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Decreasing rearrangements and $L^{p,q}$ of the Bohr group

by

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Abstract. To a complex-valued function f on a measure space (X, μ) can be associated a nonincreasing function f^* mapping the positive reals into themselves in such a way that $|f|$ and f^* are equimeasurable. When X has a topological structure, the map $f \to f^*$ is studied to see which properties of f are inherited by f^*. Continuity is often inherited if X is connected. For $X = [0, 1)$, differentiability is not inherited but the property of being $\operatorname{Lip}\alpha$, $0 < \alpha \leqslant 1$, is. The Bohr compactification $\hat{D}$ of the reals is introduced. Two natural definitions of $L^{p,q}(\hat{D})$ are given and are shown to agree for continuous functions. Considerations of $L^{p,q}(\hat{D})$ are shown to have applications to the concrete spaces $L^{p,q}(\boldsymbol{R})$.

§ 1. Introduction. To a complex-valued function f on a measure space (X, μ) can be associated a nonincreasing function f^* mapping the positive reals into themselves in such a way that $|f|$ and f^* are equimeasurable:

$$\mu\{x \in X: |f(x)| > \alpha\} = \lambda\{t \in \boldsymbol{R}^+: f^*(t) > \alpha\} \quad \text{for} \quad \alpha \in \boldsymbol{R}^+$$

where λ is Lebesgue measure. The function f^* is a "copy" of f that is often easier to work with than f. A major use of f^* is in defining a family of spaces $L^{p,q}(X)$, $p, q \in [1, \infty]$, which generalize $L^p(X)$ and are useful in the theory of interpolation operators.

In Section 2, X is assumed to have some topological structure and the map $f \to f^*$ is studied to see which properties of f are inherited by f^*. Continuity is often inherited if X is connected. However, differentiability is not, although some smoothness does pass over. For example, if $X = \boldsymbol{T}$ then $\operatorname{Lip}\alpha$, $0 < \alpha \leqslant 1$, is inherited while the property of being in the Zygmund class Λ_* is not, even though $\operatorname{Lip} 1 \subset \Lambda_* > \bigcap_{0<\alpha<1} \operatorname{Lip}\alpha$.

In Section 3 the Bohr compactification $\hat{D}$ of the reals is introduced and two obvious definitions of $L^{p,q}(\hat{D})$ are considered. The first arises by considering $\hat{D}$ as a compact abelian group with Haar measure μ and defining $L^{p,q}(\hat{D})$ using f^* where

$$f^*(t) = \inf\{\alpha: \mu\{|f| > \alpha\} \leqslant t\}.$$

The second comes from viewing a continuous function f on $\hat{D}$ as an almost periodic function on $\boldsymbol{R}$, determining the $L^{p,q}$ size of f on each interval $[-T, T]$ and then letting $T \to \infty$. The main result of Section 3 is that these two definitions agree when f is continuous. (Of course the second definition doesn't even make sense for general f on $\hat{D}$ since $\mu(\boldsymbol{R}) = 0$.) Since the continuous functions are dense in $L^{p,q}(\hat{D})$ for $p, q \in [1, \infty)$, the definitions are equivalent for all practical purposes.

In Section 4 considerations of $L^{p,q}(\hat{D})$ are shown to have applications to the concrete spaces $L^{p,q}(\boldsymbol{R})$. This is interesting in view of the rather abstract development of $\hat{D}$ which is large and nonmetrizable.

§ 2. **The decreasing rearrangement.** Let (X, μ) be a σ-finite measure space. If f is a complex-valued μ-measurable function on X, its *distribution function* f_* is given by

$$f_*(\alpha) = \mu\{x \in X: |f(x)| > \alpha\} \quad \text{for} \quad \alpha \in \boldsymbol{R}^+.$$

The *decreasing rearrangement* of f is defined by

$$f^*(t) = \inf\{\alpha: f_*(\alpha) \leqslant t\} \quad \text{for} \quad t \in \boldsymbol{R}^+.$$

Roughly speaking, f_* and f^* are mutually inverse functions. For a detailed presentation of these functions, see [8], pp. 165–169; [9], pp. 189–190; or [5], pp. 251–253. The function f^* is an equimeasurable copy of $|f|$. In particular, the map $f \to f^*$ preserves L^p spaces; in fact,

$$\|f\|_{L^p(X)} = \|f^*\|_{L^p(\boldsymbol{R}^+)} \quad \text{for} \quad 0 < p \leqslant \infty.$$

We are interested in properties preserved by $f \to f^*$ when X is also a topological space.

Before considering preservation of smoothness, a small quirk in the definition of f^* must be dealt with. Consider the function f that is identically 1 on $X = [0, 1]$. Then $f^*(t) = 1$ for $0 \leqslant t < 1$ and $f^*(t) = 0$ for $t \geqslant 1$, so that f^* is discontinuous at $t = 1$. In general, if $\mu(X) < \infty$ and f is bounded away from 0 on X, then f^* will have a jump at $t = \mu(X)$ no matter how smooth f is. Hence the best we can hope for is to have smoothness preserved by the map which takes f to $f^*|_{[0,\mu(X))}$. Call this map $*$. Note also that $\mu(X) < \infty$ is a necessary condition for any smoothness preservation since $f(x) = x$ for $x \in \boldsymbol{R}$ doesn't even have a decreasing rearrangement.

Let X be a metric space with metric d. We will say that X has *property* T if the following condition holds: whenever X is decomposed into three mutually disjoint sets A, B and N with A and B nonempty and $\mu(N) = 0$, then $\operatorname{dist}(A, B) = 0$. We will say that X has *property* T_β, $0 < \beta \leqslant 1$, if there is a constant C such that: whenever X is decomposed into three mutually disjoint sets A, B, and N with A and B nonempty, then $\operatorname{dist}(A, B) \leqslant C\mu(N)^\beta$.

Examples. Any connected metric space in which every open set has positive μ-measure satisfies property T. For fixed n, Euclidean n-space $\boldsymbol{R}^n$ and the n-torus $\boldsymbol{T}^n$ both have properties $T_{\frac{1}{n}}$ but not T_β for any $\beta > 1/n$. To get a feel for this, let $A = \{a\}$, let N be the punctured n-dimensional sphere of radius d about a, and let B be the complement of $A \cup N$ (in $\boldsymbol{R}^n$ or $\boldsymbol{T}^n$); then $\operatorname{dist}(A, B) = d$ and $[\mu(N)]^{1/n} = C_n \cdot d$. An example of a connected space without property T is

$$X = A \cup B \cup N = ([0, 1] \times [0, 1]) \cup ([2, 3] \times [0, 1]) \cup \{(x, 0): 1 < x < 2\},$$

endowed with the usual 2-dimensional measure and metric.

Theorem 1. (a) *If (X, μ, d) has property T and $\mu(X) < \infty$, then the decreasing rearrangement map $*$ preserves continuity.*

(b) *If (X, μ, d) has property T_β, $0 < \beta \leqslant 1$, and $\mu(X) < \infty$, then $*$ maps* $\operatorname{Lip}(\alpha)(X)$ *into* $\operatorname{Lip}(\alpha\beta)([0, \mu(X)))$ *for* $0 < \alpha \leqslant 1$.

Proof. Let

$$\omega(h) = \sup\{|f(x) - f(y)|: x, y \in X, d(x, y) \leqslant h\}$$

and

$$\omega^*(h) = \sup\{|f^*(s) - f^*(t)|: s, t \in [0, \mu(X)), |s - t| \leqslant h\}$$

be the moduli of continuity of f and f^*, respectively.

(a): If f^* is not continuous, there is a point $t_0 \in (0, \mu(X))$ and $\delta > 0$ such that $\lim\limits_{t \to t_0^-} f^*(t) = f^*(t_0) + \delta$. Then $X = E^+ \cup E^- \cup N$ where $E^+ = \{|f| \geqslant f^*(t_0) + \delta\}$, $E^- = \{|f| \leqslant f^*(t_0)\}$ and $N = \{f^*(t_0) < |f| < f^*(t_0) + \delta\}$. Since $\mu(E^+) = t_0 > 0$ and $\mu(E^-) = \mu(X) - t_0 > 0$, E^+ and E^- are nonempty. Also $\mu(N) = 0$. By property T we have $\operatorname{dist}(E^+, E^-) = 0$ and so $\omega(h) \geqslant \delta$ for all $h > 0$. Thus f is discontinuous on X and (a) is proved.

(b): To prove (b) we will establish the slightly stronger inequality

$$\omega^*(h) \leqslant \omega(C' h^\beta) \tag{1}$$

under the assumptions that f is continuous on X and X has property T_β with constant $C < C'$. Fix $h > 0$ and $\varepsilon > 0$. Since f^* is monotone decreasing and continuous, we can find t and $t + h$ in $(0, \mu(X))$ satisfying

$$f^*(t) - f^*(t + h) > \omega^*(h) - \varepsilon.$$

Again, let $E^+ = \{|f| \geqslant f^*(t)\}$, $E^- = \{|f| \leqslant f^*(t + h)\}$ and $N = X \setminus (E^+ \cup E^-)$. Since $\mu(E^+) \geqslant t$ and $\mu(E^-) \geqslant \mu(X) - (t + h)$, we have $0 < \mu(N) \leqslant h$. Hence, by property T_β, we can find $x \in E^+$, $y \in E^-$ such that $d(x, y) \leqslant C' h^\beta$. The definitions of E^+ and E^- imply that $|f(x) - f(y)| \geqslant \omega^*(h) - \varepsilon$ and therefore $\omega(C' h^\beta) \geqslant \omega^*(h) - \varepsilon$. Let $\varepsilon \to 0$ to obtain (1).

Corollary 1. *The map $*$ takes* $\operatorname{Lip}(\alpha)(\boldsymbol{T}^n)$ *into* $\operatorname{Lip}(\alpha/n)([0, 1))$.

Corollary 2. *The map $*$ on $\boldsymbol{T}$ satisfies $\omega^*(h) \leqslant \omega(2h)$ and thus preserves* $\operatorname{Lip}\alpha$.

Proof. The only fine point here is to notice that the constant C appearing in the proof of (b) above may be taken to be 2 in this case.

Remark. Theorem 1 is quite sharp. For part (a) consider the function $g(x, y) = x$ on

$$X = ([0, 1]\times[0, 1])\cup([2, 3]\times[0, 1])\cup\{(x, 0):\ 1 < x < 2\}.$$

This function is continuous on X while g^* has a jump at $t = 1$. For part (b) consider the function $h(x, y) = x+y$ on $[0, 1]\times[0, 1]$. The function h is in $\mathrm{Lip}\,1$ but h^* is only in $\mathrm{Lip}\frac{1}{2}$ since $h^*(t) = 2-\sqrt{2t}$ for $0 \leqslant t \leqslant \frac{1}{2}$ and $h^*(t) = \sqrt{2-2t}$ for $\frac{1}{2} \leqslant t \leqslant 1$.

Define Λ_* to be the set of all complex-valued measurable functions f on T for which there exist positive numbers h_0 and K satisfying

(2) $$|f(x+h)+f(x-h)-2f(x)| \leqslant K|h|$$

whenever $|h| \leqslant h_0$. Note that $T = [0, 1)$ and that we add in T modulo 1.

THEOREM 2. *Let f be a complex-valued measurable function on T. Then $f\in\Lambda_*$ if and only if f is continuous and satisfies* (2) *for all h.*

Proof. We need only prove the forward implication, so consider $f\in\Lambda_*$. We first show that f is bounded on T. It suffices to show that, for each $x_0\in T$, f is bounded on some neighborhood of x_0. Without loss of generality we may assume $x_0 = 0$. Since

$$[0, h_0] = \bigcup_{m=1}^{\infty} [0, h_0]\cap\{|f| \leqslant m\},$$

there exists m so that $\mu(E_m) > 0$, where

$$E_m = [0, h_0]\cap\{|f| \leqslant m\}.$$

Let $r, s\in E_m$ and apply inequality (2) three times: first with $x = (r-s)/2$, $h = (r-s)/2$; then with $x = (r-s)/2$, $h = (r+s)/2$; and finally with $x = 0$, $h = s$ to get successively

(3) $$|f(r-s)| \leqslant 2|f((r-s)/2)|+|f(0)|+Kh_0,$$

(4) $$2|f((r-s)/2)| \leqslant |f(r)|+|f(-s)|+Kh_0,$$

(5) $$|f(-s)| \leqslant |f(s)|+2|f(0)|+Kh_0.$$

Putting (5) into (4) and then (4) into (3) yields

$$\begin{aligned}|f(r-s)| &\leqslant \{|f(r)|+(|f(s)|+2|f(0)|+Kh_0)+Kh_0\}+|f(0)|+Kh_0\\ &\leqslant 2m+3|f(0)|+3Kh_0.\end{aligned}$$

This shows that f is bounded on the set E_m-E_m. Since $\mu(E_m) > 0$, Steinhaus's theorem ([10], [11]) asserts that E_m-E_m contains a neighborhood of 0. We conclude that f is bounded on T, i.e. that $\|f\|_u < \infty$. It follows that f is in $\mathrm{Lip}\,\alpha$ for every $\alpha < 1$ and, in particular, continuous on T; see [12], page 44. Finally, if $|h| > h_0$, then

$$|f(x+h)+f(x-h)-2f(x)| \leqslant 4\|f\|_u < \frac{4\|f\|_u}{h_0}|h|,$$

and so inequality (2) holds (with a different K) for all h.

Define λ_* to be the set of all complex-valued measurable functions on T with the property that

(6) $$|f(x+h)+f(x-h)-2f(x)| = o(h) \quad \text{as} \quad |h|\to 0.$$

THEOREM 3. *For $X = T$, the map $*$ does not preserve λ_*, Λ_*, C^n, $n = 1, 2, \ldots$, or C^∞.*

This is somewhat unexpected since $\mathrm{Lip}\,1 \subset \Lambda_* > \mathrm{Lip}\,\alpha$ for each α, $0 < \alpha < 1$, and $*$ does preserve each $\mathrm{Lip}\,\alpha$, $0 < \alpha \leqslant 1$, by Corollary 2.

Proof. The function g, where $g(x) = \exp\{-x^2/(1-x^2)\}$ for $|x| < 1$ and $g(x) = 0$ elsewhere is C^∞. So is $f(x) = 2g(4(x-\frac{1}{4}))+g(4(x-\frac{3}{4}))$, which we consider as a function on T. The graphs of f and f^* are below.

The inverse to g restricted to $[0, 1]$ is h, where

$$h(y) = \sqrt{\frac{\ln(1/y)}{1+\ln(1/y)}}.$$

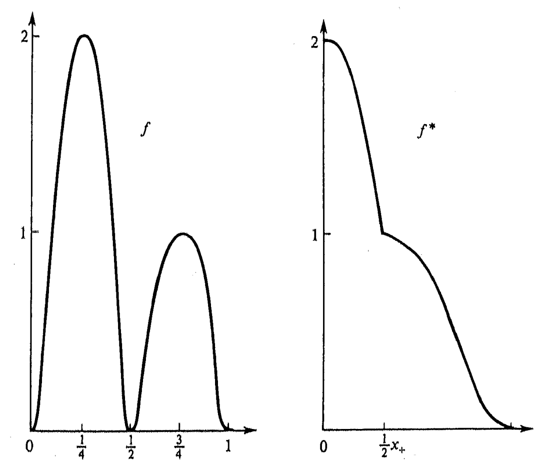

Fig. 1

Since $g(x_+) = g(x_-) = \frac{1}{2}$, where

$$x_\pm = \pm h(\tfrac{1}{2}) = \pm\sqrt{\frac{\ln 2}{1+\ln 2}},$$

it follows that $f_*(y) = (f^*)^{-1}(y) = \frac{1}{2}h(y/2)$ for $1 \leqslant y \leqslant 2$ and $f_*(y) = \frac{1}{2}h(y/2)+\frac{1}{2}h(y)$ for $0 < y \leqslant 1$. By first calculating the derivatives of f_* at $f^*(\frac{1}{2}x_+)$, we find that at $\frac{1}{2}x_+$, f^* has a left derivative of $-4\sqrt{\ln 2(1+\ln 2)^3}$ and a right derivative of 0. This corner at $\frac{1}{2}x_+$ disqualifies f^* from $C^\infty \cup C^1 \cup C^2 \cup \ldots$ and also from λ_* since relation (6) may be rewritten as

$$\frac{f(x+h)-f(x)}{h} - \frac{f(x)-f(x-h)}{h} = o(1).$$

For the part of Theorem 3 concerning Λ_*, let g be the C^∞ function given above, set $l(x) = x\ln|x|$, $x \neq 0$, $l(0) = 0$, and finally form

$$f(x) = g(4(x-\tfrac{1}{4}))\{l(x-\tfrac{1}{4})+1\}+g(4(x-\tfrac{3}{4})).$$

The graphs of f and f^* are given in Figure 2.

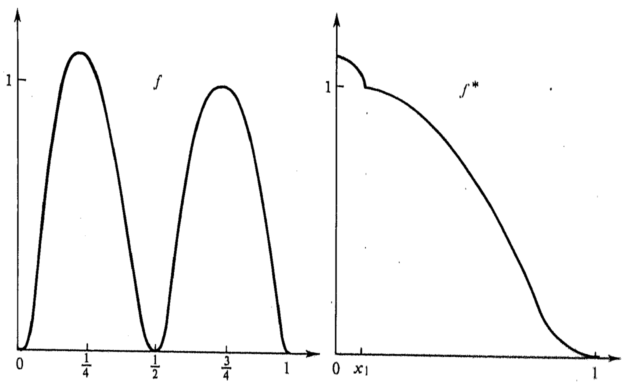

Fig. 2

We first observe that $l \in \Lambda_*$. To see this, fix $h > 0$ and form $L(x) = [l(x+h)+l(x-h)-2l(x)]h^{-1}$. Now $L(h) = \ln 4 > 0$, $l'' = x^{-1}$ and $l''' = -x^{-2}$ so that l is concave upward and l' is concave downward on $(0, \infty)$, whence L is positive and decreasing on (h, ∞). Since L is odd,

$$\sup|L| = \sup_{x\in[0,h]}|L| = |L(h/\sqrt{2})| = \ln(3+2\sqrt{2}) < \infty.$$

We compute (numerically) that $f(1/4) = f(.145\ldots) = 1$ and that $f'(1/4) = -\infty$ since $l'(0) = -\infty$. Letting $x_1 = 1/4-.145\ldots$, we have $f^*(x_1) = 1$ and f^* has a left derivative at x_1 of $-\infty$. The contribution of the latter term of f is sufficient to force the right derivative of f^* at x_1 to be 0. Hence as $h \to 0^+$,

$$\frac{f^*(x_1+h)-f^*(x_1)}{h} - \frac{f^*(x_1)-f^*(x_1-h)}{h} \to 0+\infty$$

and f^* is not in Λ_*.

§ 3. The Lorentz spaces $L^{p,q}(\hat{D})$. Let D be R with the discrete topology. The Bohr group $\hat{D}$ is the set of all functions φ of D into $\{z\in C: |z| = 1\}$ satisfying $\varphi(x+y) = \varphi(x)\varphi(y)$. With the finite-open topology, $\hat{D}$ is a compact abelian group. The reals with the usual topology are densely embedded in $\hat{D}$ via the mapping $\alpha \to \varphi_\alpha$ where $\varphi_\alpha(x) = e^{2\pi i\alpha x}$. This mapping also embeds the rationals densely in $\hat{D}$, so $\hat{D}$ is separable. However, $\hat{D}$ is not first countable and hence not metrizable. Moreover, $\hat{D}$ has cardinality 2^c, where c is the power of the continuum, by a theorem of Kakutani; see [4], 24.47.

Let μ be Haar measure on $\hat{D}$ so that $\mu(\hat{D}) = 1$. The space AP of almost periodic functions on R can be identified with the space $C(\hat{D})$ of all continuous functions on $\hat{D}$. In fact, each f in $C(\hat{D})$ is associated with its restriction to the dense image of R in $\hat{D}$. Haar measure on $\hat{D}$ is determined by the fundamental identity

(7) $$\int_{\hat{D}} f\,d\mu = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x)\,dx.$$

All this can be found, elegantly phrased, in § 41 of [7]. The fascinating concrete background calculations can be found in Bohr's book [3].

The space $L^{p,q}(\hat{D})$ is the space of all complex-valued measurable functions f on $\hat{D}$ for which $\|f\|_{p,q} < \infty$, where

(8) $$\|f\|_{p,q}^q = \frac{q}{p}\int_0^1 [f^*(t)\,t^{1/p}]^q\frac{dt}{t}, \quad 1 \leqslant p,\ q < \infty$$

and

(9) $$\|f\|_{p,\infty} = \sup_{t\in(0,1)} t^{1/p-1}\int_0^t f^*(s)\,ds, \quad 1 \leqslant p \leqslant \infty.$$

In the notation of Stein and Weiss [9], our $\|f\|_{pq}$ is $\|f\|_{pq}^*$ if $q < \infty$ and our $\|f\|_{p\infty}$ is $\|f\|_{p\infty}$.

Equation (7) suggests that $L^{p,q}(\hat{D})$ should be definable in terms of the more concrete spaces $L^{p,q}([-T, T])$. In fact, the following is possible.

0) If $f \in \mathrm{AP}$ and $T > 0$, endow $[-T, T]$ with the normalized Lebesgue measure $dx/2T$ and form $\|f\|_{p,q,T}$ using equation (8) or (9).

1) Show that, for $f \in \mathrm{AP}$, the limit

$$|f|_{p,q} = \lim_{T \to \infty} \|f\|_{p,q,T}$$

exists. This defines $|f|_{p,q}$ on $C(\hat{D})$ via the identification of AP with $C(\hat{D})$.

2) Show that $|f|_{p,q} = \|f\|_{p,q}$ for $f \in C(\hat{D})$.

The completion of this three step program provides an alternative definition of $L^{p,q}(\hat{D})$ when $q < \infty$ since the continuous functions are dense in $L^{p,q}$, $q < \infty$. (To see this, note that simple functions are dense in $L^{p,q}$, $q < \infty$; [5], page 258. Then note that a simple function can be closely approximated in L^r norm by a continuous function if $r < \infty$, [4], 12.10. Finally, this approximation is also good in $L^{p,q}$ if $r = \max\{p, q\} < \infty$ since from (8) $\|f\|_{p,q} \leqslant (r/p)^{1/q} \|f\|_{r,q}$ and from (1.8) of [5], $\|f\|_{r,q} \leqslant \|f\|_{q,q} = \|f\|_q$.) Even for $L^{p,\infty}(\hat{D})$, where $C(\hat{D})$ is *not* dense, the three step program has merit because of the duality between $L^{p,\infty}$ and $L^{p',1}$ and the density of $C(\hat{D})$ in $L^{p',1}$. See equations (2.28) and (2.29) of [1] for this duality and the proof of Theorem 5 in that paper for an application of this duality.

THEOREM 4. *Let $f \in C(\hat{D}) \approx \mathrm{AP}$ and fix $p, q \geqslant 1$. Then $|f|_{p,q}$ exists and is equal to $\|f\|_{p,q}$.*

Proof. For each natural number n, let μ_n be the measure on $\hat{D}$ corresponding to the normalized Lebesgue measure $dx/2n$ on $[-n, n]$. Identity (7) tells us that

$$\int_{\hat{D}} f\,d\mu = \lim_{n \to \infty} \int_{\hat{D}} f\,d\mu_n \quad \text{for} \quad f \in C(\hat{D}),$$

i.e., $\mu_n \to \mu$ weakly. Since Haar measure μ is regular, the present theorem follows from the next theorem.

THEOREM 5. *Let μ and μ_n be finite Borel measures on a compact space X and assume μ is regular. If $\mu_n \to \mu$ weakly, then*

$$\lim_{n \to \infty} \|f\|_{p,q,\mu_n} = \|f\|_{p,q,\mu} \quad \textit{for all } f \in C(X),$$

where for every measure ν, $\|f\|_{p,q,\nu}$ is given by equation (8) or (9) with f^* replaced by

$$f_\nu^*(t) = \inf\{\alpha\colon f_{*,\nu}(\alpha) \leqslant t\},$$

where $f_{*,\nu}(\alpha) = \nu\{x \in X\colon |f(x)| > \alpha\}$.

Proof. Consider fixed f in $C(X)$. Each of the functions $f_{\mu_n}^*, f_{*,\mu_n}, f_\mu^*, f_{*,\mu}$ is nonincreasing and right continuous. Let D be the countable set of points at which at least one of these functions is discontinuous. First we show

(10) $$\lim_{n \to \infty} f_{*,\mu_n}(\alpha) = f_{*,\mu}(\alpha) \quad \text{for all } \alpha \notin D.$$

Let $A = \{|f| > \alpha\}$; we must show $\lim_{n \to \infty} \mu_n(A) = \mu(A)$. Since f is continuous, the boundary ∂A of A is contained in $\{|f| = \alpha\}$. For each n,

$$\mu(\partial A) \leqslant \mu\{\alpha - 1/n < |f| \leqslant \alpha + 1/n\} = f_{*,\mu}(\alpha - 1/n) - f_{*,\mu}(\alpha + 1/n).$$

Since $f_{*,\mu}$ is continuous at α, we conclude that $\mu(\partial A) = 0$. This is enough to guarantee $\lim_{n \to \infty} \mu_n(A) = \mu(A)$, and hence (10) holds, by a standard argument. See, for example, Theorem 4.5.1 of [2]. In the proof of that theorem X is assumed to be metric, but the proof goes through for X merely compact Hausdorff provided the limit measure μ is regular.

Assume $q < \infty$. Next we show

(11) $$\lim_{n \to \infty} f_{\mu_n}^*(t) = f_\mu^*(t) \quad \text{for all } t \notin D.$$

Assume (11) fails for some (fixed) $t \notin D$. Passing to a subsequence, if necessary, we may suppose that there is an $\varepsilon > 0$ such that either

(12) $$f_{\mu_n}^*(t) > f_\mu^*(t) + \varepsilon \quad \text{for all } n$$

or

(13) $$f_{\mu_n}^*(t) < f_\mu^*(t) - \varepsilon \quad \text{for all } n.$$

The cases (12) and (13) seem to require separate arguments. Assume (12) holds. Since f_μ^* is continuous at t, there is $\delta > 0$ so that $f_\mu^*(t-\delta) < f_\mu^*(t) + \varepsilon/2$ and hence

(14) $$f_{\mu_n}^*(t) > f_\mu^*(t-\delta) + \varepsilon/2 \quad \text{for all } n.$$

Select $\alpha_0 \notin D$ satisfying

(15) $$f_\mu^*(t-\delta) < \alpha_0 < f_\mu^*(t-\delta) + \varepsilon/2.$$

From (15) we infer $f_{*,\mu}(\alpha_0) \leqslant t - \delta$. Since $\lim_{n \to \infty} f_{*,\mu_n}(\alpha_0) = f_{*,\mu}(\alpha_0)$ by (10), there exists N so that $f_{*,\mu_n}(\alpha_0) < t$ for $n > N$. This implies that $f_{\mu_n}^*(t) \leqslant \alpha_0$ for $n > N$. This is contradicted by (14) and (15) which together imply $f_{\mu_n}^*(t) > \alpha_0$ for all n. Now assume (13) holds. Since f_μ^* is right continuous at t, there is $\delta > 0$ so that $f_\mu^*(t+\delta) > f_\mu^*(t) - \varepsilon/2$ and hence

(16) $$f_{\mu_n}^*(t) < f_\mu^*(t+\delta) - \varepsilon/2 \quad \text{for all } n.$$

Select $\alpha_0 \notin D$ satisfying

(17) $$f_\mu^*(t+\delta) - \varepsilon/2 < \alpha_0 < f_\mu^*(t+\delta).$$

Then we have $f_{*,\mu}(\alpha_0) > t + \delta$. Again (10) shows that there is N such that

$f_{*,\mu_n}(\alpha_0) > t$ for $n > N$. A routine argument, using the continuity of f_{*,μ_n} at α_0, shows that $f_{\mu_n}^*(t) > \alpha_0$ for $n > N$. On the other hand, $f_{\mu_n}^*(t) < \alpha_0$ for all n by (16) and (17). Thus (13) and (12) are both impossible and we conclude that (11) holds.

To show

$$\lim_{n\to\infty} \|f\|_{p,q,\mu_n} = \|f\|_{p,q,\mu}$$

it suffices to show

$$(18) \qquad \lim_{n\to\infty} \int_0^\infty [f_{\mu_n}^*(t)\, t^{1/p}]^q \frac{dt}{t} = \int_0^\infty [f_\mu^*(t)\, t^{1/p}]^q \frac{dt}{t}.$$

By (11) the integrands converge almost everywhere, so we need only verify that the convergence is dominated. Weak convergence implies $\lim_{n\to\infty} \mu_n(X) = \mu(X)$ and so $M = \sup_n \mu_n(X) < \infty$. Since $f_{*,\mu}(\alpha) = 0$ for $\alpha \geqslant \|f\|_u$, we have $f_\mu^*(t) \leqslant \|f\|_u$ for $t \geqslant 0$. Also, $f_{*,\mu}(\alpha) \leqslant M$ for all $\alpha \geqslant 0$ and so. $f_\mu^*(t) = 0$ for $t \geqslant M$. The same observations apply to μ_n and so all the integrands in (18) are dominated by $g(t)$, where g is $t^{(q/p)-1}\|f\|_u$ times the characteristic function of $[0, M]$. Since $\int_0^M t^{(q/p)-1}\, dt < \infty$, g is integrable and so (18) holds.

Let $q = \infty$. Arguing as above, it is easy to show that for each $t \in (0, 1)$, $\int_0^t f_{\mu_n}^*(s)\, ds \to \int_0^t f_\mu^*(s)\, ds$. The $q = \infty$ cases of Theorem 4 follow immediately.

§ 4. **Applications.** The group $\hat{\boldsymbol{D}}$ is a large abstract object compared with $\boldsymbol{R}$, but it has one important advantage: compactness. This gives rise to a method of transference which can sometimes be used to get results for $\boldsymbol{R}$ which were previously known for compact groups. We give two examples of this, the first due to deLeeuw [6].

THEOREM. *A bounded continuous function f on $\boldsymbol{R}$ is a multiplier for the space of Fourier transforms on $L_p(\boldsymbol{R})$, $1 < p < 2$, if and only if there is a constant K satisfying the following: for each choice $\{a_j, b_j, \lambda_j\}$ of real numbers satisfying*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \Big|\sum_{j=1}^n a_j e^{i\lambda_j x}\Big|^p dx \leqslant 1$$

and

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \Big|\sum_{j=1}^n b_j e^{i\lambda_j x}\Big|^{p'} dx \leqslant 1$$

one has

$$\Big|\sum_{j=1}^n a_j b_j f(\lambda_j)\Big| \leqslant K; \qquad \text{here} \qquad 1/p + 1/p' = 1.$$

This and two companion results essentially equate the multiplier operators on $L^p(\boldsymbol{R})$ with those on $L^p(\hat{\boldsymbol{D}})$. (A typical multiplier operator is T_f: $\varphi \to (f(x)\hat{\varphi}(x))^\vee$, where f is a bounded continuous function. See [6] for exact definitions.)

For our second example, we begin by observing that if T_f is a bounded multiplier operator from $L^2(G)$ to $L^{2,1}(G) =$ weak $L^2(G)$ and if G is compact, then it is almost immediate that f is a bounded function (test f on characters to see this). Hence T_f maps $L^2(G)$ into $L^2(G)$. As an application of transference this result can be extended to the case of $G = \boldsymbol{R}$. To do this, deLeeuw's idea of identifying the multiplier operators on $L^p(\boldsymbol{R})$ with those on $L^p(\hat{\boldsymbol{D}})$ had to be generalized to identifying the multiplier operators which take $L^{p,q_1}(\boldsymbol{R}) \to L^{p,q_2}(\boldsymbol{R})$ with those that take $L^{p,q_1}(\hat{\boldsymbol{D}}) \to L^{p,q_2}(\hat{\boldsymbol{D}})$. This idea (with $p = 2$, $q_1 = 1$, $q_2 = \infty$) was proposed by Misha Zafran and executed in [1].

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