lemma a positive y_1 such that

$$\tilde{\Phi}_\tau(y^p)^\lambda \leqslant \Phi_\sigma(y)$$

for $y > y_1$. By the relation in the remark for $x < y_1^{-1}$ we get

$$\Phi_\tau(x^p)^\lambda \geqslant \Phi_\sigma(x).$$

Applying Proposition 2 to $\Phi := \Phi_\tau$ and $\psi := \Phi_\sigma$, we see that $\mathscr{E}(D_{\Phi_\sigma})$ does not have property $(\mathrm{DN}_{\Phi_\tau^{-1}})$. But, due to Proposition 1, the space $\mathscr{E}(D_{\Phi_\tau})$ has this property and so the spaces are not isomorphic. Setting $K_\tau := D_{\Phi_\tau}$ for $\tau \in [a, b]$ the corollary is proved.

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The canonical seminorm on Weak L^1

by

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Abstract. For each $f \in \mathrm{Weak}\ L^1$ let $q_1(f) = \sup_{\alpha>0} \alpha\mu(\{x \mid |f(x)| > \alpha\})$ and let $q(f)$ be the seminorm,

$$q(f) = \inf_{f=f_1+f_2+\ldots+f_n} \sum_{j=1}^{n} q_1(f_j).$$

It is known that q is equivalent to the seminorm I defined by

$$I(f) = \lim_{n\to\infty} \{\sup_{b/a>n} (\log b/a)^{-1} \int_{\{x \mid a \leqslant |f(x)| \leqslant b\}} |f(x)|\, d\mu\}.$$

It is shown here that in fact $q(f) = I(f)$ and also that the normed quotient space of WeakL^1 generated by q is not complete.

0. Introduction. This note is a sequel to [1]. We shall assume familiarity with the terminology and notation of that paper in which it was shown that, for a non-atomic underlying measure space, the canonical seminorm q on the space of measurable functions WeakL^1 is equivalent to a more "concretely" defined seminorm I. We shall show here that the seminorms q and I are in fact equal, using a refinement of the argument in [1]. We also exhibit another two seminorms which are equivalent to q and show that W, the quotient space of WeakL^1 modulo the functions f satisfying $q(f) = 0$, is not complete, as incorrectly claimed in [1].

We gratefully acknowledge correspondence with Nigel Kalton who expressed doubts about the claim in [1].

1. Equality of q and I. In order to establish that $q(f) = I(f)$ for all $f \in \mathrm{Weak}L^1$ it suffices to show that $q(f) \leqslant I(f)$ for each function of the form $f = \sum_{k=-\infty}^{\infty} \lambda^k \chi_{I_k}$, where $\lambda > 1$, and I_k are disjoint measurable sets of finite measure (cf. [1], pp. 151–152). In [1] the sets I_k were taken as intervals on the real line. However, for our purposes here it is a little simpler to consider them as (disjoint) circles. More specifically we assume that the underlying measure space (X, Σ, μ) contains each I_k and each Lebesgue measurable

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subset of I_k as elements of Σ and that μ restricted to any I_k coincides with Lebesgue measure (generated by arc length) on I_k. It is of course a routine matter to extend our results from this particular case to the case of any non-atomic measure space.

Fix an integer N. Divide each circle I_k into N arcs of equal length by N equally spaced points t_{kj}, $j = 1, 2, \ldots, N$. Define

$$g_j(t) = \sum_{k=-\infty}^{\infty} \frac{\lambda^k |I_k|}{d_k(t, t_{kj}) + |I_k|/N} \chi_{I_k}(t),$$

where $|I_k| = \mu(I_k)$ is the circumference of I_k and, for any two points s, t of I_k, $d_k(s, t)$ is the length of the shorter arc in I_k joining s and t.

We now claim that

$$(1) \qquad f(t) \leqslant \big(2N(\log(N/2)-1)\big)^{-1} \sum_{j=1}^{N} g_j(t)$$

for all $t \in X$.

To prove (1) suppose that $t \in I_k$ for some k. If, for example, t lies in the arc (of length $|I_k|/N$) from t_{kN} to t_{k1}, then

$$d_k(t, t_{kj}) \leqslant \frac{|I_k|}{N} \min(j, N+1-j)$$

and

$$\sum_{j=1}^{N} g_j(t) \geqslant 2 \sum_{m=1}^{[N/2]} \frac{\lambda^k N}{m+1} \geqslant 2Nf(t)\big[\log([N/2]+1)-1\big],$$

where $[N/2]$ is the integer part of $N/2$. By symmetry the same estimate holds for t between t_{kj} and $t_{k,j+1}$ for any j, i.e. for all t in any I_k. This establishes (1), which in turn shows that

$$q(f) \leqslant \big(2N(\log(N/2)-1)\big)^{-1} \sum_{j=1}^{N} q_1(g_j) = q_1(g_1)/2\big(\log(N/2)-1\big)$$

since all of the functions g_j have the same distribution function. For $\alpha > 0$ the set $E_k(\alpha) = I_k \cap \{t|\ g_1(t) > \alpha\}$ coincides with I_k if $\alpha < \lambda^k(1/N+1/2)^{-1}$ and is empty if $\alpha \geqslant \lambda^k N$. If $\lambda^k(1/2+1/N)^{-1} \leqslant \alpha < \lambda^k N$, then $\mu(E_k(\alpha)) = 2|I_k|(\lambda^k/\alpha - 1/N)$. It follows that

$$\alpha\mu(\{t|\ g_1(t) > \alpha\}) \leqslant \alpha \sum_{\alpha < \lambda^k(1/2+1/N)^{-1}} |I_k| + \alpha \sum_{\lambda^k(1/2+1/N)^{-1} \leqslant \alpha < \lambda^k N} 2|I_k|\lambda^k/\alpha$$
$$\leqslant \alpha\mu(\{t|\ f(t) > \alpha(1/2+1/N)\}) + 2 \int_{\{t|\alpha/N \leqslant f(t) \leqslant \alpha(1/2+1/N)\}} f(t)\, d\mu(t).$$

We conclude that

$$q_1(g_1) \leqslant 2q_1(f) + 2 \sup_{b/a = N/2+1} \int_{\{t|a \leqslant f(t) \leqslant b\}} f(t)\, d\mu(t).$$

Dividing both sides of this inequality by $2(\log(N/2)-1)$ and letting N tend to infinity we obtain that $q(f) \leqslant I(f)$.

2. An auxiliary estimate. Below we shall need an estimate for $q(f)$ for f of the above form $f = \sum_{k=-\infty}^{\infty} \lambda^k \chi_{I_k}$ in the special case where, for a fixed integer N, $|I_k| = (2/(2+N))^k$ and $\lambda = ((2+N)/2)$. Using the notation and decomposition above we again have $q(f) \leqslant q_1(g_1)/2(\log(N/2)-1)$. A routine calculation shows that $q_1(g_1) = 2$ (in fact $g_1^*(t) = 2/t$) so $q(f) \leqslant 1/(\log(N/2)-1)$.

3. Further equivalent seminorms. Let

$$K(f) = \lim_{h \downarrow 1} \Big\{ \sup_{\substack{\alpha > 0 \\ 1 < p < h}} \frac{(1-1/p)^{1/p}}{\alpha^{1-1/p}} \big(\textstyle\int \min(\alpha, |f|^p\, d\mu)^{1/p}\Big\}$$

and

$$L(f) = \lim_{N \to \alpha} \{\sup_{\alpha t \geqslant N} (1+\log \alpha t)^{-1} \int_0^t \min(\alpha, f^*(s))\, ds\}.$$

For each $f \in \mathrm{Weak}\,L^1$ it is not difficult to verify that $K(f)$ and $L(f)$ are seminorms dominated by $q_1(f)$ and thus by $q(f)$. Using arguments similar to those of [1], p. 152, we can also show that $q(f) \leqslant eK(f)$ and in fact $q(f) = L(f)$.

4. The space W. As in [1] we let W denote the quotient space of Weak L^1 modulo the subspace of functions f satisfying $q(f) = 0$. This is a normed space (normed by q) whose dual coincides with the dual of WeakL^1. We shall show here that W is not complete, thus correcting a false assertion in [1].

LEMMA. *Let a, b, c, d be positive constants such that $b < 1$ and $c < d$. Let $V(t)$ be a function assuming the constant value d/ab^k on the interval $[ab^{k+1}, ab^k)$ for each $k = 0, \pm 1, \pm 2, \ldots$*

If $U(t) \geqslant \max(V(t) - c/t, 0)$, then $q(U) > 0$.

Proof. $U(t) \geqslant (d - \sqrt{cd})/ab^k$ on the interval $[\sqrt{c/d}\, ab^k, ab^k)$ for each integer k. It follows that for each $\alpha > 0$ the set $\{t|\ U(t) > \alpha\}$ has measure exceeding D/α for some constant $D > 0$. Therefore $q(U) \geqslant D$. □

For each integer N we define a function V_N as in the preceding lemma by taking $a = d = (2+N)/N$ and $b = 2/(2+N)$, i.e. $V_N = ((2+N)/2)^k$ on the interval $I_k = \big[((2+N)/N)(2/(2+N))^{k+1}, ((2+N)/N)(2/(2+N))^k\big]$. Since $|I_k| = (2/(2+N))^k$ the calculation of Section 2 shows that $q(V_N) \leqslant 1/(\log(N/2)-1)$. Now let

$(m_N)_{N=1}^{\infty}$ be an increasing sequence of integers such that $\log(m_N/2)-1 \geqslant N^3$. Let $Y_N(t) = NV_{m_N}(t)$ so that $q(Y_N) \leqslant 1/N^2$. The sequence $(\Phi_n)_{n=1}^{\infty}$ defined by $\Phi_n = \sum_{N=1}^{n} Y_N$ is thus a Cauchy sequence with respect to the seminorm q.

If W is complete, then there exists a function Φ in WeakL^1 such that $q(\Phi-\Phi_n) \to 0$. For each $t>0$, $\Phi_n(t) = \Phi_n^*(t) \leqslant (\Phi_n-\Phi)^*(t/2)+\Phi^*(t/2)$. (See e.g. [2], p. 253.)

Consequently $\max(\Phi_n(t)-\Phi^*(t/2), 0) \leqslant (\Phi_n-\Phi)^*(t/2)$ and since $\Phi^*(t/2) \leqslant c/t$ for some constant c we deduce that $q\big(\max(Y_N(t)-c/t, 0)\big) = 0$ for all N. However, for N sufficiently large, $N(2+m_N)/m_N > c$ and, by the preceding lemma, $q\big(\max(Y_N(t)-c/t, 0)\big) > 0$. This contradiction shows that W cannot be complete.

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A further generalization of Ky Fan's minimax inequality and its applications

by

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Abstract. The celebrated 1972 Ky Fan's minimax inequality is slightly generalized simultaneously to non-compact convex settings and to a pair of functions. This extension includes Brézis–Nirenberg–Stampacchia's minimax inequality. Applying the generalized minimax inequality, Dugundji–Granas' variational inequality in reflexive Banach spaces, which is an extension of Hartman–Stampacchia variational inequality, is generalized simultaneously to set-valued maps and to non-compact convex sets in topological vector spaces. The generalized variational inequality in the single-valued case is in turn used to obtain fixed point theorems for pseudo-contractive and non-expansive maps on a non-weakly compact subset of a Hilbert space, generalizing the well-known Browder's fixed point theorem.

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1. Introduction. We begin with the celebrated 1972 Ky Fan's minimax inequality [11].

[KY FAN'S MINIMAX INEQUALITY]. *Let X be a non-empty compact convex set in a Hausdorff topological vector space. Let φ be a real-valued function defined on $X \times X$ such that:*

(a) *For each fixed $x \in X$, $\varphi(x, y)$ is a lower semicontinuous function of y on X.*

(b) *For each fixed $y \in X$, $\varphi(x, y)$ is a quasi-concave function of x on X. Then the minimax inequality*

$$\min_{y\in X} \sup_{x\in X} \varphi(x, y) \leqslant \sup_{x\in X} \varphi(x, x)$$

holds.

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