[6] N. Dunford, J. T. Schwartz, *Linear operators*, Part III, *Spectral operators*, Interscience, New York 1971.
[7] P. Masani, M. Rosenberg, *When is an operator the integral of a given spectral measure*, J. Funct. Anal. 21 (1976), 88–121.
[8] A. R. Sourour, *Unbounded operators generated by a given spectral measure*, ibidem 29 (1978), 16–22.

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Weak type inequalities for the maximal ergodic function and the maximal ergodic Hilbert transform in weighted spaces

by

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Abstract. In this paper we show that the maximal ergodic function associated to an invertible, measure preserving ergodic transformation on a probability space is of weak type (1,1) with respect to $wd\mu$, where w is a positive integrable function, if and only if w satisfies Muckenhoupt condition A_1. We also prove the same result for the maximal ergodic Hilbert transform.

1. Introduction. Let $(X, \mathfrak{F}, \mu)$ be a non-atomic probability space and T an ergodic, invertible measure preserving point transformation from X onto itself. We will denote by f^* the non-centered maximal ergodic function

$$f^*(x) = \sup_{n,m\geqslant 0} (n+m+1)^{-1} \sum_{i=-n}^{m} |f(T^i x)|, \quad n, m \in \boldsymbol{Z}, \tag{1.1}$$

and by

$$Hf(x) = \sup_{s,t\geqslant 0} \left| \sum_{s<|h|<t} \frac{f(T^h x)}{h} \right|, \quad s, t \in \boldsymbol{Z},$$

the maximal ergodic Hilbert transform.

In [1] and [2] it was shown that the operators $f \to f^*$ and $f \to Hf$ are bounded on $L^p(wd\mu)$, $p > 1$, if and only if the positive integrable function w satisfies the condition:

(A'_p) There exists a constant M such that for a.e. x

$$k^{-1} \sum_{i=0}^{k-1} w(T^i x) \cdot \left[k^{-1} \sum_{i=0}^{k-1} (w(T^i x))^{-1/(p-1)}\right]^{p-1} \leqslant M \tag{1.2}$$

for all positive integers k.

Condition A'_p is the natural analogue of Muckenhoupt condition for the Hardy–Littlewood maximal operator [4].

In this paper our main result is given by the following theorem.

(1.1) THEOREM. *Let w be a positive integrable function. Then*

(i) *The operator $f \to f^*$ is of weak type* (1.1) *with respect to $wd\mu$ if and only if $w \in A_1$.*

(ii) *The operator $f \to Hf$ is of weak type* (1.1) *with respect to $wd\mu$ if and only if $w \in A_1$.*

By A_1 we mean the well-known Muckenhoupt condition, i.e., $w \in A_1$ if there exists a constant C such that $w^*(x) \leqslant C \cdot w(x)$ a.e. As usual, in this paper C will denote an absolute constant, not necessarily the same at each occurrence.

In the proof of Theorem (1.1) the concept of ergodic rectangle will be used.

(1.2) DEFINITION. Let B be a subset of X with positive measure and k a positive integer such that

$$T^i B \cap T^j B = \emptyset, \quad i \neq j,\ 0 \leqslant i, j \leqslant k-1.$$

The set $R = \bigcup_{i=0}^{k-1} T^i B$ will be called an *ergodic rectangle* of base B and length k.

We shall also use the following two results, for a proof see [1].

(1.3) PROPOSITION. *Let k be a positive integer and let $A \subset X$ be a subset of positive measure. Then there exists $B \subset A$ such that B is base of a rectangle of length k.*

(1.4) LEMMA. *X can be written as a countable union of bases of rectangles of length k.*

2. Proof of theorem (1.1) for the maximal ergodic function. We firstly assume that $f \to f^*$ is of weak type (1.1) with respect to $wd\mu$. Let n, m be non-negative integers and choose a base B of a rectangle of length $n+m+1$. Let $A \subset B$ with positive measure. The set

$$R = \bigcup_{i=-n}^{m} T^i A$$

is clearly a rectangle of length $n+m+1$.

Let $f = \chi_A$ be the characteristic function of the set A and consider $T^j x$, $-n \leqslant j \leqslant m$, $x \in A$. Then

$$(\chi_A)^*(T^j x) \geqslant (n+m+1)^{-1} \sum_{i=-n-j}^{m-j} \chi_A(T^i(T^j x)) = (n+m+1)^{-1}.$$

Therefore

$$R \subset \{x\colon (\chi_A)^*(x) \geqslant (n+m+1)^{-1}\}.$$

This inclusion and our assumption allow us to write

$$\int_R w\,d\mu \leqslant \int_{\{x:(\chi_A)^*(x) \geqslant (n+m+1)^{-1}\}} w\,d\mu \leqslant C(n+m+1) \int_X \chi_A w\,d\mu = C(n+m+1) \int_A w\,d\mu.$$

Thus we have

$$\int_A (n+m+1)^{-1} \sum_{i=-n}^{m} w(T^i x)\,d\mu \leqslant C \int_A w\,d\mu.$$

Since this holds for every A measurable subset of B with positive measure, we obtain

$$(2.1) \qquad (n+m+1)^{-1} \sum_{i=-n}^{m} w(T^i x) \leqslant Cw(x) \quad \text{a.e. } x \in B.$$

A straightforward application of Lemma (1.4) gives us (2.1) for almost all x in X which immediately implies that $w \in A_1$.

The converse will be a consequence of the following theorem.

(2.2) THEOREM. *Let $p > 1$. Then there exists a constant C_p, $0 < C_p < \infty$, such that*

$$(2.3) \qquad \int_X (f^*)^p w\,d\mu \leqslant C_p \int_X |f|^p w^*\,d\mu$$

for all measurable functions f. Furthermore, for $\lambda > 0$

$$(2.4) \qquad \int_{\{x:f^*(x) > \lambda\}} w\,d\mu \leqslant \frac{C}{\lambda} \int_X |f|\,w^*\,d\mu.$$

Proof. Since $f \to f^*$ is a bounded operator from $L^\infty(w^*\,d\mu)$ to $L^\infty(wd\mu)$, it will suffice to show the weak type estimation (2.4) to obtain, using the Marcinkiewicz interpolation theorem, the strong type inequality (2.3). Then, let $\lambda > 0$. If

$$\lambda \leqslant \int_X |f|\,w^*\,d\mu \cdot \Big(\int_X w\,d\mu\Big)^{-1},$$

(2.4) is clear. Suppose

$$(2.5) \qquad \lambda > \int_X |f|\,w^*\,d\mu \cdot \Big(\int_X w\,d\mu\Big)^{-1}$$

and call f^{**} the one-sided maximal function defined by

$$f^{**}(x) = \sup_{k>0} k^{-1} \sum_{i=0}^{k-1} |f(T^i x)|.$$

Let O_λ be the set

$$O_\lambda = \{x\colon f^{**}(x) > \lambda\}$$

and let

$$B_i = \{x\colon x, Tx, \ldots, T^{i-1} x \in O_\lambda,\ T^{-1} x,\ T^i x \notin O_\lambda\},$$
$$B = \{x\colon T^i x \in O_\lambda,\ i \geqslant 0,\ T^{-1} x \notin O_\lambda\},$$

$$B' = \{x:\ T^{-i}x \in O_\lambda,\ i \geqslant 0,\ Tx \notin O_\lambda\},$$

$$C = \{x:\ T^i x \in O_\lambda,\ i \in \mathbf{Z}\}.$$

Then $O_\lambda = \bigcup\limits^{\infty} R_i \cup R \cup R' \cup C$, where $R_i = B_i \cup TB_i \cup \ldots \cup T^{i-1}B_i$, $R = \bigcup\limits_{i \geqslant 0} T^i B$, $R' = \bigcup\limits_{i \geqslant 0} T^{-i}B'$. Clearly, $\{R_i\}_{i=1}^{\infty}$ is a collection of disjoint ergodic rectangles and R and R' are *rectangles* of infinite length. Since $\mu(X) < \infty$, $\mu(R) = \mu(R') = 0$. Now let us prove that $\mu(C) = 0$. Let $x \in C$. There exists $r_0 \geqslant 1$ such that

$$r_0^{-1} \sum_{i=0}^{r_0-1} |f(T^i x)| > \lambda,$$

$T^{r_0}x \in C$. There exists $r_1 \geqslant 1$ such that

$$r_1^{-1} \sum_{i=0}^{r_1-1} |f(T^{i+r_0}x)| > \lambda.$$

Then

$$(r_0+r_1)^{-1} \sum_{i=0}^{r_0+r_1-1} |f(T^i x)| > \lambda.$$

Continuing this process and fixing M, we can find $k > M$ such that

$$k^{-1} \sum_{i=0}^{k-1} |f(T^i x)| > \lambda$$

and therefore

$$k^{-1} \sum_{i=0}^{k-1} |f(T^i x)|\, w^*(T^i x) > \lambda k^{-1} \sum_{i=0}^{k-1} w(T^i x) \quad (x \in C).$$

If $\mu(C) > 0$, applying the ergodic individual theorem, we obtain

$$\int_X |f|\, w^*\, d\mu \geqslant \lambda \int_X w\, d\mu$$

against (2.5). Thus $\mu(C) = 0$ and consequently $\mu(O_\lambda - \bigcup\limits_{i=1}^{\infty} R_i) = 0$. We shall consider two sets *are equal* if they agree up to a set of measure zero. The following proposition, which is an ergodic analog of the Calderón–Zygmund decomposition, will be needed.

(2.6) Proposition. *The following inequalities are valid:*

(2.7) $$|f(x)| \leqslant \lambda \quad \text{if} \quad x \notin O_\lambda,$$

(2.8) $$\lambda < i^{-1} \sum_{h=0}^{i-1} |f(T^h x)| \leqslant 2\lambda \quad (x \in B_i).$$

Proof. Inequality (2.7) is clear. In order to prove (2.8) suppose that for some $x \in B_i$ we have

$$\sum_{h=0}^{i-1} |f(T^h x)| \leqslant i\lambda.$$

If $r \geqslant 0$

$$\sum_{h=0}^{i+r} |f(T^h x)| = \sum_{h=0}^{i-1} |f(T^h x)| + \sum_{h=i}^{i+r} |f(T^h x)| \leqslant (i+r+1)\lambda,$$

where the fact that $T^i x \notin O_\lambda$ has been used.

We thus have

$$(m+1)^{-1} \sum_{h=0}^{m} |f(T^h x)| \leqslant \lambda \quad \text{if} \quad m \geqslant i.$$

Since $x \in O_\lambda$, there exists an s such that

$$(s+1)^{-1} \sum_{h=0}^{s} |f(T^h x)| > \lambda.$$

Obviously, $0 \leqslant s < i$.

Now if

$$q = \sup\{s:\ (s+1)^{-1} \sum_{h=0}^{s} |f(T^h x)| > \lambda\}$$

let us prove that $q = i-1$. If $q < i-1$, then $T^{q+1}x \in O_\lambda$ and this implies that there exists $t \geqslant q+1$ such that

$$\sum_{h=q+1}^{t} |f(T^h x)| > \lambda(t-q)$$

and clearly

$$\sum_{h=0}^{t} |f(T^h x)| > \lambda(t+1).$$

Consequently we have $t \leqslant i-1$ and $t \geqslant q+1$, against q being the maximum. The right-hand side inequality in (2.8) follows from the fact that if $x \in B_i$, then $T^{-1}x \notin O_\lambda$ which implies

$$(i+1)^{-1} \sum_{h=-1}^{i-1} |f(T^h x)| \leqslant \lambda$$

and then

$$\sum_{h=0}^{i-1} |f(T^h x)| \leqslant (i+1)\lambda.$$

Dividing the two members of the former inequality by i, we infer the proof of the proposition.

From (2.8), if $x \in B_i$ we obtain

$$\sum_{j=0}^{i-1} |f(T^j x)|\, w^*(T^j x) \geqslant \sum_{j=0}^{i-1} |f(T^j x)| \cdot \Big(i^{-1} \sum_{h=0}^{i-1} w(T^h x)\Big) > \lambda \sum_{h=0}^{i-1} w(T^h x).$$

Now, integrating over B_i and adding up in i,

$$\lambda^{-1} \int_{O_\lambda} |f|\, w^*\, d\mu \geqslant \int_{O_\lambda} w\, d\mu.$$

Using the fact that

$$\{x\colon f^*(x) > \lambda\} \subset \{x\colon \sup_{k>0} k^{-1} \sum_{i=0}^{k-1} |f(T^i x)| > \tfrac{1}{2}\lambda\} \cup$$

$$\cup \{x\colon \sup_{k>0} k^{-1} \sum_{i=0}^{k-1} |f(T^{-i} x)| > \tfrac{1}{2}\lambda\},$$

we get inequality (2.4). Theorem (1.1) for f^* follows immediately since our assumption now is that w satisfies A_1.

Note. (a) A similar result to Theorem (2.2), but with f^* being the maximal Hardy–Littlewood function, can be found in [3].

(b) Observe that Theorem (2.2), used in the proof of part (i) of Theorem (1.1), shows that if w satisfies condition A_1, then $f \to f^*$ is bounded on $L^p(w\,d\mu)$ $(1 < p < \infty)$. Incidentally, keeping in mind the above-mentioned result of [1], we also obtain that condition A_1 implies A_p, $p > 1$.

3. Proof of Theorem (1.1) for the maximal ergodic Hilbert transform. Assume that $f \to H_f^*$ is of weak type (1.1) with respect to $w\,d\mu$. Let n, m be non-negative integers and choose a base B of a rectangle of length $2n+2m+1$. Let $A \subset B$ and $\mu(A) > 0$.

The set $\bigcup_{i=-2n}^{2m} T^i A$ is a rectangle of length $2n+2m+1$.

Consider $T^j x$, $x \in A$, $-n \leqslant j \leqslant m$, $j \neq 0$. Then it is clear that

$$H\chi_A(T^j x) \geqslant \Big| \sum_{s<|h|<t} \frac{\chi_A(T^{h+j} x)}{h} \Big|.$$

Choosing $s = |j|-1$, $t = |j|+1$ we have

$$\Big| \sum_{s<|h|<t} \frac{\chi_A(T^{h+j} x)}{h} \Big| = |j|^{-1} \cdot (n+m+1)^{-1}.$$

Therefore

$$H\chi_A(y) \geqslant (n+m+1)^{-1} \quad \text{if} \quad y \in R' = \bigcup_{\substack{i=-n\\ i\neq 0}}^{m} T^i A$$

which implies

$$R' \subset \{x\colon H\chi_A(x) \geqslant (n+m+1)^{-1}\}.$$

Integrating and using our initial assumption, we get

$$\int_{R'} w\,d\mu \leqslant \int_{\{x: H\chi_A(x) \geqslant (n+m+1)^{-1}\}} w\,d\mu \leqslant C(n+m+1) \int_A w\,d\mu;$$

this inequality can be written as

$$(n+m+1)^{-1} \int_A \sum_{\substack{i=-n\\ i\neq 0}}^{m} w(T^i x)\,d\mu \leqslant C \int_A w\,d\mu.$$

Adding up $(n+m+1)^{-1} \int_A w\,d\mu$:

$$\int_A (n+m+1)^{-1} \sum_{i=-n}^{m} w(T^i x)\,d\mu < (C+1) \int_A w\,d\mu.$$

Since this holds for every A, arbitrary measurable subset of positive measure of B, we have

$$(n+m+1)^{-1} \sum_{i=-n}^{m} w(T^i x) \leqslant (C+1)\, w(x) \qquad \text{a.e. } x \in B.$$

Using (1.4), we obtain that w satisfies condition A_1.

To prove the converse we need to use again the subset O_λ with

$$\lambda > \int_X |f|\, w\,d\mu \Big(\int_X w\,d\mu\Big)^{-1}$$

and the decomposition

$$O_\lambda = \bigcup_{i=1}^{\infty} R_i$$

obtained in Section 2.

As in the classical case (see [5]), we now proceed to decompose the function $f \in L^1(w\,d\mu)$ into a sum:

$$f(x) = g(x) + b(x),$$

where g is in $L^2(w\,d\mu)$ and b is supported on a small set. More precisely, we define $g(x) = f(x)$ if $x \in F = X - O_\lambda$ and if $x \in O_\lambda$, then there exists i such that $x \in R_i = \bigcup_{j=0}^{i-1} T^j B_i$; in that case we set

$$g(x) = i^{-1} \sum_{j=0}^{i-1} f(T^{j-h} x),$$

where h is such that $0 \leqslant h \leqslant i-1$ and $T^{-h} x \in B_i$.

The function b is defined by

$$b(x) = f(x) - g(x).$$

Obviously, $b(x) = 0$ if $x \in F$ and

$$\sum_{j=0}^{i-1} b(T^j x) = 0 \quad \text{if} \quad x \in B_i. \tag{3.1}$$

As in the classical case g is in $L^2(w d\mu)$

(3.2) THEOREM. *The function g is in $L^2(w d\mu)$ and*

$$\int_X |g|^2 w\, d\mu \leqslant C\lambda \int_X |f|\, w\, d\mu.$$

Proof.

$$\int_X |g|^2 w\, d\mu = \int_F |f|^2 w\, d\mu + \int_{O_\lambda} |g|^2 w\, d\mu \leqslant \lambda \int_F |f|\, w\, d\mu + \int_{O_\lambda} |g|^2 w\, d\mu.$$

In the former inequality (2.7) has been used. By (2.8) $|g(x)| \leqslant 2\lambda$ if $x \in O_\lambda$, therefore

$$\int_X |g|^2 w\, d\mu \leqslant \lambda \int_F |f|\, w\, d\mu + 4\lambda^2 \int_{O_\lambda} w\, d\mu.$$

The assumption $w \in A_1$ and Section 2 tell us that $f \to f^*$ has weak type (1.1) with respect to $w d\mu$; thus we obtain

$$\int_X |g|^2 w\, d\mu \leqslant \lambda \int_F |f|\, w\, d\mu + 4\lambda C \int_X |f|\, w\, d\mu \leqslant \lambda(1+4C) \int_X |f|\, w\, d\mu.$$

The following theorem shows us that $b \in L^1(w d\mu)$ and provides an integral inequality that will be used later.

(3.3) THEOREM. *The function b is in $L^1(w d\mu)$ and*

$$\int_X |b|\, w\, d\mu \leqslant C \int_X |f|\, w\, d\mu.$$

Proof.

$$\int_X |b|\, w\, d\mu \leqslant \int_{O_\lambda} |f|\, w\, d\mu + \int_{O_\lambda} |g|\, w\, d\mu,$$

$$\begin{aligned}
\int_{O_\lambda} |g|\, w\, d\mu &= \sum_{i=1}^{\infty} \int_{B_i} \sum_{j=0}^{i-1} |g(T^j x)|\, w(T^j x)\, d\mu \\
&= \sum_{i=1}^{\infty} \int_{B_i} \sum_{j=0}^{i-1} \Big|i^{-1} \sum_{h=0}^{i-1} f(T^h x)\Big|\, w(T^j x)\, d\mu \\
&\leqslant \sum_{i=1}^{\infty} \int_{B_i} \sum_{h=0}^{i-1} \Big[|f(T^h x)| \cdot i^{-1} \sum_{j=0}^{i-1} w(T^j x)\Big]\, d\mu \\
&\leqslant \sum_{i=1}^{\infty} \int_{B_i} \sum_{h=0}^{i-1} |f(T^h x)| \cdot w^*(T^h x)\, d\mu \\
&= \int_{O_\lambda} |f|\, w^*\, d\mu \leqslant C \int_X |f|\, w\, d\mu.
\end{aligned}$$

Thus we finally have

$$\int_X |b|\, w\, d\mu \leqslant (1+C) \int_X |f|\, w\, d\mu.$$

We now need to find a constant C, independent of f and λ, so that

$$\int_{\{x: Hf(x) > \lambda\}} w\, d\mu \leqslant \frac{C}{\lambda} \int_X |f|\, w\, d\mu. \tag{3.4}$$

Since $Hf \leqslant Hg + Hb$, it follows that

$$\int_{\{x: Hf(x) > \lambda\}} w\, d\mu \leqslant \int_{\{x: Hg(x) > \lambda/2\}} w\, d\mu + \int_{\{x: Hb(x) > \lambda/2\}} w\, d\mu \tag{3.5}$$

and it suffices to establish separately for both terms of the right-hand side inequalities analogous to (3.4).

Estimate for Hg. $w \in A_1$ implies that $w \in A_2'$ and therefore, as it was shown in [2], $f \to Hf$ has strong type (2.2) and consequently weak type (2.2). Thus

$$\int_{\{x: Hg(x) > \lambda/2\}} w\, d\mu \leqslant (2C/\lambda)^2 \int_X |g|^2 w\, d\mu.$$

Applying theorem (3.2), we obtain

$$\int_{\{x: Hg(x) > \lambda/2\}} w\, d\mu \leqslant \frac{C}{\lambda} \int_X |f|\, w\, d\mu. \tag{3.6}$$

Estimate for Hb. Denote by $\tilde{R}_i$ the set R_i expanded 3 times, i.e., $\tilde{R}_i = T^{-i} R_i \cup R_i \cup T^i R_i$. Let $\tilde{O}_\lambda$ and $\tilde{F}$ be the sets

$$\tilde{O}_\lambda = \bigcup_{i=1}^{\infty} \tilde{R}_i, \qquad \tilde{F} = X - \tilde{O}_\lambda.$$

Then

$$\int_{\{x \in X: Hb(x) > \lambda/2\}} w\, d\mu = \int_{\{x: Hb(x) > \lambda/2\}} w\, d\mu + \int_{\{x \in \tilde{F}: Hb(x) > \lambda/2\}} w\, d\mu. \tag{3.7}$$

The first integral on the right-hand side is bounded by

$$\sum_{i=1}^{\infty} \int_{B_i} \sum_{j=-i}^{2i-1} w(T^j x)\, d\mu$$

and by Proposition (2.6) this is not bigger than

$$3\lambda^{-1} \sum_{i=1}^{\infty} \int_{B_i} (3i)^{-1} \sum_{j=-i}^{2i-1} w(T^j x) \cdot \sum_{h=0}^{i-1} |f(T^h x)|\, d\mu.$$

Therefore we have

$$\int_{\{x\in\tilde{O}_\lambda : Hb(x)>\lambda/2\}} w\,d\mu \leqslant 3\lambda^{-1}\sum_{i=1}^{\infty}\int_{B_i}\sum_{h=0}^{i-1}|f(T^h x)|\,w^*(T^h x)\,d\mu$$
$$= 3\lambda^{-1}\int_{O_\lambda}|f(x)|\,w^*(x)\,d\mu.$$

Since w satisfies condition A_1, we finally obtain

(3.8)
$$\int_{\{x\in\tilde{O}_\lambda : Hb(x)>\lambda/2\}} w\,d\mu \leqslant \frac{C}{\lambda}\int_X |f|\,w\,d\mu.$$

To estimate $\int_{\{x\in\tilde{F} : Hb(x)>\lambda/2\}} w\,d\mu$ we need to work harder.

The following lemma will be used.

(3.9) LEMMA. *If $1<p<\infty$ and w satisfies the A'_p condition with constant C, then there is a constant K, depending only on p and C, such that for every interval I in the integers of the form $\{-i, \ldots, i\}$*

(3.10)
$$\sum_{h\notin I}(\# I)^{p-1}\,w(T^h x)\,|h|^{-p} \leqslant K(\# I)^{-1}\sum_{h\in I} w(T^h x) \quad \text{a.e.}$$

where $\# I$ stands for the number of elements of I.

Proof. There exists r, $1<r<p$, such that $w\in A'_r$ with constant D depending only on p and C. For a proof see [1].

It will be sufficient to show that

(3.11)
$$\sum_{h>i}(i+1)^{p-1}\,w(T^h x)\,|h|^{-p} \leqslant K(i+1)^{-1}\sum_{h=0}^{i} w(T^h x) \quad \text{a.e.}$$

Since w satisfies condition A'_r with constant D, we have for any positive integer k

$$\sum_{h=0}^{k} w(T^h x)\cdot\Big(\sum_{h=0}^{k} w(T^h x)^{-1/(r-1)}\Big)^{r-1} \leqslant D(k+1)^r.$$

If $k \geqslant i+1$, it is clear that

(3.12)
$$(k+1)^{-p-1}\sum_{h=i+1}^{k} w(T^h x)\cdot\Big(\sum_{h=0}^{i} w(T^h x)^{-1/(r-1)}\Big)^{r-1} \leqslant D(k+1)^{r-p-1}.$$

By Hölder's inequality applied to

$$\sum_{h=0}^{i} w(T^h x)^{1/r}\cdot w(T^h x)^{-1/r}$$

we have

$$(i+1)^r\Big(\sum_{h=0}^{i} w(T^h x)\Big)^{-1} \leqslant \Big(\sum_{h=0}^{i} w(T^h x)^{-1/(r-1)}\Big)^{r-1}.$$

Using this in (3.12) leads to

$$(k+1)^{-p-1}\sum_{h=i+1}^{k} w(T^h x)(i+1)^r\Big(\sum_{h=0}^{i} w(T^h x)\Big)^{-1} \leqslant D(k+1)^{r-p-1},$$

an inequality that holds for $k \geqslant i+1$. Adding up in k:

(3.13)
$$\sum_{k=i+1}^{\infty}(k+1)^{-p-1}\sum_{h=i+1}^{k} w(T^h x) \leqslant D(i+1)^{-r}\sum_{k=i+1}^{\infty}(k+1)^{r-p-1}\sum_{h=0}^{i} w(T^h x).$$

Now (3.11) is an immediate consequence of the following inequalities:

(3.14)
$$\sum_{h=i+1}^{\infty} w(T^h x)\,p^{-1}(h+1)^{-p} \leqslant \sum_{k=i+1}^{\infty}(k+1)^{-p-1}\sum_{h=i+1}^{k} w(T^h x),$$

(3.15)
$$\sum_{k=i+1}^{\infty}(k+1)^{r-p-1} \leqslant (p-r)^{-1}(i+1)^{-p+r}.$$

(3.16) Note. Since $w\in A_1$ implies $w\in A'_2$, (3.10) will hold, for $p=2$, if $w\in A_1$.

The sufficiency of condition A_1 will be a consequence of the following proposition.

(3.17) PROPOSITION. *If $x\in\tilde{F}$ and $Hb(x)>a$, then*

$$a < \sum_{i=1}^{\infty} i \sum_{\substack{h=-\infty \\ h\neq 0}}^{+\infty}(|b|\,\chi_{R_i})(T^h x)\,h^{-2} + 8b^*(x).$$

Proof. There exist s and t such that

$$H_{s,t}b(x) = \Big|\sum_{s<|h|<t}\frac{b(T^h x)}{h}\Big| > a.$$

Associated with the orbit of x in O_λ we consider the set

$$\{k\in \mathbf{Z}:\ T^k x\in O_\lambda\}.$$

This set can be written as a union of a sequence $\{I_k\}$ of finite and disjoint intervals in the integers.

We will consider those intervals I_k that have a non-empty intersection with $\{s+1, s+2, \ldots, t-1\}$, $\{-(t-1), \ldots, -(s+2), -(s+1)\}$. We will denote by K' the finite set of indices corresponding to these intervals.

Call K'' the set of indices corresponding to those intervals of the sequence $\{I_k\}$ such that contain some integers of the set $\{\pm(s+1), \pm(t-1)\}$. Clearly, $K''\subset K'$ and $\# K''\leqslant 4$.

It is obvious that

$$H_{s,t}\, b(x) \leqslant \sum_k \Bigg| \sum_{\substack{s<|h|<t \\ h\in I_k}} \frac{b(T^h x)}{h} \Bigg| .$$

This sum is bounded by

$$\sum_{k\in K''} \Bigg| \sum_{\substack{s<|h|<t \\ h\in I_k}} \frac{b(T^h x)}{h} \Bigg| + \sum_{k\in K'} \Bigg| \sum_{h\in I_k} \frac{b(T^h x)}{h} \Bigg| . \tag{3.18}$$

We are going to bound the first sum of (3.18). Suppose that $s+1\in I_k$. The treatment of the other cases is similar. Let I_k be of the form

$$\{m, m+1, \ldots, m+q-1\}.$$

Since $x\in \tilde F$, we have $m>0$ and $m>q$.
Then we have

$$\Bigg| \sum_{\substack{h=m \\ s<h<t}}^{m+q-1} \frac{b(T^h x)}{h} \Bigg| \leqslant \sum_{h=m}^{m+q-1} \frac{|b(T^h x)|}{h} \leqslant m^{-1} \sum_{h=0}^{m+q-1} |b(T^h x)| < 2b^*(x).$$

Therefore the first sum of (3.18) is bounded by $8b^*(x)$ and the second by

$$\sum_k \Bigg| \sum_{h\in I_k} \frac{b(T^h x)}{h} \Bigg| .$$

Consider now an interval of length i, $\{m, m+1, \ldots, m+i-1\}$, with $m>0$.
Using the mean value property of b (3.1), we get

$$\Bigg| \sum_{h=m}^{m+i-1} \frac{b(T^h x)}{h} \Bigg| = \Bigg| \sum_{h=m}^{m+i-1} \left(\frac{1}{h} - \frac{1}{m+i-1} \right) b(T^h x) \Bigg| \leqslant \sum_{h=m}^{m+i-1} i\, |b(T^h x)|\, h^{-2}.$$

Note that if $m<0$ we would have subtracted m^{-1} to obtain the same bound. Therefore the sum corresponding to all the intervals of length i is bounded by

$$i \sum_{h\neq 0} |(b\chi_{R_i})(T^h x)|\, h^{-2}.$$

Thus (3.18) is bounded by

$$\sum_{i=1}^{\infty} i \sum_{h\neq 0} |(b\chi_{R_i})(T^h x)|\, h^{-2} + 8b^*(x).$$

This completes the proof of the proposition.
Since

$$\{x\in\tilde F\colon Hb(x) > \tfrac{1}{2}\lambda\} \subset \{x\in \tilde F\colon \sum_{i=1}^{\infty} i \sum_{h\neq 0} |(b\chi_{R_i})(T^h x)|\, h^{-2} + 8b^*(x) > \tfrac{1}{2}\lambda\},$$

we have

$$\int\limits_{\{x\in\tilde F: Hb(x)>\lambda/2\}} w\,d\mu \leqslant \int\limits_{\{x\in\tilde F: 8b^*(x)>\lambda/4\}} w\,d\mu + \int\limits_{\{x\in\tilde F: \sum_{i=1}^{\infty} i \sum_{h\neq 0} |(b\chi_{R_i})(T^h x)|\, h^{-2} > \lambda/4\}} w\,d\mu. \tag{3.19}$$

The part of Theorem (1.1) proved in Section 2 gives us a bound for the first integral of the right-hand side:

$$\frac{C}{\lambda} \int_X |b|\, w\,d\mu. \tag{3.20}$$

The second integral is bounded by

$$\frac{4}{\lambda} \sum_{i=1}^{\infty} \int_{\tilde F} i \sum_{h\neq 0} |(b\chi_{R_i})(T^h x)|\, h^{-2}\, w\,d\mu.$$

Since $\tilde F \subset -\tilde R_i$, this is not bigger than

$$\frac{4}{\lambda} \sum_{i=1}^{\infty} \int_{-\tilde R_i} i \sum_{h\neq 0} |(b\chi_{R_i})(T^h x)|\, h^{-2}\, w\,d\mu. \tag{3.21}$$

Note that $x\in -\tilde R_i$ implies $T^j x\notin R_i$ for $j\in \{-i, -(i-1), \ldots, i-1, i\} = J_i$.
As a consequence, (3.21) can be written in the form

$$\begin{aligned} &\frac{4}{\lambda} \sum_{i=1}^{\infty} \int_{-\tilde R_i} i \sum_{h\notin J_i} |(b\chi_{R_i})(T^h x)|\, h^{-2}\, w\,d\mu \\ &\qquad \leqslant \frac{4}{\lambda} \sum_{i=1}^{\infty} i \sum_{h\notin J_i} \int_{R_i} |b(x)|\, h^{-2}\, w(T^{-h} x)\, d\mu \\ &\qquad \leqslant \frac{2}{\lambda} \sum_{i=1}^{\infty} \int_{R_i} (2i+1)\, |b(x)| \sum_{h\notin J_i} w(T^{-h} x)\, h^{-2}\, d\mu. \end{aligned}$$

By (3.16) we have that this is bounded by

$$\frac{C}{\lambda} \int_{O_\lambda} |b(x)|\, w^*(x)\, d\mu.$$

Since $w\in A_1$, we finally get

$$\int\limits_{\{x\in\tilde F: \sum_{i=1}^{\infty} i \sum_{h\neq 0} |(b\chi_{R_i})(T^h x)|\, h^{-2} > \lambda/4\}} w\,d\mu \leqslant \frac{C}{\lambda} \int_X |b|\, w\,d\mu. \tag{3.22}$$

Now Theorem (1.1) follows from (3.5), (3.6), (3.7), (3.8), (3.19), (3.20) and (3.22).

References

[1] E. Atencia and A. de la Torre, *A dominated ergodic estimate for L_p spaces with weights*, Studia Math. 74 (1982), 35–47.

[2] E. Atencia and F. J. Martin-Reyes, *The maximal ergodic Hilbert transform with weights*, Pacific J. Math. 108 (2) (1983), 257–263.

[3] C. Fefferman and E. M. Stein, *Some maximal inequalities*, Amer. J. Math. 1 (1971), 107–115.

[4] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), 207–226.

[5] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, 1970.

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On a generalized Carleson inequality

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Abstract. In this note we prove a generalized Carleson inequality

$$\Big|\iint_{R^2_+} F(x,t)\,v(x,t)\,dx\,dt\Big| \leqslant C\int_R A_p(F)(x)\,v_{*p'}(x)\,dx,$$

where $1/p+1/p' = 1$, $1 \leqslant p \leqslant \infty$,

$$A_p(F)(x) = \Big(\iint_{\Gamma(x)} |F(y,t)|^p \frac{dy\,dt}{t}\Big)^{1/p}, \qquad v_{*p'}(x) = \sup_{x\in I}\Big(\frac{1}{|I|}\iint_{\hat I} |v(y,t)|^{p'}\,dy\,dt\Big)^{1/p'}.$$

Moreover, $v_{*p'}$ belongs to the Muckenhoupt class A_1 for $p' > 1$.

1. Introduction. The inequality

$$\Big|\iint_{R^2_+} F(x,t)\,v(x,t)\,dx\,dt\Big| \leqslant C\int_R F^*(x)\,dx \ (*) \tag{1}$$

is known as the Carleson inequality ([4], [5], p. 236), where $F^*(x)$ is the non-tangential maximal function of $F(x,t)$, i.e.,

$$F^*(x) = \sup_{|y-x|<t} |F(y,t)|,$$

and $v(x,t)\,dx\,dt$ is a Carleson measure on R^2_+, i.e., $v(x,t) \geqslant 0$ and

$$\frac{1}{|I|}\int_{I\times[0,|I|]} v(x,t)\,dx\,dt \leqslant C$$

for any interval I on R. The purpose of this note is to give a more general form of inequality (1). To prove this we need to prove that a new kind of a maximal function gives rise to weights in A_1. This is of independent interest. Our inequality incorporates various inequalities proved by C. Fefferman and E. M. Stein and easily extends to R^n or, more generally, to the spaces of homogeneous type.

(*) As usual, throughout this note C will denote a constant not necessarily the same at each occurrence.