$(m_N)_{N=1}^{\infty}$ be an increasing sequence of integers such that $\log(m_N/2)-1 \geqslant N^3$. Let $Y_N(t) = NV_{m_N}(t)$ so that $q(Y_N) \leqslant 1/N^2$. The sequence $(\Phi_n)_{n=1}^{\infty}$ defined by $\Phi_n = \sum_{N=1}^{n} Y_N$ is thus a Cauchy sequence with respect to the seminorm q.

If W is complete, then there exists a function Φ in WeakL^1 such that $q(\Phi-\Phi_n) \to 0$. For each $t > 0$, $\Phi_n(t) = \Phi_n^*(t) \leqslant (\Phi_n-\Phi)^*(t/2)+\Phi^*(t/2)$. (See e.g. [2], p. 253.)

Consequently $\max(\Phi_n(t)-\Phi^*(t/2), 0) \leqslant (\Phi_n-\Phi)^*(t/2)$ and since $\Phi^*(t/2) \leqslant c/t$ for some constant c we deduce that $q\big(\max(Y_N(t)-c/t, 0)\big) = 0$ for all N. However, for N sufficiently large, $N(2+m_N)/m_N > c$ and, by the preceding lemma, $q\big(\max(Y_N(t)-c/t, 0)\big) > 0$. This contradiction shows that W cannot be complete.

References

[1] M. Cwikel and C. Fefferman, *Maximal seminorms on* WeakL^1, Studia Math. 69 (1980), 149–154.

[2] R. A. Hunt, *On L(p, q) spaces*, Enseignement Math. (2) 12 (1966), 249–276.

DEPARTMENT OF MATHEMATICS
TECHNION, I. I. T., HAIFA 32000, ISRAEL

DEPARTMENT OF MATHEMATICS
PRINCETON UNIVERSITY, PRINCETON N. J. 08540, U.S.A.

Received August 17, 1982 (1791)

STUDIA MATHEMATICA, T. LXXVIII. (1984)

A further generalization of Ky Fan's minimax inequality and its applications

by

MAU-HSIANG SHIH* and KOK-KEONG TAN**

Abstract. The celebrated 1972 Ky Fan's minimax inequality is slightly generalized simultaneously to non-compact convex settings and to a pair of functions. This extension includes Brézis–Nirenberg–Stampacchia's minimax inequality. Applying the generalized minimax inequality, Dugundji–Granas' variational inequality in reflexive Banach spaces, which is an extension of Hartman–Stampacchia variational inequality, is generalized simultaneously to set-valued maps and to non-compact convex sets in topological vector spaces. The generalized variational inequality in the single-valued case is in turn used to obtain fixed point theorems for pseudo-contractive and non-expansive maps on a non-weakly compact subset of a Hilbert space, generalizing the well-known Browder's fixed point theorem.

1980 Mathematical Subject Classifications. Primary 49A29, 49A40; Secondary 47H10, 52A07.

Key words and phrases. KKM-map, minimax inequality, variational inequality, pseudo-contractive map, non-expansive map, monotone operator, Hausdorff topological vector space, fixed point, set-valued maps.

1. Introduction. We begin with the celebrated 1972 Ky Fan's minimax inequality [11].

[KY FAN'S MINIMAX INEQUALITY]. *Let X be a non-empty compact convex set in a Hausdorff topological vector space. Let φ be a real-valued function defined on $X \times X$ such that:*

(a) *For each fixed $x \in X$, $\varphi(x, y)$ is a lower semicontinuous function of y on X.*

(b) *For each fixed $y \in X$, $\varphi(x, y)$ is a quasi-concave function of x on X. Then the minimax inequality*

$$\min_{y\in X} \sup_{x\in X} \varphi(x, y) \leqslant \sup_{x\in X} \varphi(x, x)$$

holds.

---

* The work was done while the author was visiting at Dalhousie University, and was partially supported by NSERC of Canada.

** The author was partially supported by NSERC of Canada under grant number A-8096.

Here, a real-valued function φ defined on a convex set X is said to be *quasi-concave* if for every real number t the set $\{x \in X\colon \varphi(x) > t\}$ is convex. Ky Fan's minimax inequality is an important tool in non-linear functional analysis [11], game theory and economic theory [1]. On the other hand, among the various extensions of Ky Fan's minimax inequality, an important one is due to Brézis, Nirenberg and Stampacchia [3]. In the present paper, we shall follow Brézis–Nirenberg–Stampacchia's idea to prove a general minimax inequality in Theorem 1. As an intrinsic application of Theorem 1, Dugundji–Granas' variational inequality [8] and [9], pp. 75–76, which is an extension of Hartman–Stampacchia's variational inequality [12], is generalized simultaneously to set-valued maps and to non-compact convex sets in a topological vector space. The generalized variational inequality in the single-valued case is in turn used to obtain fixed point theorems for pseudo-contractive maps and non-expansive maps on a non-weakly compact convex subset of a Hilbert space, generalizing the well-known Browder's fixed point theorem for non-expansive maps [4], Theorem 1.

2. Minimax inequalities. Let E be a vector space. We shall denote by 2^E the set of all subsets of E and by $\operatorname{conv}(A)$ the convex hull of $A \in 2^E$. Let X be an arbitrary non-empty subset of E. A map $F\colon X \to 2^E$ is called a *KKM-map* [8] and [9], p. 72, if $\operatorname{conv}\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^{n} F(x_i)$ for each finite subset $\{x_1, \ldots, x_n\}$ of X.

We shall establish the following:

THEOREM 1. *Let X be a non-empty closed convex set in a Hausdorff topological vector space E. Let φ and ψ be two real-valued functions on $X \times X$ having the following properties:*

(a) *$\varphi(x, y) \leqslant \psi(x, y)$ for all $(x, y) \in X \times X$.*

(b) *For each fixed $x \in X$, $\varphi(x, y)$ is a lower semicontinuous function of y on the intersection of X with any finite-dimensional subspace of E.*

(c) *For each fixed $y \in X$, the set $\{x \in X\colon \psi(x, y) > 0\}$ is convex.*

(d) *Whenever $x, y \in X$ and $(y_\alpha)_{\alpha \in \Gamma}$ is a net in X converging to y, then the inequalities $\varphi(tx + (1-t)y, y_\alpha) \leqslant 0$ for all $\alpha \in \Gamma$ and for all $t \in [0, 1]$ imply $\varphi(x, y) \leqslant 0$.*

(e) *There exist a non-empty compact (not necessarily convex) subset K of E and $x_0 \in X \cap K$ such that $\varphi(x_0, y) > 0$ for all $y \in X \setminus K$.*

Then either there exists a point $\hat{x} \in X$ such that $\psi(\hat{x}, \hat{x}) > 0$ or there exists a point $\hat{y} \in X \cap K$ such that $\varphi(x, \hat{y}) \leqslant 0$ for all $x \in X$.

The proof of Theorem 1 is based on the following important lemma of Brézis–Nirenberg–Stampacchia [3] which is an extension of Ky Fan's generalization [10] of Knaster–Kuratowski–Mazurkiewicz's geometric result [13]:

LEMMA (Brézis–Nirenberg–Stampacchia). *Let X be an arbitrary non-empty set in a Hausdorff topological vector space E. Let $F\colon X \to 2^E$ be a KKM-map such that:*

(a) *$\overline{F(x_0)}$ is compact for some $x_0 \in X$.*

(b) *For every $x \in X$, the intersection of $F(x)$ with any finite-dimensional subspace is closed.*

(c) *For any convex subset D of E we have*

$$\Big(\overline{\bigcap_{x \in X \cap D} F(x)}\Big) \cap D = \Big(\bigcap_{x \in X \cap D} F(x)\Big) \cap D.$$

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

Proof of Theorem 1. For each $x \in X$, let

$$F(x) = \{y \in X\colon \varphi(x, y) \leqslant 0\}, \qquad G(x) = \{y \in X\colon \psi(x, y) \leqslant 0\}.$$

If $G\colon X \to 2^E$ is not a KKM-map, then for some choice $\{u_1, \ldots, u_n\} \subset X$ and $\alpha_j \geqslant 0$ $(1 \leqslant j \leqslant n)$ with $\sum_{j=1}^{n} \alpha_j = 1$, we have $\sum_{j=1}^{n} \alpha_j u_j \notin \bigcup_{i=1}^{n} \psi(u_i)$, i.e., $\psi(u_i, \sum_{j=1}^{n} \alpha_j u_j) > 0$ for $1 \leqslant i \leqslant n$, so that by (c), $\psi(\sum_{j=1}^{n} \alpha_j u_j, \sum_{j=1}^{n} \alpha_j u_j) > 0$ and hence the conclusion of Theorem 1 holds by taking $\hat{x} = \sum_{j=1}^{n} \alpha_j u_j$. Thus we may assume $G\colon X \to 2^E$ is a KKM-map. By (a), $G(x) \subset F(x)$ for each $x \in X$, so that $F\colon X \to 2^E$ is also a KKM-map, and we have:

(i) By (e), $F(x_0) \subset X \cap K$; hence $\overline{F(x_0)} \subset K$ and so $\overline{F(x_0)}$ is compact.

(ii) Let $x \in X$ and L be any finite-dimensional subspace of E. By (b),

$$F(x) \cap L = \{y \in X \cap L\colon \varphi(x, y) \leqslant 0\}$$

is closed.

(iii) Let D be any convex subset of E. Let $y \in \overline{\bigcap_{x \in X \cap D} F(x)} \cap D$; then $y \in D$ and there exists a net $(y_\alpha)_{\alpha \in \Gamma}$ in $\bigcap_{x \in X \cap D} F(x)$ such that $y_\alpha \to y$. Thus

$$(*) \qquad \varphi(x, y_\alpha) \leqslant 0 \text{ for all } \alpha \in \Gamma \text{ and for all } x \in X \cap D.$$

Since X is closed (*this is an essential condition for our proof*), $y \in X \cap D$. As $x, y \in X \cap D$ and $X \cap D$ is convex, $tx + (1-t)y \in X \cap D$ for all $t \in [0, 1]$; it follows from $(*)$ that

$$\varphi(tx + (1-t)y, y_\alpha) \leqslant 0 \text{ for all } \alpha \in \Gamma \text{ and for all } x \in X \cap D.$$

By (d), $\varphi(x, y) \leqslant 0$ for all $x \in X \cap D$, so that $y \in F(x)$ for all $x \in X \cap D$ and hence $y \in \left(\bigcap_{x \in X \cap D} F(x)\right) \cap D$. Therefore,

$$\overline{\left(\bigcap_{x \in X \cap D} F(x)\right)} \cap D = \left(\bigcap_{x \in X \cap D} F(x)\right) \cap D.$$

Now applying Brézis–Nirenberg–Stampacchia's lemma to F, we have

$$\bigcap_{x \in X} F(x) \neq \emptyset.$$

Choose an $\hat{y} \in \bigcap_{x \in X} F(x)$; then $\varphi(x, \hat{y}) \leqslant 0$ for all $x \in X$, and the proof is complete. □

Observe that in the case where X is compact, condition (e) in Theorem 1 is satisfied with $K = X$. In the case $\varphi = \psi$, Theorem 1 reduces to Brézis–Nirenberg Stampacchia's inequality [3]. Note that Theorem 1 implicitly implies the following minimax inequality:

THEOREM 2. *Let X be a non-empty closed convex set in a Hausdorff topological vector space E. Let φ_1 and φ_2 be two real-valued functions on $X \times X$ having the following properties:*

(a) *$\varphi_1(x, y) \leqslant \varphi_2(x, y)$ for all $(x, y) \in X \times X$.*

(b) *For each fixed $x \in X$, $\varphi_1(x, y)$ is a lower semicontinuous function of y on the intersection of X with any finite-dimensional subspace of E.*

(c) *For each fixed $y \in X$, the set $\{x \in X: \varphi_2(x, y) > 0\}$ is convex.*

(d) *Whenever $x, y \in X$ and $(y_\alpha)_{\alpha \in \Gamma}$ is a net in X converging to y, then the inequalities $\varphi_1(tx + (1-t)y, y_\alpha) \leqslant 0$ for all $\alpha \in \Gamma$ and for all $t \in [0, 1]$ imply $\varphi_1(x, y) \leqslant 0$.*

(e) *There exist a non-empty compact (not necessarily convex) subset K of E and $x_0 \in X \cap K$ such that whenever $\sup_{x \in X} \varphi_2(x, x) < \infty$, $\varphi_1(x_0, y) > \sup_{x \in X} \varphi_2(x, x)$ for all $y \in X \setminus K$.*

Then the minimax inequality

$$\inf_{y \in K} \sup_{x \in X} \varphi_1(x, y) \leqslant \sup_{x \in X} \varphi_2(x, x)$$

holds.

Proof. Let $t = \sup_{x \in X} \varphi_2(x, x)$. Clearly, we may assume that $t < +\infty$. Applying Theorem 1 to

$$\varphi(x, y) = \varphi_1(x, y) - t, \qquad \psi(x, y) = \varphi_2(x, y) - t,$$

the conclusion follows. □

Observe that if $\varphi_1(x, y)$ is a lower semicontinuous function of y on X, then $\sup_{x \in X} \varphi_1(x, y)$ is also a lower semicontinuous function of y on X, and

therefore its minimum $\min_{y \in K} \sup_{x \in X} \varphi_1(x, y)$ on the compact set K exists. In the case where X is compact and $\varphi_1(x, y)$ is a lower semicontinuous function of y on X, by setting $K = X$ and $\varphi_1 = \varphi_2$, Theorem 2 reduces to Ky Fan's minimax inequality, and by setting $K = X$, Theorem 2 reduces to [14], Theorem 1.

3. Variational inequalities. In [8] and [9], pp. 75–76, Dugundji and Granas gave a fairly general version of Hartman–Stampacchia's variational inequality [12]. Below, by using Theorem 1 directly, Dugundji–Granas' variational inequality in reflexive Banach spaces is generalized simultaneously to set-valued maps and to non-compact sets in a topological vector space.

Let E be a Hausdorff topological vector space. We shall denote by E' the dual space of E (i.e., the vector space of all continuous linear functionals on E). We denote the pairing between E' and E by $\langle w, x\rangle$ for $w \in E'$ and $x \in E$. Let X be any non-empty subset of E; a (single-valued) map $f: X \to E'$ is said to be *monotone* on X if $\operatorname{Re}\langle f(y) - f(x), y - x\rangle \geqslant 0$ for all $x, y \in X$. A set-valued map $f: X \to 2^{E'}$ is said to be *monotone* on X [7], p. 79, if for all x and y in X, each u in $f(x)$, and each w in $f(y)$, $\operatorname{Re}\langle w - u, y - x\rangle \geqslant 0$. Let X and Y be two topological spaces. A set-valued map $f: X \to 2^Y$ is said to be *lower semicontinuous* on X [2], p. 109, if for every $x_0 \in X$ and any open set G in Y such that $f(x_0) \cap G \neq \emptyset$, there is a neighbourhood U of x_0 in X such that $f(x) \cap G \neq \emptyset$ for every $x \in U$. In other words, $f: X \to 2^Y$ is lower semicontinuous on X if for every open set G in Y, the set $\{x \in X: f(x) \cap G \neq \emptyset\}$ is open in X.

THEOREM 3. *Let X be a non-empty closed convex set in a Hausdorff topological vector space E and $f: X \to 2^{E'}$ be a set-valued map such that for each $x \in X$, $f(x)$ is a non-empty subset of E', and that f is monotone. Assume that for each one-dimensional flat $L \subset E$, $f | L \cap X$ is lower semicontinuous from the topology of E to the weak*-topology $\sigma(E', E)$ of E' and that there exist a non-empty weakly compact (not necessarily convex) subset K of E and $y_0 \in K \cap X$ such that for each $x \in X \setminus K$, there exists $u \in f(x)$ with $\operatorname{Re}\langle u, y_0 - x\rangle > 0$. Then there exists a point $\hat{y} \in X \cap K$ such that*

$$\sup_{w \in f(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x\rangle \leqslant 0 \quad \textit{for all } x \in X.$$

Proof. By monotonicity of f, for each $x, y \in X$, $u \in f(x)$ and $w \in f(y)$ we have

$$\operatorname{Re}\langle u, y - x\rangle \leqslant \operatorname{Re}\langle w, y - x\rangle.$$

Thus

$$\sup_{u \in f(x)} \operatorname{Re}\langle u, y - x\rangle \leqslant \inf_{w \in f(y)} \langle w, y - x\rangle \quad \text{for all } x, y \in X.$$

For each $x, y \in X$, define

$$\varphi(x, y) = \sup_{u \in f(x)} \operatorname{Re}\langle u, y-x\rangle, \qquad \psi(x, y) = \inf_{w \in f(y)} \operatorname{Re}\langle w, y-x\rangle.$$

Then

(i) $\varphi(x, y) \leqslant \psi(x, y)$ for all $x, y \in X$, and $\psi(x, x) = 0$ for all $x \in X$.

(ii) It is easy to check that for each fixed $x \in X$, $\varphi(x, y)$ is a weakly lower semicontinuous functions of y on X.

(iii) For each fixed $y \in X$, it is easy to see that $\{x \in X: \psi(x, y) > 0\}$ is convex.

(iv) By hypothesis, there exist a non-empty weakly compact subset K of E and $y_0 \in K \cap X$ such that, for each $x \in X \setminus K$, there exists $u \in f(x)$ with $\operatorname{Re}\langle u, y_0 - x\rangle > 0$; it follows that for each $x \in X \setminus K$

$$\varphi(x, y_0) = \sup_{u \in f(x)} \operatorname{Re}\langle u, y_0 - x\rangle > 0.$$

(v) Suppose that $x, y \in X$, $(y_\alpha)_{\alpha \in \Gamma}$ is a net in X with $y_\alpha \to y$ weakly such that

$$\varphi(tx + (1-t)y, y_\alpha) \leqslant 0 \quad \text{for all } \alpha \in \Gamma \text{ and for all } t \in [0, 1].$$

Then $\varphi(x, y_\alpha) \leqslant 0$ for all $\alpha \in \Gamma$; by (ii), $\varphi(x, y) \leqslant 0$.

We now equip E with the weak topology and we find that all the conditions in Theorem 1 are satisfied; therefore there exists a point $\hat{y} \in K \cap X$ with $\varphi(x, \hat{y}) \leqslant 0$ for all $x \in X$; in other words,

$$(**) \qquad \sup_{u \in f(x)} \operatorname{Re}\langle u, \hat{y} - x\rangle \leqslant 0 \quad \text{for all} \quad x \in X.$$

Let $x \in X$ be arbitrarily fixed, let $z_t = tx + (1-t)\hat{y} \equiv \hat{y} - t(\hat{y} - x)$ for $t \in [0, 1]$. As X is convex, we have $z_t \in X$ for $t \in [0, 1]$. Therefore, by $(**)$ we have

$$\sup_{u \in f(z_t)} \operatorname{Re}\langle u, \hat{y} - z_t\rangle \leqslant 0 \quad \text{for all} \quad t \in [0, 1],$$

so that $t \cdot \sup_{u \in f(z_t)} \operatorname{Re}\langle u, \hat{y} - x\rangle \leqslant 0$ for all $t \in [0, 1]$ and it follows that

$$(*{*}*) \qquad \sup_{u \in f(z_t)} \operatorname{Re}\langle u, \hat{y} - x\rangle \leqslant 0 \quad \text{for all} \quad t \in (0, 1].$$

Let $w_0 \in f(\hat{y})$ be arbitrarily fixed. For each $\varepsilon > 0$, let

$$U_{w_0} = \{w \in E': |\langle w_0 - w, \hat{y} - x\rangle| < \varepsilon\};$$

then U_{w_0} is a $\sigma(E', E)$-neighbourhood of w_0. Since f is lower semicontinuous, and $U_{w_0} \cap f(\hat{y}) \neq \emptyset$, there exists a neighbourhood $N(\hat{y})$ of $\hat{y}$ such that $z \in N(\hat{y})$ implies $f(z) \cap U_{w_0} \neq \emptyset$. Note that $z_t \to \hat{y}$ as $t \to 0^+$, thus there exists $\delta \in (0, 1)$ such that for all $t \in (0, \delta)$, we have $z_t \in N(\hat{y})$. But then $f(z_t) \cap U_{w_0} \neq \emptyset$ for $t \in (0, \delta)$; take any $u \in f(z_t) \cap U_{w_0}$, we have $|\langle w_0 - u, \hat{y} - x\rangle| < \varepsilon$. This implies

$$\operatorname{Re}\langle w_0, \hat{y} - x\rangle < \operatorname{Re}\langle u, \hat{y} - x\rangle + \varepsilon.$$

By $(*{*}*)$, $\operatorname{Re}\langle w_0, \hat{y} - x\rangle < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\operatorname{Re}\langle w_0, \hat{y} - x\rangle \leqslant 0$. As $w_0 \in f(\hat{y})$ is arbitrary,

$$\sup_{w \in f(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x\rangle \leqslant 0 \quad \text{for all} \quad x \in X.$$

This completes the proof. □

It would be of some interest to compare Theorem 3 with the following Browder's variational inequality [6], Theorem 6:

THEOREM 4 (Browder). *Let X be a non-empty compact convex subset of a locally convex Hausdorff topological vector space E and let $f: X \to 2^{E'}$ be upper semicontinuous such that for each $x \in X$, $f(x)$ is a non-empty compact convex subset of E'. Then there exist $\hat{y} \in X$ and $\hat{w} \in f(\hat{y})$ such that*

$$\operatorname{Re}\langle \hat{w}, \hat{y} - x\rangle \geqslant 0 \quad \textit{for all} \quad x \in X.$$

Here, given topological spaces X and Y, a set-valued map $f: X \to 2^Y$ is said to be *upper semicontinuous* [2] if for any point $x_0 \in X$ and any open set U in Y such that $f(x_0) \subset U$, there exists a neighbourhood W of x_0 such that $f(x) \subset U$ for all $x \in W$.

The following result is the single-valued case of Theorem 3 which we shall need in Section 4:

THEOREM 5. *Let X be a non-empty closed convex set in a Hausdorff topological vector space E and let $f: X \to E'$ be monotone. Assume that for each one-dimensional flat $L \subset E$, $f|L \cap X$ is continuous from the topology of E to the weak*-topology $\sigma(E', E)$ of E' and that there exist a non-empty weakly compact (not necessarily convex) subset K of E and $y_0 \in K \cap X$ such that for each $x \in X \setminus K$, $\operatorname{Re}\langle f(x), y_0 - x\rangle > 0$. Then there exists a point $\hat{y} \in X \cap K$ such that*

$$\operatorname{Re}\langle f(\hat{y}), \hat{y} - x\rangle \leqslant 0 \quad \textit{for all} \quad x \in X.$$

4. Fixed point theorems. Let E be a normed linear space with norm $\|\cdot\|$ and let X be a non-empty subset of E. A map $f: X \to E$ is said to be *pseudo-contractive* [5] if for all $x, y \in X$ and for all $r > 0$,

$$\|x - y\| \leqslant \|(1+r)(x-y) - r(f(x) - f(y))\|.$$

A map $f: X \to E$ is said to be *non-expansive* if for all $x, y \in X$,

$$\|f(x) - f(y)\| \leqslant \|x - y\|.$$

It is obvious that if f is non-expansive, then f is pseudo-contractive since

$$\|(1+r)(x-y) - r(f(x) - f(y))\| \geqslant (1+r)\|x - y\| - r\|f(x) - f(y)\|.$$

The main interest in pseudo-contractive maps stems from the firm connection which exists between these maps and the important class of accretive operators [5], Proposition 1.

We can now establish the following new fixed point theorem by applying Theorem 5.

THEOREM 6. *Let X be a non-empty closed convex subset of a Hilbert space $(H, \langle\cdot,\cdot\rangle)$ and let $f\colon X \to H$ be pseudo-contractive. Suppose that $f|L \cap X$ is continuous for each one-dimensional flat $L \subset H$ and that there exist a non-empty weakly compact (not necessarily convex) subset K of E and $y_0 \in K \cap X$ such that for each $x \in X \setminus K$, $\mathrm{Re}\langle x-f(x), y_0-x\rangle > 0$. Then there exists a point $\hat{y} \in K \cap X$ such that*

$$\mathrm{Re}\langle \hat{y}-f(\hat{y}), \hat{y}\rangle = \operatorname*{Min}_{x\in X} \mathrm{Re}\langle \hat{y}-f(\hat{y}), x\rangle.$$

In particular, if f is a self-map on X, then $\hat{y}$ is a fixed point of f.

Proof. According to a result [5], Proposition 1, of Browder, $I-f$ is monotone. Applying Theorem 5 to $I-f$, there exists a point $\hat{y} \in K \cap X$ such that

$$\mathrm{Re}\langle \hat{y}-f(\hat{y}), \hat{y}-x\rangle \leqslant 0 \quad \text{for all} \quad x \in X.$$

Hence we have

$$\mathrm{Re}\langle \hat{y}-f(\hat{y}), \hat{y}\rangle = \operatorname*{Min}_{x\in X} \mathrm{Re}\langle \hat{y}-f(\hat{y}), x\rangle.$$

In particular, if f is a self-map on X, it follows that

$$\mathrm{Re}\langle \hat{y}-f(\hat{y}), \hat{y}-f(\hat{y})\rangle \leqslant 0,$$

so that $\hat{y}$ is a fixed point of f. This completes the proof. □

As an immediate consequence, we obtain

THEOREM 7. *Let X be a non-empty closed convex subset of a Hilbert space $(H, \langle\cdot,\cdot\rangle)$ and let $f\colon X \to H$ be non-expansive. If there exist a non-empty weakly compact (not necessarily convex) subset K of E and $y_0 \in K \cap X$ such that for each $x \in X \setminus K$, $\mathrm{Re}\langle x-f(x), y_0-x\rangle > 0$. Then there exists a point $\hat{y} \in K \cap X$ such that*

$$\mathrm{Re}\langle \hat{y}-f(\hat{y}), \hat{y}\rangle = \operatorname*{Min}_{x\in X} \mathrm{Re}\langle \hat{y}-f(\hat{y}), x\rangle.$$

In particular, if f is a self-map on X, then $\hat{y}$ is a fixed point of f.

In the case X is bounded and f is a self-map on X, by setting $K = X$, Theorem 7 reduces to Browder's fixed point theorem [4], Theorem 1.

References

[1] J. P. Aubin, *Mathematical methods of game and economic theory*, North-Holland, Amsterdam 1979.

[2] C. Berge, *Topological spaces*, Oliver and Boyd, Edinburgh and London 1963.

[3] H. Brézis, L. Nirenberg and G. Stampacchia, *A remark on Ky Fan's minimax principle*, Boll. Unione Mat. Ital. (4) 6 (1972), 293–300.

[4] F. E. Browder, *Fixed-point theorems for noncompact mappings in Hilbert space*, Proc. Nat. Acad. Sci. U.S.A. 53 (1965), 1272–1279.

[5] —, *Nonlinear mappings of nonexpansive and accretive type in Banach spaces*, Bull. Amer. Math. Soc. 73 (1967), 875–882.

[6] —, *The fixed point theory of multi-valued mappings in topological vector spaces*. Math. Ann. 177 (1968), 283–301.

[7] —, *Nonlinear operators and nonlinear equations of evolution in Banach spaces*, Proc. Sympos. in Pure Math., Vol. 18, Part 2, Amer. Math. Soc., Providence, R.I., 1976.

[8] J. Dugundji and A. Granas, *KKM maps and variational inequalities*, Ann. Scuola Norm. Sup. Pisa (Ser. 4) 5 (1978), 679–682.

[9] —, —, *Fixed point theory*, Vol. 1, Warszawa 1982.

[10] K. Fan, *A generalization of Tychonoff's fixed point theorem*, Math. Ann. 142 (1961), 305–310.

[11] —, *A minimax inequality and applications*, *Inequalities III*, (Edited by O. Shisha), Academic Press, New York and London 1972.

[12] P. Hartman and G. Stampacchia, *On some nonlinear elliptic differential equations*, Acta Math. 115 (1966), 271–310.

[13] B. Knaster, C. Kuratowski and S. Mazurkiewicz, *Ein Beweis des Fixpunktsatzes für n-dimensionale Simplexe*, Fund. Math. 14 (1929), 132–137.

[14] C. L. Yen, *A minimax inequality and its applications to variational inequalities*, Pacific J. Math. 97 (1981), 477–481.

DEPARTMENT OF MATHEMATICS, CHUNG YUAN UNIVERSITY, CHUNG-LI, TAIWAN 320, TAIWAN

DEPARTMENT OF MATHEMATICS, DALHOUSIE UNIVERSITY, HALIFAX, NOVA SCOTIA, CANADA B3H 4H8

Received September 30, 1982 (1803)