1)^{1-t}}.$$

Now we are ready to prove the following:

LEMMA 2. $\|\lambda(\chi_m)\|_{C^p} = Q_m(1/p)$ *for* $m = 0, 1, 2, \ldots$ *and* $p \geqslant 1$.

Proof. Observe that the number $(2k-1)^{1/p} + (2k-1)^{1-1/p}$ belongs to the ellipse E_p. Thus

$$\|\lambda(\chi_m)\|_{C^p} \geqslant \sup_{z\in E_p} |\lambda(\chi_m)^\wedge(z)| \geqslant P_m\big((2k-1)^{1/p} + (2k-1)^{1-1/p}\big) = Q_m(1/p).$$

To get the converse inequality $\|\lambda(\chi_m)\|_{C^p} \leqslant Q_m(1/p)$ we shall show that

$$\|\chi_m * \varphi\|_p \leqslant Q_m(1/p)\,\|\varphi\|_p$$

for any function φ on G with finite support. We may assume that $\operatorname{supp}\varphi\cap G_k = \emptyset$ for $k = 0, 1, 2, \ldots, 2m-1$, so that any word y in the support of φ has length at least $2m$. Indeed, $\|\chi_m * \varphi\|_p = \|\chi_m * (\varphi * \delta_x)\|_p$ for any $x \in G$, and the function $\varphi * \delta_x$ has the desired property if $|x|$ is big enough.

For any $y \in G$, $|y| \geqslant m$ and any $x \in G_m$ put $P(x, y) = P(x, \omega)$, where ω is an element in Ω such that $\omega_n = y$ when $|y| = n$. Fix a real number t. Then

$$|\chi_m * \varphi(y)|^p = \Big|\sum_{x\in G_m} \varphi(xy)\Big|^p \leqslant \Big(\sum_{x\in G_m} P^t(x, y)\, P^{-t}(x, y)\, |\varphi(xy)|\Big)^p.$$

Since the function $s \to s^p$ is convex on $(0, \infty)$, we have for any numbers $s_1, s_2, \ldots, s_r, \alpha_1, \alpha_2, \ldots, \alpha_r$ in $(0, \infty)$

$$(\alpha_1 s_1 + \alpha_2 s_2 + \ldots + \alpha_r s_r)^p \leqslant (\alpha_1 + \alpha_2 + \ldots + \alpha_r)^{p-1} (\alpha_1 s_1^p + \alpha_2 s_2^p + \ldots + \alpha_r s_r^p).$$

In particular

$$|\chi_m * \varphi(y)|^p \leqslant \Big(\sum_{x\in G_m} P^t(x, y)\Big)^{p-1} \Big(\sum_{x\in G_m} P^{(1-p)t}(x, y)\, |\varphi(xy)|^p\Big).$$

But by (6) $\sum_{x\in G_m} P^t(x, y) = Q_m(t)$. Thus

$$\|\chi_m * \varphi\|_p^p = \sum_{y} |\chi_m * \varphi(y)|^p \leqslant Q_m^{p-1}(t) \sum_{y} \sum_{x\in G_m} P^{(1-p)t}(x, y)\, |\varphi(xy)|^p.$$

If the order of summation on the right-hand side of the last inequality is reversed, variable y replaced by $x^{-1}y$ and the order of summation changed again, the inner sum becomes

$$\sum_{x\in G_m} P^{(1-p)t}(x, x^{-1}y),$$

which by the cocycle identity and by (6) is equal to $Q_m\big((p-1)t\big)$. Therefore

$$\|\chi_m * \varphi\|_p^p \leqslant Q_m^{p-1}(t)\, Q_m\big((p-1)t\big) \sum_{y} |\varphi(y)|^p = Q_m^{p-1}(t)\, Q_m\big((p-1)t\big)\, \|\varphi\|_p^p.$$

In particular for $t = 1/p$ we have $Q_m\big((p-1)t\big) = Q_m(1-1/p) = Q_m(1/p)$, so

$$\|\chi_m * \varphi\|_p^p \leqslant Q_m^p(1/p)\, \|\varphi\|_p^p.$$

This proves the lemma.

THEOREM. *Let G be a free group on finitely many generators and let $1 \leqslant p < 2$. Then*

$$l_r^{p,1}(G) \subset C_r^p(G) \subset l_r^p(G).$$

Proof. The only non-trivial inclusion $l_r^{p,1}(G) \subset C_r^p(G)$ has to be shown.

Take an arbitrary function f in $l_r^{p,1}(G)$ and write it in the form

$$f = \sum_{m=0}^{\infty} \alpha_m \chi_m.$$

By Lemma 2 we have

$$\|\lambda(f)\|_{C^p} \leqslant \sum_{m=0}^{\infty} |\alpha_m|\, \|\lambda(\chi_m)\|_{C^p} = \sum_{m=0}^{\infty} |\alpha_m|\, Q_m(1/p).$$

Also by Lemma 1

$$\|f\|_{p,1} \geqslant C_p^{-1} \sum_{m=0}^{\infty} |\alpha_m| (\operatorname{card} G_m)^{1/p}.$$

Since $A(1-1/p) < 0$ for $p < 2$, we have in (7)

$$Q_m(1/p) \leqslant A(1/p)(2k-1)^{m/p} \leqslant A(1/p)(\operatorname{card} G_m)^{1/p}.$$

Thus

$$\|\lambda(f)\|_{C^p} \leqslant A(1/p)\, C_p \|f\|_{p,1} < \infty$$

and so $f \in C_r^p(G)$.

COROLLARY. *A non-negative radial function f belongs to $C^p(G)$, $1 \leqslant p < 2$, if and only if $f \in l^{p,1}(G)$. This implies that $l_r^{p,1}(G)$ is a convolution algebra.*

Proof. Let $f = \sum_{m=0}^{\infty} \alpha_m \chi_m$ be a non-negative function in $C_r^p(G)$, $1 \leqslant p < 2$. Then

$$\|\lambda(f)\|_{C^p} \geqslant \sup_{z\in E_p} |\hat{f}(z)| \geqslant \sum_{m=0}^{\infty} \alpha_m Q_m(1/p).$$

But $Q_m(1/p) \geqslant (\operatorname{card} G_m)^{1/p}$. Thus by Lemma 1

$$\|\lambda(f)\|_{C_p} \geqslant \sum_{m=0}^{\infty} \alpha_m (\operatorname{card} G_m)^{1/p} \geqslant C \|f\|_{p,1},$$

and so $f \in l_r^{p,1}(G)$.

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INSTITUTE OF MATHEMATICS, UNIVERSITY OF WROCŁAW

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