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Removability of ideals in commutative Banach algebras

by

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Abstract. A countable family of removable ideals in a commutative Banach algebra is removable.

Introduction. Let A be a commutative Banach algebra with unit. An ideal I in A is called *removable* if there exists a superalgebra $B \supset A$ (i.e., B is a commutative Banach algebra with unit and there is an isometric unit preserving isomorphism $f\colon A \to B$) such that I is not contained in a proper ideal in B. A family $\{I_j\}_{j\in J}$ is called *removable* if there is a superalgebra $B \supset A$ such that I_j is not contained in a proper ideal in B for every $j \in J$.

These notion were introduced by R. Arens [1] where also the following question was raised: Is every (every finite) family of removable ideals removable?

In general the answer is negative as was shown by B. Bollobás [2]. He presented an example of an uncountable family of removable ideals which is not removable. There was also shown that we can adjoin inverses to any countable family of elements of A which are not permanently singular (i.e., which are not topological divisors of zero).

Removable ideals were further studied, e.g. in [4] and [5].

For finite families the answer to the question of R. Arens is affirmative. This was shown in [3] as a consequence of the characterization of non-removable ideals: an ideal I is non-removable if and only if it consists of joint topological divisors of zero (i.e., for every $x_1, \ldots, x_n \in I$ there exists a sequence $\{z_k\}_{k=1}^{\infty} \subset A$, $\|z_k\| = 1$, $\lim_{k\to\infty} \sum_{i=1}^{n} \|z_k x_i\| = 0$).

The aim of this paper is to fill the gap, namely to consider the countable case (see also Problem 3 of [4]). We show that any countable family of removable ideals is removable.

THEOREM 1. *Let A be a commutative Banach algebra with unit, let $p_1, p_2, \ldots$ be positive integers and $K_1, K_2, \ldots$ positive real numbers such that $2 \leqslant p_l \leqslant l+1$, $K_l^{p_1} \leqslant l$, $p_l^{p_1} \leqslant 4\cdot l$ $(l = 1, 2, \ldots)$ (these conditions are only technical). Let $u_{rs} \in A$, $\|u_{rs}\| = 1$ $(r = 1, 2, \ldots, 1 \leqslant s \leqslant p_r)$ and*

$$\|x\| \leqslant K_r \sum_{s=1}^{p_r} \|u_{rs} x\| \quad (r = 1, 2, \ldots, x \in A). \tag{1}$$

Then there exist a superalgebra $B \supset A$ and elements $b_{rs} \in B$ $(r = 1, 2, \ldots, 1 \leqslant s \leqslant p_r)$ such that $\sum_{s=1}^{p_r} u_{rs} b_{rs} = 1$ $(r = 1, 2, \ldots)$.

Proof. Denote by N the set of non-negative integers, and $T = \{(r, s), r = 1, 2, \ldots, 1 \leqslant s \leqslant p_r\}$, $D = \{\boldsymbol{k}: T \to N, \boldsymbol{k}((r, s)) \neq 0$ only for finite number of $(r, s) \in T\}$. For $\boldsymbol{k}, \boldsymbol{j} \in D$, $l \in \{1, 2, \ldots\}$ and $(r, s) \in T$ denote $k_{rs} = \boldsymbol{k}((r, s))$, $|\boldsymbol{k}|_l = \sum_{t=1}^{p_l} k_{lt}$ and $(\boldsymbol{k}+\boldsymbol{j}) \in D$, $(\boldsymbol{k}+\boldsymbol{j})_{rs} = k_{rs}+j_{rs}$. We write $\boldsymbol{k} \leqslant \boldsymbol{j}$ if $k_{rs} \leqslant j_{rs}$ $((r, s) \in T)$.

Put $R_l = 2^{2^{l+6}}$ $(l = 1, 2, \ldots)$.

Let V be the l^1 algebra over A and adjoined elements b_{rs} $((r, s) \in T)$ such that $\|b_{rs}\|_V = R_r$, i.e., the elements of V are of the form $y = \sum_{k \in D} a_k \boldsymbol{b}^k$, such that $\|y\|_V = \sum_{k \in D} \|a_k\| \boldsymbol{R}^{|k|}$, where $a_k \in A$ $(k \in D)$, $\boldsymbol{b}^k$ stands for $\prod_{(r,s) \in T} b_{rs}^{k_{rs}}$ and $\boldsymbol{R}^{|k|} = \prod_{l=1}^{\infty} R_l^{|k|_l}$ (all products are finite).

Multiplication in V is defined by

$$\Big(\sum_{i \in D} a_i \boldsymbol{b}^i\Big)\Big(\sum_{j \in D} a_j' \boldsymbol{b}^j\Big) = \sum_{k \in D} \boldsymbol{b}^k \Big(\sum_{i \cdot j = k} a_i a_j'\Big).$$

Then V is a commutative Banach algebra, $A \subset V$. Let I be the closed ideal in V generated by the elements $z_r = 1 - \sum_{s=1}^{p_r} u_{rs} b_{rs}$ $(r = 1, 2, \ldots)$. Write $B = V/I$. Obviously, $\bar{1} = \sum_{s=1}^{p_r} \bar{u}_{rs} \bar{b}_{rs}$ $(r = 1, 2, \ldots)$, where $\bar{v} = v + I \in B$ for $v \in V$. It is sufficient to prove that A is a subalgebra of B, i.e., $\|a\|_A = \|\bar{a}\|_B = \inf_{v \in I} \|a+v\|_V$ for each $a \in A$. Putting $v = 0$, we get $\|a\|_A \geqslant \|\bar{a}\|_B$. Finite sums of the form $\sum_{l=1}^{\infty} z_l \sum_{i \in D} a_i^{(l)} \boldsymbol{b}^i$ are dense in I, so we are to prove that $\|a\|_A \leqslant \|a + \sum_{l=1}^{\infty} z_l (\sum_{i \in D} a_i^{(l)} \boldsymbol{b}^i)\|_V$ $(a, a_i^{(l)} \in A$ and both sums are finite). We have

$$\Big\|a + \sum_{l=1}^{\infty} z_l \sum_{i \in D} a_i^{(l)} \boldsymbol{b}^i\Big\|_V = \Big\|a + \sum_{l=1}^{\infty} \sum_{i \in D} \Big(1 - \sum_{s=1}^{p_r} u_{ls} b_{ls}\Big) a_i^{(l)} \boldsymbol{b}^i\Big\|_V$$

$$= \Big\|a + \sum_{l=1}^{\infty} a_0^{(l)} + \sum_{\substack{i \in D \\ i \neq 0}} \boldsymbol{b}^i f_i\Big\|_V = \Big\|a + \sum_{l=1}^{\infty} a_0^{(l)}\Big\|_A + \sum_{i \neq 0} \|f_i\|_A \boldsymbol{R}^{|i|}$$

$$\geqslant \|a\|_A - \Big\|\sum_{l=1}^{\infty} a_0^{(l)}\Big\| + \sum_{i \neq 0} \|f_i\| \boldsymbol{R}^{|i|},$$

where

$$f_i = \sum_{l=1}^{\infty} a_i^{(l)} - \sum_{l=1}^{\infty} \sum_{\substack{1 \leqslant t \leqslant p_l \\ i_{lt} \neq 0}} a_j^{(l)} u_{lt} \quad \text{and} \quad j_{lt} = i_{lt} - 1, \quad j_{rs} = i_{rs},$$

for $(r, s) \neq (l, t)$ (j depends on l and t).

It is sufficient to prove $\|\sum_{l=1}^{\infty} a_0^{(l)}\|_A \leqslant \sum_{i \neq 0} \|f_i\|_A \boldsymbol{R}^{|i|}$ (all sums are finite).

Suppose the contrary. Let $a_i^{(l)} \in A$, $l = 1, 2, \ldots$, $\boldsymbol{i} \in D$. Suppose that only finite number of them is non-zero, $\|\sum_{l=1}^{\infty} a_0^{(l)}\|_A = 1$ and $\sum_{i \neq 0} \|f_i\|_A \boldsymbol{R}^{|i|} < 1$. In particular, $\|f_i\|_A < R^{-|i|} = \prod_{l=1}^{\infty} R_l^{-|i|_l}$ $(\boldsymbol{i} \in D, \boldsymbol{i} \neq 0)$ and $a_i^{(l)} = 0$ if either $l > l_0$ or $|\boldsymbol{i}|_r \neq 0$ for some $r > l_0$.

In the rest of the proof we are going to prove that this leads to a contradiction. The proof is divided into seven parts.

I. First we need some results of [3]. For $c \in N$, $c \geqslant 1$ and $k \in N^n$ $(n \geqslant 1)$, write

$$\alpha_{c,k} = \binom{|\boldsymbol{k}|+c-1}{c-1} \frac{|\boldsymbol{k}|!}{k_1! \ldots k_n!},$$

where $|\boldsymbol{k}| = \sum_{t=1}^{n} k_t$. The following lemmas were proved in [3] (see Lemmas 3 and 4):

LEMMA 1.1. $\sum_{c=1}^{d} \sum_{\substack{k \in N^n \\ |k| \leqslant l}} \alpha_{c,k} \leqslant 8^d n^d$ $(d = 1, 2, \ldots)$.

LEMMA 1.2. $\alpha_{c,k} = \sum_{j \leqslant k} \alpha_{c-1,j} \cdot \alpha_{1,k-j}$ $(c \geqslant 2, \boldsymbol{k} \in N^n)$, *where, for* $\boldsymbol{j}, \boldsymbol{k} \in N^n$, $\boldsymbol{j} \leqslant \boldsymbol{k}$ *means that* $j_t \leqslant k_t$ $(1 \leqslant t \leqslant n)$ *and* $(\boldsymbol{k}-\boldsymbol{j}) \in N^n$ *is defined by* $(k-j)_t = k_t - j_t$.

To apply these results to our situation, write

$$o_{ls}(\boldsymbol{i}, \boldsymbol{j}) = (i_{l,1} - j_{l,1}, \ldots, i_{l,s-1} - j_{l,s-1}, i_{l,s+1} - j_{l,s+1}, \ldots, i_{l,p_l} - j_{l,p_l}) \in N^{p_l - 1}$$

$$(l \in \{1, 2, \ldots\},\ 1 \leqslant s \leqslant p_l,\ \boldsymbol{i}, \boldsymbol{j} \in D,\ i_{lt} \geqslant j_{lt} \text{ for } t \neq s).$$

II. It will be convenient to consider linear combinations of $a_i^{(l)}$'s as formal expressions. From this reason we introduce the following notations:

Let W be the free additive group with generators $\hat{a}_i^{(l)} \hat{\boldsymbol{u}}^j$ $(\boldsymbol{i}, \boldsymbol{j} \in D$, $l \in \{1, 2, \ldots, l_0\})$. (Here we consider $\hat{a}_i^{(l)} \hat{\boldsymbol{u}}^j$ as one symbol; there is no multiplication in W.)

Define the additive mapping P: $W \to A$ by $P(\hat{a}_i^{(l)} \hat{\boldsymbol{u}}^j) = a_i^{(l)} \boldsymbol{u}^j$. Let I: $W \to W$

be the identical mapping. Define further the following additive mappings acting in W:

Let $\boldsymbol{i}$, $\boldsymbol{k}\in D$, $\boldsymbol{k}\geqslant\boldsymbol{i}$, l, $m\in\{1, 2, \ldots, l_0\}$, $d\in N$. Put

$$H_{md}(\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{i}})=\begin{cases}\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{i}} & \text{if } |\boldsymbol{i}|_m=d,\\ 0 & \text{otherwise,}\end{cases}$$

$$\pi_{lm}(\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{i}})=\hat{a}_i^{(m)}\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{i}},\quad \pi_{lm}(\hat{a}_i^{(r)}\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{i}})=0\quad\text{for } r\neq l,$$

$$F_{lm}(\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{i}})=\sum_{\boldsymbol{j}\in M_{\boldsymbol{i},m}}\hat{a}_j^{(l)}\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{j}},$$

where

(2) $\quad M_{\boldsymbol{i},m}=\{\boldsymbol{j}=\{j_{rs}\}\in D$, there exist t, $1\leqslant t\leqslant p_m$, $j_{mt}=i_{mt}-1$, $j_{rs}=i_{rs}$ for $(r, s)\neq(m, t)\}$,

$F_{lm}(\hat{a}_i^{(r)}\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{i}})=0$ for $r\neq l$ and, for $1\leqslant s\leqslant p_l$, $k_{ls}\geqslant i_{ls}+|\boldsymbol{i}|_l+1$,

$$G_{ls}(\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{i}})=\sum_{\boldsymbol{j}\in J_1}(-1)^{j_{ls}-i_{ls}-1}\cdot\alpha_{1,o_{ls}(\boldsymbol{i},\boldsymbol{j})}\hat{a}_j^{(l)}\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{j}},$$

where

(3) $\quad J_1=\{\boldsymbol{j}=\{j_{rs}\}\in D,\ j_{rt}=i_{rt}$ for $r\neq l$, $j_{lt}\leqslant i_{lt}$ for $t\neq s$ and $|\boldsymbol{j}|_l=|\boldsymbol{i}|_l+1\}$

and $\alpha_{1,o_{ls}(\boldsymbol{i},\boldsymbol{j})}$ are the numbers defined in part I.

We put $G_{ls}(\hat{a}_i^{(r)}\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{i}})=0$ if either $r\neq l$ or $k_{ls}<i_{ls}+|\boldsymbol{i}|_l+1$.

Lemma 2. *Let m, m', $l\in\{1, 2, \ldots, l_0\}$, d, $d'\in N$. Then*

(a) $H_{md}H_{md'}=H_{md'}H_{md}$,

(b) $H_{md}\pi_{lm'}=\pi_{lm'}H_{md}$,

(c) $F_{lm}F_{lm'}=F_{lm'}F_{lm}$,

(d) $F_{mm'}\pi_{lm}=\pi_{lm}F_{lm'}$,

(e) $H_{md}G_{ls}=G_{ls}H_{md}$ $(l\neq m,\ 1\leqslant s\leqslant p_l)$,

(f) $F_{lm}G_{ls}=G_{ls}F_{lm}$ $(l\neq m,\ 1\leqslant s\leqslant p_l)$,

(g) $F_{lm}H_{m'd}=H_{m'd}F_{lm}$ $(l\neq m,\ m\neq m')$,

(h) $F_{ll}G_{ls}(\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{i}})=\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{i}}$ $(1\leqslant s\leqslant p_l$, $\boldsymbol{i}$, $\boldsymbol{k}\in D$, $\boldsymbol{k}\geqslant\boldsymbol{i}$, $k_{ls}\geqslant i_{ls}+|\boldsymbol{i}|_l+1)$.

Proof. Relations (a)–(g) easily follow from the definitions.

(h) We have

$$F_{ll}G_{ls}(\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{i}})=\sum_{\boldsymbol{j}\in J_1}(-1)^{j_{ls}-i_{ls}-1}\cdot\alpha_{1,o_{ls}(\boldsymbol{i},\boldsymbol{j})}\sum_{\boldsymbol{j}'\in M_{\boldsymbol{j},l}}\hat{a}_{j'}^{(l)}\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{j}'},$$

where

$$\alpha_{1,o_{ls}(\boldsymbol{i},\boldsymbol{j})}=\frac{(j_{ls}-i_{ls}-1)!}{\prod\limits_{\substack{1\leqslant t\leqslant p_l\\ t\neq s}}(i_{lt}-j_{lt})!},$$

the index sets J_1, $M_{\boldsymbol{j},l}$ are defined in (2) and (3). Let $\boldsymbol{j}'\in M_{\boldsymbol{j},l}$, where $\boldsymbol{j}\in J_1$.

Then $|\boldsymbol{j}'|_l=|\boldsymbol{j}|_l-1=|\boldsymbol{i}|_l$, $j'_{lt}\leqslant j_{lt}\leqslant i_{lt}$ for $t\neq s$. For such a $\boldsymbol{j}'$, $\boldsymbol{j}'\neq\boldsymbol{i}$, the coefficient at $\hat{a}_{j'}^{(l)}\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{j}'}$ is equal to

$$(-1)^{j'_{ls}-i_{ls}}\frac{(j'_{ls}-i_{ls})!}{\prod\limits_{t\neq s}(i_{lt}-j'_{lt})!}+\sum_{\substack{t\neq s\\ j'_{lt}<i_{lt}}}(-1)^{j'_{ls}-i_{ls}-1}\frac{(j'_{ls}-i_{ls}-1)!}{\prod\limits_{t'\neq t,s}(i_{lt'}-j'_{lt'})!\,(i_{lt}-j'_{lt}-1)!}$$

$$=(-1)^{j'_{ls}-i_{ls}}\frac{(j'_{ls}-i_{ls}-1)!}{\prod\limits_{t'\neq s}(i_{lt'}-j_{lt'})!}\Big[j'_{ls}-i_{ls}-\sum_{\substack{t\neq s\\ j'_{lt}<i_{lt}}}(i_{lt}-j'_{lt})\Big]=0.$$

This gives (h) since one can easily see that for $\boldsymbol{j}'=\boldsymbol{i}$ the corresponding coefficient is equal to 1.

III. Define

(4) $$\hat{f}_i\hat{\boldsymbol{u}}^{\boldsymbol{j}}=\sum_{m=1}^{l_0}\hat{a}_i^{(m)}\hat{\boldsymbol{u}}^{\boldsymbol{j}}-\sum_{m=1}^{l_0}F_{mm}\hat{a}_i^{(m)}\hat{\boldsymbol{u}}^{\boldsymbol{j}}\qquad(\boldsymbol{i},\boldsymbol{j}\in D)$$

so that $P(\hat{f}_i\hat{\boldsymbol{u}}^{\boldsymbol{j}})=f_i\boldsymbol{u}^{\boldsymbol{j}}$.

Lemma 3.1. *Let $\boldsymbol{i}$, $\boldsymbol{k}\in D$, $\boldsymbol{k}\geqslant\boldsymbol{i}$, $l\in\{1, 2, \ldots, l_0\}$, $1\leqslant s\leqslant p_l)$, $k_{ls}\geqslant i_{ls}+|\boldsymbol{i}|_l+1$. Then*

$$\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{i}}=-\sum_{\boldsymbol{j}\in J_1}(-1)^{j_{ls}-i_{ls}-1}\alpha_{1,o_{ls}(\boldsymbol{i},\boldsymbol{j})}\hat{f}_j\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{j}}+G_{ls}(\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{i}})+$$
$$+\sum_{l'\neq l}\pi_{ll'}(I-F_{l'l'})G_{ls}(\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{i}}).$$

Proof. Using (4) and the definition of G_{ls}, we have

$$\sum_{\boldsymbol{j}\in J_1}(-1)^{j_{ls}-i_{ls}-1}-\alpha_{1,o_{ls}(\boldsymbol{i},\boldsymbol{j})}\hat{f}_j\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{j}}$$
$$=\sum_{m=1}^{l_0}\pi_{lm}G_{ls}(\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{i}})-\sum_{\boldsymbol{j}\in J_1}(-1)^{j_{ls}-i_{ls}-1}\alpha_{1,o_{ls}(\boldsymbol{i},\boldsymbol{j})}\sum_{m=1}^{l_0}\pi_{lm}F_{lm}(\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{i}})$$
$$=G_{ls}(\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{i}})+\sum_{l'\neq l}\pi_{ll'}G_{ls}(\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{i}})-$$
$$-\sum_{m\neq l}\pi_{lm}F_{lm}G_{ls}(\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{i}})-F_{ll}G_{ls}(\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{i}}).$$

Lemma 3.1 now follows from Lemma 2.1 (h).

Define now Z_{ls}: $W\to W$ $(l\in\{1, 2, \ldots, l_0\}$, $1\leqslant s\leqslant p_l)$ by $Z_{ls}(\hat{v})=G_{ls}\hat{v}+\sum_{m\neq l}\pi_{lm}(I-F_{lm})G_{ls}\hat{v}+\sum_{m\neq l}\pi_{mm}\hat{v}$ $(\hat{v}\in W)$. Then we can rewrite Lemma 3.1 as follows:

$$\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{i}}=Z_{ls}(\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{i}})-\sum_{\boldsymbol{j}\in J_1}(-1)^{j_{ls}-i_{ls}-l'}\alpha_{1,o_{ls}(\boldsymbol{i},\boldsymbol{j})}\hat{f}_j\hat{\boldsymbol{u}}^{\boldsymbol{k}-\boldsymbol{j}};$$

hence

COROLLARY 3.2. $\|P[\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{k-i}-Z_{ls}(\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{k-i})]\| \leqslant \sum_{j\in J_1} \alpha_{1,o_{ls}(i,j)} R_l^{-|i|_l-1}$.

Remark 3.3. If $|\boldsymbol{i}|_l=0$, then $\sum_{j\in J_1}\alpha_{1,o_{ls}(i,j)}=1$ and there is only R_l^{-1} on the right-hand side.

IV. LEMMA 4.1. *Let* $\boldsymbol{k}, \boldsymbol{i}\in D$, $\boldsymbol{k}\geqslant \boldsymbol{i}$, $l\in\{1, 2, \ldots, l_0\}$, $1\leqslant s\leqslant p_l$, $c\in\{1, 2, \ldots\}$, $k_{ls}\geqslant i_{ls}+|\boldsymbol{i}|_l+c$. *Then*

$$G_{ls}^c(\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{k-i}) = \sum_{j\in J_{c,i}} (-1)^{j_{ls}-i_{ls}-c}\alpha_{c,o_{ls}(i,j)}\,\hat{a}_j^{(l)}\hat{\boldsymbol{u}}^{k-j},$$

where $J_{c,i}=\{\boldsymbol{j}=\{j_{rt}\}\in D,\ j_{rt}=i_{rt}\ (r\neq l),\ j_{lt}\leqslant i_{lt}\ (t\neq s)$ *and* $|\boldsymbol{j}|_l=|\boldsymbol{i}|_l+c\}$ (G_{ls}^c *is the c-th power of* G_{ls}).

Proof. For $c=1$, the statement of Lemma 4.1 is the definition of G_{ls}. Suppose that the statement is true for some c. Then

$$G_{ls}^{c+1}(\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{k-i}) = G_{ls}\Big[\sum_{j\in J_{c,i}} (-1)^{j_{ls}-i_{ls}-c}\alpha_{c,o_{ls}(i,j)}\,\hat{a}_j^{(l)}\hat{\boldsymbol{u}}^{k-j}\Big]$$
$$= \sum_{j\in J_{c,i}} (-1)^{j_{ls}-i_{ls}-c}\alpha_{c,o_{ls}(i,j)} \sum_{j'\in J_{1,j}} (-1)^{j'_{ls}-j_{ls}-1}\cdot\alpha_{1,o_{ls}(j,j')}\,\hat{a}_{j'}^{(l)}\hat{\boldsymbol{u}}^{k-j'}$$
$$= \sum_{j''\in D} \beta_{j''}\,\hat{a}_{j''}^{(l)}\hat{\boldsymbol{u}}^{k-j''}$$

for some integers $\beta_{j''}$. Obviously, $\beta_{j''}=0$ for $j''\notin J_{c+1,i}$. If $\boldsymbol{j}''\in J_{c+1,i}$, then

$$\beta_{j''} = \sum_{j\in J} (-1)^{j_{ls}-i_{ls}-c}\alpha_{c,o_{ls}(i,j)}(-1)^{j''_{ls}-j_{ls}-1}\alpha_{1,o_{ls}(j,j'')},$$

where

$$J = \{\boldsymbol{j}=\{j_{rs}\}\in D,\ j_{rt}=j''_{rt}\ (r\neq l),\ i_{lt}\geqslant j_{lt}\geqslant j''_{lt}\ (t\neq s) \text{ and } |\boldsymbol{j}|_l+1=|\boldsymbol{j}''|_l\}.$$

By Lemma 1.2 we have

$$\sum_{j\in J}\alpha_{c,o_{ls}(i,j)}\alpha_{1,o_{ls}(j,j'')} = \alpha_{c+1,o_{ls}(i,j)} \quad\text{and}\quad \beta_{j''} = (-1)^{j''_{ls}-i_{ls}-c-1}\alpha_{c+1,o_{ls}(i,j'')}.$$

This finishes the induction step.

LEMMA 4.2. *Let* $\boldsymbol{i}$, $\boldsymbol{k}$, l, s, c *be as above. Then*

$$Z_{ls}^c(\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{k-i}) = \sum_{j\in J_{c,i}} (-1)^{j_{ls}-i_{ls}-c}\alpha_{c,o_{ls}(i,j)}\,\hat{a}_j^{(l)}\hat{\boldsymbol{u}}^{k-j}+$$
$$+\sum_{m\neq l}\sum_{c'=1}^{c}\sum_{j\in J_{c',i}} (-1)^{j_{ls}-i_{ls}-c'}\alpha_{c',o_{ls}(i,j)}(I-F_{mm})(\hat{a}_j^{(m)}\hat{\boldsymbol{u}}^{k-j}).$$

Proof. For $\hat{v}=\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{k-i}$ we can easily prove by induction on c that

$$Z_{ls}^c\hat{v} = G_{ls}^c\hat{v}+\sum_{m\neq l}\pi_{lm}(I-F_{lm})\sum_{c'=1}^{c} G_{ls}^{c'}\hat{v}$$

and the statement follows from the previous lemma and Lemma 2.1 (f).

LEMMA 4.3. *Let* $\boldsymbol{i}$, $\boldsymbol{k}$, l, s, c *be as above. Then, for* $\hat{v}=\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{k-i}$,

$$Z_{ls}^c\hat{v}-\hat{v} = \sum_{c'=1}^{c}\sum_{j\in J_{c',i}} (-1)^{j_{ls}-i_{ls}-c'}\alpha_{c',o_{ls}(i,j)}\hat{f}_j\hat{\boldsymbol{u}}^{k-j}.$$

Proof. Substitute $\hat{f}_j\hat{\boldsymbol{u}}^{k-j}=\sum_{m=1}^{l_0}(I-F_{mm})(\hat{a}_j^{(m)}\hat{\boldsymbol{u}}^{k-j})$ into the right-hand side. Comparing with Lemma 4.2 it is sufficient to prove

$$\sum_{j\in J_{c,i}} (-1)^{j_{ls}-i_{ls}-c}\alpha_{c,o_{ls}(i,j)}\,\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{k-j}-\hat{v}$$
$$= \sum_{c'=1}^{c}\sum_{j\in J_{c',i}} (-1)^{j_{ls}-i_{ls}-c'}\alpha_{c',o_{ls}(i,j)}(I-F_{ll})\,\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{k-j};$$

in other words,

$$(G_{ls}^c-I)\hat{v} = \sum_{c'=1}^{c}(I-F_{ll})G_{ls}^{c'}\hat{v}.$$

This follows from Lemma 2.1 (h).

COROLLARY 4.4. *Let* $\boldsymbol{i}$, $\boldsymbol{k}$, l, s, c *be as above. Then, for* $\hat{v}=\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{k-i}$,

1° $\|P(\hat{v}-Z_{ls}^c\hat{v})\| \leqslant \sum_{c'=1}^{c}\sum_{j\in J_{c',i}}\alpha_{c',o_{ls}(i,j)}R_l^{-|i|_l-c'}$.

2° *If* $|\boldsymbol{i}|_l\neq 0$, $c=|\boldsymbol{i}|_l$, *then*

$$\|P(\hat{v}-Z_{ls}^c\hat{v})\| \leqslant 8^{|i|_l}(p_1-1)^{|i|_1}R_l^{-|i|_l-1}.$$

Proof. 1° follows from the estimate $\|P(\hat{f}_j\hat{\boldsymbol{u}}^{k-j})\| = \|f_j\boldsymbol{u}^{k-j}\| \leqslant \|f_j\| < R_l^{-|j|_l}$.

2° follows from Lemma 1.1.

V. Let $\boldsymbol{k}\in D$, $d\in N$, $m\in\{1, 2, \ldots\}$ and $\hat{v}\in W$,

$$\hat{v}=\sum_{l=1}^{l_0}\hat{v}^{(l)},\qquad \hat{v}^{(l)}=\sum_{i\leqslant k}\gamma_i^{(l)}\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{k-i}, \tag{5}$$

where $\gamma_i^{(l)}$'s are integers. Then $\deg_m\hat{v}=d$ ($\deg_m\hat{v}\leqslant d$) means $|\boldsymbol{i}|_m=d$ ($|\boldsymbol{i}|_m\leqslant d$) whenever $\gamma_i^{(l)}\neq 0$ for some $l\in\{1, 2, \ldots, l_0\}$.

LEMMA 5.1. *Let* $\boldsymbol{k}\in D$, $l\in\{1, 2, \ldots, l_0\}$, $1\leqslant s\leqslant p_l$, $n_1, n_2, \ldots, n_{l_0}\in N$. *Let* $\hat{v}\in W$ *be of form* (5), $k_{ls}\geqslant i_{ls}+|\boldsymbol{i}|_l+1$ *whenever* $\gamma_i^{(l)}\neq 0$ *and the following conditions are satisfied:*

1° $\deg_m\hat{v}\leqslant n_m$, $\deg_m\hat{v}^{(m)}=n_m$ ($m=1, 2, \ldots, l_0$), $\deg_m\hat{v}=0$ ($m>l_0$);

2° $(I-F_{mm'})\hat{v}^{(m)}=H_{m',n_{m'}}\hat{v}^{(m)}$, *i.e.*, $\deg_{m'}(I-F_{mm'})\hat{v}^{(m)}=n_{m'}$ *for every* $m\neq m'$;

3° $H_{m,n_m}\hat{v}^{(m')}=\pi_{mm'}H_{m',n_{m'}}\hat{v}^{(m)}$ ($1\leqslant m, m'\leqslant l_0$) (*i.e., the parts of* $\hat{v}^{(m)}$ *and* $\hat{v}^{(m')}$ *with maximal both* $\deg_m$ *and* $\deg_{m'}$ *are equal up to the upper index*).

Put $\hat{w} = Z_{ls}\hat{v}$, i.e., $\hat{w} = \sum_{m=1}^{l_0} \hat{w}^{(m)}$, $\hat{w}^{(l)} = G_{ls}\hat{v}^{(l)}$, $\hat{w}^{(m)} = \hat{v}^{(m)} + \pi_{lm}(I-F_{lm})G_{ls}\hat{v}^{(l)}$ *for* $m \neq l$.

Then $\hat{w}$ *satisfies conditions* 1°–3° *of this lemma with* n_l *replaced by* n_l+1 (n_m *does not change for* $m \neq l$).

Proof. 1° $\deg_l \hat{w}^{(l)} = \deg_l G_{l_s}\hat{v}^{(l)} = n_l+1$ by the definition of G_{ls}. For $m \neq l$ we have $\deg_m \hat{v}^{(m)} = n_m$ and

$$\begin{aligned}\deg_m \pi_{lm}(I-F_{lm})G_{ls}\hat{v}^{(l)} &= \deg_m G_{ls}(I-F_{lm})\hat{v}^{(l)}\\ &= \deg_m G_{ls}H_{m,n_m}\hat{v}^{(l)} = \deg_m H_{m,n_m}G_{ls}\hat{v}^{(l)} = n_m\end{aligned}$$

by property 2° and Lemma 2.1.

The rest of 1° is clear.

2° (a) Let $m, m' \neq l$, $m \neq m'$. Then

$$\begin{aligned}(I-F_{mm'})\hat{w}^{(m)} &= (I-F_{mm'})[\hat{v}^{(m)} + \pi_{lm}(I-F_{lm})G_{ls}\hat{v}^{(l)}]\\ &= H_{m',n_{m'}}\hat{v}^{(m)} + \pi_{lm}(I-F_{lm'})(I-F_{lm})G_{ls}v^{(l)}\\ &= H_{m',n_{m'}}\hat{v}^{(m)} + \pi_{lm}(I-F_{lm})G_{ls}H_{m',n_{m'}}v^{(l)} = H_{m',n_{m'}}\hat{w}^{(m)}.\end{aligned}$$

(b) Let $m \neq l$. Then

$$(I-F_{lm})\hat{w}^{(l)} = (I-F_{lm})G_{ls}\hat{v}^{(l)} = G_{ls}H_{m,n_m}\hat{v}^{(l)} = H_{m,n_m}G_{ls}\hat{v}^{(l)} = H_{m,n_m}\hat{w}^{(l)}.$$

(c) Let $m \neq l$. Then

$$\begin{aligned}(I-F_{ml})\hat{w}^{(m)} &= (I-F_{ml})[\hat{v}^{(m)} + \pi_{lm}(I-F_{lm})G_{ls}\hat{v}^{(l)}]\\ &= H_{l,n_l}\hat{v}^{(m)} + \pi_{lm}(I-F_{ll})(I-F_{lm})G_{ls}v^{(l)}\\ &= H_{l,n_l}\hat{v}^{(m)} + \pi_{lm}(I-F_{lm})(G_{ls}-I)\hat{v}^{(l)}\\ &= \pi_{lm}H_{m,n_m}\hat{v}^{(l)} + \pi_{lm}(G_{ls}-I)H_{m,n_m}\hat{v}^{(l)}\\ &= \pi_{lm}G_{ls}H_{m,n_m}\hat{v}^{(l)}\end{aligned}$$

and

$$\begin{aligned}H_{l,n_l+1}\hat{w}^{(m)} &= H_{l,n_l+1}[\hat{v}^{(m)} + \pi_{lm}(I-F_{lm})G_{ls}v^{(l)}]\\ &= \pi_{lm}(I-F_{lm})H_{l,n_l+1}G_{ls}v^{(l)} = \pi_{lm}(I-F_{lm})G_{ls}\hat{v}^{(l)}\\ &= \pi_{lm}G_{ls}H_{m,n_m}\hat{v}^{(l)}\end{aligned}$$

(we used the fact that G_{ls} increases $\deg_l$).

3° (a) $m \neq m'$, $m, m' \neq l$. Then

$$\begin{aligned}\pi_{mm'}H_{m',n_{m'}}\hat{w}^{(m)} &= \pi_{mm'}H_{m',n_{m'}}[\hat{v}^{(m)} + \pi_{lm}(I-F_{lm})G_{ls}\hat{v}^{(l)}]\\ &= H_{m,n_m}\hat{v}^{(m')} + \pi_{mm'}\pi_{lm}G_{ls}H_{m',n_{m'}}(I-F_{lm})\hat{v}^{(l)}\\ &= H_{m,n_m}\hat{v}^{(m')} + \pi_{lm}G_{ls}H_{m',n_{m'}}H_{m,n_m}\hat{v}^{(l)}\\ &= H_{m,n_m}[\hat{v}^{(m')} + \pi_{lm'}G_{ls}H_{m',n_{m'}}\hat{v}^{(l)}] = H_{m,n_m}\hat{w}^{(m')}.\end{aligned}$$

(b) $m \neq l$. Then

$$\begin{aligned}\pi_{ml}H_{l,n_l+1}\hat{w}^{(m)} &= \pi_{ml}H_{l,n_l+1}[\hat{v}^{(m)} + \pi_{lm}(I-F_{lm})G_{ls}\hat{v}^{(l)}]\\ &= \pi_{ml}\pi_{lm}H_{l,n_l+1}G_{ls}(I-F_{lm})\hat{v}^{(l)} = H_{l,n_l+1}G_{ls}H_{m,n_m}\hat{v}^{(l)}\\ &= H_{m,n_m}G_{ls}\hat{v}^{(l)} = H_{m,n_m}\hat{w}^{(l)}.\end{aligned}$$

VI. For $\hat{v} \in W$, $\hat{v} = \sum_{l=1}^{l_0} \hat{v}^{(l)}$, $\hat{v}^{(l)} = \sum_{i,j \in D} \gamma_{ij}^{(l)}\hat{a}_i^{(l)}\hat{\boldsymbol{u}}^{\boldsymbol{j}}$ write $|\hat{v}| = \max_l \sum_{i,j \in D} |\gamma_{ij}^{(l)}|$. Further, let, for $q \geqslant 1$ and $n_1, n_2, \ldots, n_{l_0} \in \boldsymbol{N}$, $s_{n_1,n_2,\ldots,n_{l_0}} = \max\|P\hat{v}\|$, where the maximum is taken over all elements $\hat{v} \in W$ of form (5) satisfying conditions 1°–3° of Lemma 5.1 (with the numbers $n_1, n_2, \ldots, n_{l_0}$ and with some $\boldsymbol{k} \in D$, $\boldsymbol{k} \geqslant \boldsymbol{i}$ whenever $\gamma_{\boldsymbol{i}}^{(l)} \neq 0$ for some $l \in \{1, 2, \ldots, l_0\}$) such that $|\hat{v}| \leqslant q$.

Lemma 6.1. *We have*

$$s^{(q)}_{n_1,\ldots,n_{l-1},0,n_{l+1},\ldots,n_{l_0}} \leqslant K_l p_l [qR_l^{-1} + s^{(2q)}_{n_1,\ldots,n_{l-1},1,n_{l+1},\ldots,n_{l_0}}].$$

Proof. Let $\hat{v} \in W$, $|\hat{v}| \leqslant q$ satisfies conditions 1°–3° of Lemma 5.1 with the numbers $n_1, n_2, \ldots, n_{l-1}, 0, n_{l+1}, \ldots, n_{l_0}$ and with some $\boldsymbol{k} \in D$. Then, by (1),

$$\|P\hat{v}\| \leqslant K_l \sum_{t=1}^{p_l} \|(P\hat{v})u_{lt}\| \leqslant K_l \cdot \sum_{t=1}^{p_l} (\|P[\hat{v}\hat{u}_{lt} - Z_{lt}(\hat{v}\hat{u}_{lt})]\| + \|PZ_{lt}(\hat{v}\hat{u}_{lt})\|),$$

where $\|P[\hat{v}\hat{u}_{lt} - Z_{lt}(\hat{v}\hat{u}_{lt})]\| \leqslant qR_l^{-1}$ by Remark 3.3 and $Z_{lt}(\hat{v}\hat{u}_{lt})$ satisfies conditions 1°–3° of Lemma 5.1 with the numbers $n_1, \ldots, n_{l-1}, 1, n_{l+1}, \ldots, n_{l_0}$. It remains to prove that $|Z_{lt}(\hat{v}\hat{u}_{lt})| \leqslant 2q$. Since

$$Z_{lt}(\hat{v}\hat{u}_{lt}) = G_{lt}(\hat{v}^{(l)}\hat{u}_{lt}) + \sum_{m \neq l} [\hat{v}^{(m)}\hat{u}_{lt} + \pi_{lm}(I-F_{lm})G_{ls}(\hat{v}^{(l)}\hat{u}_{lt}],$$

$$|G_{lt}\hat{v}^{(l)}\hat{u}_{lt}| \leqslant |\hat{v}^{(l)}\hat{u}_{lt}| \leqslant |\hat{v}|$$

and, for $m \neq l$, $|\hat{v}^{(m)}| \leqslant |\hat{v}|$,

$$|\pi_{lm}(I-F_{lm})G_{lt}\hat{v}^{(l)}\hat{u}_{lt}| = |G_{lt}H_{m,n_m}\hat{v}^{(l)}\hat{u}_{lt}| \leqslant |G_{lt}\hat{v}^{(l)}\hat{u}_{lt}| \leqslant |\hat{v}|,$$

we conclude that $|Z_{lt}(\hat{v}\hat{u}_{lt})| \leqslant 2|\hat{v}| \leqslant 2q$.

Lemma 6.2.

$$s^{(q)}_{n_1,\ldots,n_{l_0}} \leqslant K_l^{2n_lp_l} p_l^{2n_lp_l} [q8^{n_l}(p_l-1)^{n_l} R_l^{-n_l-1} + s^{(2q8^{n_l}(p_l-1)^{n_l})}_{n_1,\ldots,n_{l-1},2n_l,n_{l+1},\ldots,n_{l_0}}].$$

Proof. Let $\hat{v} \in W$ satisfy 1°–3° of Lemma 5.1 with the numbers $n_1, \ldots, n_{l_0}$. Then, using (1) repeatedly, we infer

$$\|P\hat{v}\| \leqslant K_l^{2n_l p_l} \sum \|(P\hat{v})\, u_{l1}^{e_1} u_{l2}^{e_2} \ldots u_{l,p_l}^{e_{p_l}}\| \frac{(2n_l p_l)!}{e_1! \ldots e_{p_l}!},$$

where the sum is taken over all $e_1, \ldots, e_{p_l}$ with $\sum_{t=1}^{p_l} e_t = 2n_l p_l$. It follows from this that $e_s \geqslant 2n_l$ for some s, $1 \leqslant s \leqslant p_l$, in each of $p_l^{2n_l p_l}$ summands and

$$\|(P\hat{v})\, u_{l1}^{e_1} \ldots u_{l,p_l}^{e_{p_l}}\| \leqslant \|(P\hat{v})\, u_{ls}^{2n_l}\| = \|P\hat{w}\| \leqslant \|PZ_{ls}^{n_l}\hat{w}\| + \|P(\hat{w} - Z_{ls}^{n_l}\hat{w})\|,$$

where $\hat{w} = \hat{v}\hat{u}_{ls}^{2n_l}$. Further, $\|P(\hat{w} - Z_{ls}^{n_l}\hat{w})\| \leqslant q \cdot 8^{n_l}(p_l - 1)^{n_l} R_l^{-n_l-1}$ by Corollary 4.4 and $Z_{ls}^{n_l}\hat{w}$ satisfies conditions 1°–3° of Lemma 5.1 for $n_1, \ldots, n_{l-1}, 2n_l, n_{l+1}, \ldots, n_{l_0}$. We have to estimate $|Z_{ls}^{n_l}\hat{w}|$. We have

$$Z_{ls}^{n_l}\hat{w} = G_{ls}^{n_l}\hat{w}^{(l)} + \sum_{m \neq l} \Big[\hat{w}^{(m)} + \pi_{lm}(I - F_{lm}) \sum_{c=1}^{n_l} G_{ls}^{c}\hat{w}^{(l)}\Big]$$

$$= G_{ls}^{n_l}\hat{w}^{(l)} + \sum_{m \neq l} \Big[\hat{w}^{(m)} + \pi_{lm} H_{m,n_m} \sum_{c=1}^{n_l} G_{ls}^{c}\hat{w}^{(l)}\Big],$$

where $\hat{w}^{(m)} = \pi_{mm}\hat{w}$ is the part of $\hat{w}$ with the upper index m. Thus

$$|Z_{ls}^{n_l}\hat{w}| \leqslant |\hat{w}| + \sum_{c=1}^{n_l} |G_{ls}^{c}\hat{w}| \leqslant q + 8^{n_l}(p_l - 1)^{n_l} q \leqslant 2\, 8^{n_l}(p_l - 1)^{n_l} q.$$

Hence $\|PZ_{ls}^{n_l}\hat{w}\| \leqslant s_{n_1,\ldots,n_{l-1},2n_l,n_{l+1},\ldots,n_{l_0}}^{(2q8^{n_l}(p_l-1)^{n_l})}$.

Corollary 6.3. $s_{n_1,\ldots,n_{l_0}}^{(q)} \leqslant 4^{4n_l} l^{4n_l} \big[q 8^{n_l} l^{n_1} R_l^{-n_l-1} + s_{n_1,\ldots,n_{l-1},2n_l,n_{l+1},\ldots,n_{l_0}}^{(2q8^{n_l} l^{n_l})}\big]$.

VII. Define $h_r = s_{2^{r-1},2^{r-2},\ldots,2,1,0,\ldots,0}^{(q_r)}$ for $0 \leqslant r \leqslant l_0$ and $h_r = s_{2^{r-1},2^{r-2},\ldots,2^{r-l_0}}^{(q_r)}$ for $r > l_0$, where $q_r = 2^{2^{r+3}}$.

Lemma 7.1. $h_r \leqslant 2^{2^{r+4}} \cdot h_{r+1} + 2^{2^{r+5}} \big[R_1^{-2^{r-1}-1} + R_2^{-2^{r-2}-1} + \ldots + R_r^{-2} + R_{r+1}^{-1}\big]$.

Proof. This estimate can easily be proved by using Lemmas 6.2 and 6.1 repeatedly (we also use the estimate $1 \cdot 2^{1/2} \cdot 3^{1/4} \ldots r^{(1/2)^{r-1}} \leqslant 8$ for every r).

Corollary 7.2. $h_r \leqslant 2^{2^{r+4}} h_{r+1} + 2^{-2^{r+5}}$.

Proof. Substitute $R_l = 2^{2^{l+6}}$.

Proof of Theorem 1. By induction we get $h_0 \leqslant 2^{-2^5} + 2^{-2^6} + \ldots \leqslant 2^{-31}$ (as $h_r = 0$ for $r > l_0$). On the other hand, $h_0 = s_{0,\ldots,0}^{(2^8)} = \max \|P\hat{w}\|$, where $\hat{w} \in W$ satisfies conditions 1°–3° of Lemma 5.1 (with $n_1 = \ldots = n_{l_0} = 0$ and $|\hat{w}| \leqslant 2^8$). Put $\hat{w} - \sum_{l=1}^{l_0} \hat{a}_0^{(l)}$. Clearly, $h_0 \geqslant \|P\hat{w}\| = \|\sum_{l=1}^{l_0} a_0^{(l)}\| = 1$, a contradiction.

Theorem 2. *A countable family of removable ideals in a commutative Banach algebra with unit is removable.*

Proof. Let A be a commutative Banach algebra with unit, $I_1, I_2, \ldots$ removable ideals in A. Then, for $r = 1, 2, \ldots$, there exist a positive integer p_r, a positive constant K_r and elements $u_{r1}, u_{r2}, \ldots, u_{rp_r} \in I_r$ such that $\|x\| \leqslant K_r \sum_{t=1}^{p_r} \|xu_{rt}\|$ $(x \in A)$. We may assume without loss of generality that the conditions $2 \leqslant p_r \leqslant r+1$, $K_r^{p_r} \leqslant r$ and $p_r^{p_r} \leqslant 4r$ are satisfied for each r. Then, by Theorem 1, $\sum_{t=1}^{p_r} u_{rt} b_{rt} = 1$ in some $B \supset A$. This means that I_r is removed in B for $r = 1, 2, \ldots$

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