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On the Rényi theory of conditional probabilities

by

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Abstract. The Rényi theory of conditional probability spaces is studied. Various concepts are introduced and discussed. In particular, the convolution of Rényi probability distributions and their convergence as well as the convergence of the respective Fourier transforms are defined and described by some equivalent conditions. Connections with the theory of distributions are shown. In particular, characterizations for the convolution of (tempered) distributions to exist (and to be a tempered distribution) are given in terms of supports. The concept of a distributor and operations on distributors are considered.

Introduction. In the fifties, Alfred Rényi gave (see paper [16] and book [17]) an axiomatic approach to probability theory which is a generalization of the classical theory of Kolmogorov and creates some new mathematical objects: unbounded probability distributions. An example of such an unbounded distribution is the uniform distribution on the whole line, determined by the density function, constant on $(-\infty, \infty)$. Such a concept, interesting in itself, appears to be useful in some situations in probability theory and not only, particularly when the limit probability distributions are concerned (for examples of applications see [16]). A mathematical justification of this concept is possible on the base of the notion of conditional probability, being a starting point in Rényi's approach.

The theory originated by A. Rényi is still in an initial phase of its development. It seems there are at least two reasons of this situation. First, the development of Rényi's theory met with obstacles connected with the necessity of solving some problems in the theory of distributions. Secondly, probability theory based on Kolmogorov's axioms has stood for decades an entirely satisfactory fundament for descriptions of random phenomena.

There are, however, arguments for further investigations in Rényi's theory. Apart from purely cognitive reasons, we can indicate just connections with the theory of distributions, which are very interesting and stimulating for both theories.

The common point of the theories is the Fourier transform. In the classical probability theory, Fourier transforms (characteristic functions) are defined for all probability distributions and are continuous functions. For Rényi's probability distributions which are, in general, unbounded, Fourier transforms do not exist in the classical sense, but they do, if Rényi distributions are of polynomial growth, in the sense of tempered distributions of L. Schwartz (see [18]).

As is well known, the sum of random variables in Kolmogorov probability spaces corresponds to the convolution of their probability distributions and with the product of their characteristic functions.

The same can be expected in Rényi probability spaces. However neither the product of Schwartz (tempered) distributions nor the convolution of unbounded Rényi distributions exist, in general. The question of feasibility of the operations of convolution and product for tempered distributions as well as connections between the operations with respect to the Fourier transform are interesting in themselves and were studied in several papers (see, e.g., [4], [20], [5], [6], [11] and Section 5).

But in the particularly interesting case of the uniform probability distributions on $(-\infty, \infty)$ neither the convolution $1 * 1$ nor the product $\delta \cdot \delta$ of the respective Schwartz distributions exist, provided the operations are defined in a natural way (some authors define the product of distributions in such a way that $\delta \cdot \delta = 0$, but those definitions are somewhat artificial).

A natural solution of this problem appears owing to the fact that Rényi probability distributions and their Fourier transforms are not singular distributions, but equivalent classes with respect to the relation: $f \sim g$ if $f = \alpha g$ for some $\alpha > 0$; we call them distributors. The convolution and the product of distributors are defined by using a modification of the Mikusiński method of irregular operations and the quotient convergence of distributors. Both operations exist in this sense for all tempered distributors, though the result can be the zero distributor. For many pairs of distributors, however, the operations lead to non-trivial results.

In particular, we get

$$[1] * [1] = [1] \quad \text{and} \quad [\delta] \cdot [\delta] = [\delta],$$

where $[1]$ and $[\delta]$ are distributors represented by the distributions 1 and δ, respectively (see Section 6). These results have the following probabilistic interpretation: the sum of independent random variables with the uniform distribution on $(-\infty, \infty)$ has also the uniform distribution on $(-\infty, \infty)$.

The results of this paper stand only for a fragment of Renyi's theory being a natural continuation of his ideas and a completion of some of his results (see [16] and [17]); the paper develops also ideas presented in [8].

In Section 1, concepts of Rényi probability distributions, distribution functions and characteristic distributors are introduced and discussed.

In Section 2, the convergence and the tempered convergence of Rényi probability distributions (Rényi distribution functions) and the convergence of characteristic distributors are introduced and described in some equivalent conditions.

In Section 3, the convolution of Rényi probability distributions (Rényi

distribution functions) is defined; it is shown that, if ξ and η are independent random variables, then $[F_{\xi+\eta}] = [F_\xi] * [F_\eta]$, where $[F_\xi]$, $[F_\eta]$, $[F_{\xi+\eta}]$ are the Rényi distribution functions of ξ, η, $\xi+\eta$, respectively.

In Sections 4 and 5, the convolution of non-negative measures and distributions are discussed. In particular, characterizations for the convolution of distributions (and tempered distributions) to exist (and to be a tempered distribution again) are given in terms of supports.

Section 6 is an outline of the theory of operations on distributors.

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1. Rényi probability distributions. By a *Rényi space*, we mean a system $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$, where Ω is an arbitrary set, $\mathscr{A}$ is a σ-algebra of subsets of Ω, $\mathscr{B}$ is a non-empty subfamily of $\mathscr{A}$ and P is a non-negative function on $\mathscr{A} \times \mathscr{B}$ (*conditional probability*) satisfying the following axioms:

(I) $P(B|B) = 1$ for every $B \in \mathscr{B}$;

(II) $P(\bigcup_{i=1}^{\infty} A_i|B) = \sum_{i=1}^{\infty} P(A_i|B)$ for any disjoint sets $A_i \in \mathscr{A}$ and $B \in \mathscr{B}$;

(III) If $A \in \mathscr{A}$, B, $B' \in \mathscr{B}$, $B \subset B'$ and $P(B|B') > 0$, then

$$P(A|B) = \frac{P(A \cap B|B')}{P(B|B')}.$$

These axioms were given by A. Rényi in [17], p. 70 (see also [16]). Various properties of conditional probability following from (I)–(III) can be found in [16] or in [12]. In particular, it follows from (I)–(III) that the system $[\Omega, \mathscr{A}, P_B]$, where $P_B(A) = P(A|B)$ for each $B \in \mathscr{B}$ and $A \in \mathscr{A}$, is the usual probability space in the sense of Kolmogorov.

Axioms (I)–(III) imply, in particular, that $\emptyset \notin \mathscr{B}$ and it can be said not much more about the family $\mathscr{B}$, in general. Therefore, it is sometimes convenient to adopt some additional axioms, e.g:

(IV) $$P(A^1|B^1) \cdot P(A^2|B^1) = P(A^2|B^1) \cdot P(A^1|B^2)$$

for $A^1, A^2 \in \mathscr{A}$, $B^1, B^2 \in \mathscr{B}$ such that $A^1, A^2 \subset B^1 \cap B^2$;

(IV′) If $B^1, B^2 \in \mathscr{B}$ and $P(B^1|B^2) + (P(B^2|B^1) > 0$, then $B^1 \cap B^2 \in \mathscr{B}$.

Note that (IV′) implies (IV) (see [12]).

As it was shown in [12], these axioms guarantee possibility of extending the family $\mathscr{B}$ by joining to it some sets from $\mathscr{A}$ and defining in a proper way conditional probability P for this extended family. For example, under (I)–(IV), the family $\mathscr{B}$ can be extended by adding arbitrary sets $B \in \mathscr{A}$ such that $B \subset B'$ and $P(B|B') > 0$ for some $B' \in \mathscr{B}$. In particular, we can join to $\mathscr{B}$ all such sets; then the extended Rényi space fulfils axiom (IV′).

For examples of Rényi spaces see [16] and [17] (p.248). In [2], characterizations of Rényi spaces are given to be represented by a family of non-negative measures (bounded or unbounded) with various conditions of compatibility.

Let $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ be a fixed Rényi space. By *random variables* in $\mathscr{R}$ we mean, similarly to the usual definition in the Kolmogorov approach, functions defined on Ω, measurable with respect to $\mathscr{A}$. In the sequel, we shall consider random variables with values in R^q or in R.

In further considerations the symbol K will be reserved for a fixed set in R^q of the form:

$$K = K^1 \times \ldots \times K^q,$$

where, for each $i = 1, 2, \ldots, q$,

(1) $$K^i = (a_i, b_i) \quad \text{with} \quad -\infty \leqslant a_i < b_i \leqslant \infty$$

or

(2) $$K^i = [a_i, b_i) \quad \text{with} \quad -\infty < a_i < b_i \leqslant \infty.$$

In particular, one can put $K = R^q$.

By *intervals* in K, we shall mean sets of the form $I = I^1 \times \ldots \times I^q \subset K$, where $I^i = [\alpha_i, \beta_i)$ with $-\infty < \alpha_i < \beta_i < \infty$ for $i = 1, \ldots, q$.

For $A \subset R^q$, we shall write $A \Subset K$ if the closure of A is compact and contained in K.

For a given random variable ξ with values in the set K, denote by $\mathscr{M}_\xi$ the set of all intervals $I \Subset K$ such that $\xi^{-1}(I) \in \mathscr{B}$.

We assume that

(d_0) $$\mathscr{M}_\xi \neq \emptyset.$$

By $B(K)$, denote the σ-algebra of all Borel subsets of K and by $B_0(K)$ the family of all $E \in B(K)$ such that $E \Subset K$.

We shall use the following notation:

$$P(\xi^{-1}(E_1)|\xi^{-1}(E_2)) = P(\xi \in E_1|\xi \in E_2)$$

for any Borel subsets E_1, E_2 of R^q.

By a *Rényi probability distribution* of ξ in $K \subset R^q$, we mean the class of all non-trivial (i.e., not identically equal to 0), non-negative measures μ_ξ, defined on $B(K)$, finite on $B_0(K)$ and satisfying the identity

(3) $$P(\xi \in I_1|\xi \in I_2) = \frac{\mu_\xi(I_1)}{\mu_\xi(I_2)}$$

for all intervals I_1, I_2 such that $I_1 \subset I_2$, $I_2 \in \mathscr{M}_\xi$ and $\mu_\xi(I_2) > 0$.

In view of axiom (III), every measure μ satisfying (3) in K fulfils also the equality

$$P(\xi \in E_1|\xi \in E_2) = \frac{\mu_\xi(E_1)}{\mu_\xi(E_2)}$$

for arbitrary Borel subsets E_1, E_2 of K such that $\mu_\xi(E_2) > 0$ and $E_1 \subset E_2 \subset J$ for some interval $J \in \mathscr{M}_\xi$.

Note that if a measure μ_ξ satisfies (3), then the measure $a\mu_\xi$ for every $a > 0$ also satisfies (3). The converse is true if the random variable ξ fulfils the following condition, stronger than (d_0) (cf. [17], p. 245):

(d_1) there exists a non-decreasing sequence of intervals $K_n \in \mathscr{M}_\xi$ such that $K = \bigcup_{n=1}^{\infty} K_n$.

Namely, we have

THEOREM 1.1. *Suppose that a random variable ξ with values in K fulfils condition (d_1). If μ_ξ and ν_ξ are two representatives of the Rényi probability distribution in K, then*

(4) $$\nu_\xi(E) = \alpha\mu_\xi(E) \quad (E \in B(K))$$

for some constant $\alpha > 0$.

Proof. Let I_1, I_2 be two intervals such that $I_1, I_2 \Subset K$. Since $\mu = \mu_\xi$ and $\nu = \nu_\xi$ are non-trivial, measures, we have $\mu(I_1) > 0$, $\nu(I_2) > 0$ for some intervals $J_1, J_2 \Subset K$. By (d_1) we have

$$I_1 \cup I_2 \cup J_1 \cup J_2 \subset K_{n_0}$$

for some $n_0 \in N$ and thus $\mu(K_{n_0}) > 0$ and $\nu(K_{n_0}) > 0$. In view of (3), we get

(5) $$\frac{\mu(I_i)}{\mu(K_{n_0})} = \frac{\nu(I_i)}{\nu(K_{n_0})} \quad (i = 1, 2).$$

If $\mu(I_i) > 0$ $(i = 1, 2)$, then (5) yields

$$\frac{\nu(I_1)}{\mu(I_1)} = \frac{\nu(K_{n_0})}{\mu(K_{n_0})} = \frac{\nu(I_2)}{\mu(I_2)},$$

which implies (4) for all intervals $E \Subset K$ such that $\mu(E) > 0$. But if $\mu(E) = 0$ for any interval $E \Subset K$, then we can apply the above arguments for $I_i = E$ ($i = 1$ or 2) and (5) gives $\nu(E) = 0$. This means, equality (4) holds for all intervals $E \Subset K$ and, consequently, for all $E \in B(K)$.

In the sequel, the Rényi probability distribution of a random variable ξ in K will be denoted by $[\mu_\xi]$ or $[\mu]$, where $\mu_\xi = \mu$ is an arbitrary of representants of the class. Under condition (d_1), we have $[\mu] = \{\alpha\mu : \alpha > 0\}$, in view of Theorem 1.1.

For a given point function $F\colon K \to R$ and arbitrary points $a = (a_1, \ldots, a_q)$ and $b = (b_1, \ldots, b_q)$ in K, we denote

(6) $$\Delta_{ab} F = \sum_{\varepsilon} (-1)^{q-\varepsilon_1-\ldots-\varepsilon_q} \cdot F(a+\varepsilon(b-a)),$$

where the sum extends over all systems $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_q)$ such that $\varepsilon_1 = 0$ or 1 for $i = 1, \ldots, q$ and

$$\varepsilon(b-a) = (\varepsilon_1(b_1-a_1), \ldots, \varepsilon_q(b_q-a_q)).$$

Of course,

(7) $$\Delta_{ab} F = \Delta'_{a_1b_1} \Delta_{a_2b_2} \ldots \Delta'_{a_qb_q} F,$$

where the operator $\Delta'_{a_ib_i}$ $(i = 1, \ldots, q)$ to functions F of i variables assigns functions of $i-1$ variables in the following way:

(8) $$\Delta'_{a_ib_i} F(x_1, \ldots, x_{i-1}) = F(x_1, \ldots, x_{i-1}, b_i) - F(x_1, \ldots, x_{i-1}, a_i).$$

If $a \leqslant b$ (i.e., $a_i \leqslant b_i$ for $i = 1, \ldots, q$) and $I = [a, b)$, then we apply the notation:

(9) $$\Delta_I F = \Delta_{ab} F.$$

For a fixed $x_0 = (x_1^0, \ldots, x_q^0) \in K$ and an arbitrary $x = (x_1, \ldots, x_q) \in K$, let

(10) $$I_x = [\alpha_1, \beta_1) \times \ldots \times [\alpha_q, \beta_q),$$

where

(11) $$\alpha_i = \min(x_1^0, x_i), \qquad \beta_i = \max(x_i^0, x_i).$$

By (7) and (8), we have

(12) $$\Delta_{I_x} F = (-1)^k \Delta_{x_0x} F,$$

where k is the number of those indices i for which $x_i < x_i^0$.

Given a point function $F\colon K \to R$, we denote by $\{F\}$ the class of all functions G of the form:

(13) $$G(x) = F(x) + \sum_{i=1}^{q} F_i(y_i),$$

where $x = (x_1, \ldots, x_q) \in K$,

(14) $$y_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_q)$$

and F_i are arbitrary functions of $q-1$ variables.

It is clear that $\Delta_I G = \Delta_I F$ for each $I \Subset K$. That means, to a given class $\{F\}$ of point functions in K the interval function $\Psi(I) = \Delta_I F$ in K corresponds.

Conversely, given an interval function $\Psi(I)$ in K and a fixed point $x_0 \in K$, let

(15) $$F_{x_0}(x) = (-1)^k \Psi(I_x),$$

where I_x and k have the same meaning as in (12). It is easy to see that if x_0' is another fixed point in K, then $F_{x_0'} \in \{F_{x_0}\}$. Note that, for a given function $F\colon K \to R$, if $\Psi(I) = \Delta_I F$ for $I \Subset K$, then

$$\Delta_I G = \Delta_I F \qquad (I \Subset K)$$

for every $G \in \{F_{x_0}\}$, where F_{x_0} is given by formula (15).

LEMMA 1.1. *The relation*

$$\Psi(I) = \Delta_I F$$

establishes a one-to-one correspondence between classes $\{F\}$ of point functions in K and interval functions Ψ in K.

Proof. According to the remarks above, it suffices to show that if

(16) $$\Delta_I F = \Psi(I) = \Delta_I G$$

for all $I \Subset K$, then (13) holds for some functions F_i $(i = 1, \ldots, q)$. Fix $x_0 \in K$. By (16) and (12), we have

(17) $$\Delta_{x_0x}(G-F) = 0$$

for each $x \in K$. Hence, by using definition (6), we deduce (13) with the following functions F_i:

$$F_i(y_i) = \sum_{\varepsilon} (-1)^{q-i-\varepsilon_{i+1}-\ldots-\varepsilon_q} \cdot H(x_0+\varepsilon(x-x_0)),$$

where y_i is defined in (14), $H = G-F$ and the sum is taken over all systems $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_q)$ such that $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_{i-1} = 1$, $\varepsilon_i = 0$ and $\varepsilon_j = 0$ or 1 for j: $i < j \leqslant q$. The assertion is proved.

We say that a point function F in K is (a) *nondecreasing* if $\Delta_I F \geqslant 0$ for each interval $I \Subset K$, (b) *left-continuous* if

$$F(x_1^n, \ldots, x_q^n) \to F(x_1, \ldots, x_q),$$

provided $x_1^n \nearrow x_1, \ldots, x_q^n \nearrow x_q$.

Let $\Psi(I)$ be an interval function in K and consider the following conditions:

(i) $\Psi(I) \geqslant 0$ for each $I \Subset K$;

(ii) If $\bigcup_{i=1}^{n} I_i = I \Subset K$ for disjoint intervals I_i, then $\Psi(I) = \sum_{i=1}^{q} \Psi(I_i)$;

(iii) $\Psi([a_n, b_n)) \to \Psi([a, b))$ for any $a_n, b_n, a, b \in K$ such that $a_n \nearrow a$, $b_n \nearrow b$.

One can check that the following relations hold:

LEMMA 1.2. *If in a class $\{F\}$ of point functions in K there exists a left-continuous and nondecreasing representative, then the corresponding interval function Ψ in K fulfils* (i)–(iii).

Conversely, if an interval function in K fulfils (i)–(iii), *then all representatives of the corresponding class $\{F\}$ are nondecreasing and there exists a representative of $\{F\}$ (given by formula* (15)) *which is left-continuous.*

As in [13] (Correspondence Theorem, p. 96), one can prove that

LEMMA 1.3. *The relation*

$$\mu(I) = \Psi(I) \qquad (I \Subset K)$$

establishes a one-to-one correspondence between non-negative measures μ on $B(K)$ (finite on $B_0(K)$) and interval functions in K satisfying conditions (i)–(iii).

By the *Rényi distribution function* of a random variable in K, we mean the class of all functions F_ξ: $K \to R$ for which there exist representatives μ_ξ of the Rényi probability distribution of ξ such that

$$\Delta_I F_\xi = \mu_\xi(I) \tag{18}$$

for each interval $I \Subset K$ or, equivalently, the class of all nondecreasing functions F_ξ (which are non-trivial, i.e., $\Delta_I F_\xi > 0$ for some $I \Subset K$) such that

$$P(\xi \in I \mid \xi \in J) = \Delta_I F_\xi / \Delta_J F_\xi \tag{19}$$

for any intervals $I \subset J \Subset K$, $J \in \mathcal{M}_\xi$, provided $\Delta_J F_\xi > 0$.

In the sequel, the Rényi distribution function of a random variable ξ in K will be denoted by $[F_\xi]$ or $[F]$, where $F_\xi = F$ is an arbitrary representative of the class.

THEOREM 1.2. *Let $\{F\}$ be the Rényi distribution function in K of some random variable ξ and let $\Psi(I) = \Delta_I F$ for $I \Subset K$. Then Ψ fulfils conditions* (i)–(iii).

Proof. Properties (i)–(ii) follow immediately from the equality $\Psi(I) = \mu(I)$, where μ is the corresponding representative of the Rényi probability distribution of ξ.

To prove (iii) notice that

$$\Psi([a_n, b_n)) = \mu(A_n \setminus B_n) = \mu(A_n) - \mu(B_n),$$

where $A_n = [a_n, b)$ and $B_n = [a_n, b) \setminus [a_n, b_n)$. Since $A_n \searrow [a, b)$ and $B_n \searrow \emptyset$, we have

$$\mu(A_n) \to \mu([a, b)), \qquad \mu(B_n) \to 0,$$

i.e.,

$$\Psi([a_n, b_n)) \to \Psi([a, b)),$$

as desired.

As a consequence of Theorems 1.1 and 1.2 and the preceding lemmas, we obtain

THEOREM 1.3. *The relation*

$$\mu(I) = \Delta_I F \qquad (I \Subset K)$$

establishes a one-to-one correspondence between the Rényi probability distribution and the Rényi distribution function of a given random variable ξ.

Moreover, under condition (d$_1$), *the Rényi distribution function $[F]$ of ξ is the class of all functions G of the form*

$$G(x) = \alpha F(x) + \sum_{i=1}^{q} F_i(y_i), \tag{20}$$

where $\alpha > 0$, F_i are arbitrary functions of $q-1$ variables and y_i are defined in (14). *In particular, if $K \subset R$, then $[F]$ consists of all the functions*

$$G(x) = \alpha F(x) + \beta \qquad (x \in K \subset R),$$

where $\alpha > 0$ and $\beta \in R$.

THEOREM 1.4. *Every non-negative measure on $B(K)$ (finite on $B_0(K)$) is a Rényi probability distribution of some random variable (in some Rényi space).*

Every class of nondecreasing functions defined by formula (20), *containing a left-continuous representative, is a Rényi distribution function of some random variable (in some Rényi space).*

Proof. Let μ be a non-negative measure on $B(K)$, finite on $B_0(K)$. Put $\Omega = K$, $\mathcal{A} = B(K)$, $\mathcal{B} = \{B \in B_0(K)\colon \mu(B) > 0\}$ and

$$P(A \mid B) = \mu(A \cap B)/\mu(B)$$

for $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Let $\xi(\omega) = \omega$ for $\omega \in \Omega$. Evidently, ξ is a random variable in the Rényi space $\mathcal{R} = [\Omega, \mathcal{A}, \mathcal{B}, P]$ and $\mu_\xi = \mu$.

The second part of the theorem follows by the first one and Theorem 1.3.

If a Rényi distribution function $[F]$ of a random variable ξ is absolutely continuous (i.e., its all representatives are absolutely continuous) in $K \subset R^q$, then the class of the functions

$$g(x) = G'(x) = \frac{\partial^q}{\partial x_1 \ldots \partial x_q} G(x_1, \ldots, x_q) \tag{21}$$

for $x = (x_1, \ldots, x_q) \in K$, where $G \in [F]$, will be called the *Rényi density function* of ξ in K.

Remark 1.1. If a random variable ξ fulfils condition (d$_1$) and its Rényi distribution function $[F]$ is absolutely continuous, then its Rényi density function $[f]$ is the class of the functions of the form $g = \alpha f$, where $f = F'$ and $\alpha > 0$.

EXAMPLE 1.1. Let $\Omega = R$, let $\mathscr{A}$ be the σ-algebra of all Borel subsets of R, let $\mathscr{B}$ be the family of all $B \in \mathscr{A}$ such that $0 < |B| < \infty$, where $|\cdot|$ is the Lebesgue measure and let $P(A|B) = |A \cap B|/|B|$ for $A \in \mathscr{A}$, $B \in \mathscr{B}$. The system $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ is a Rényi space.

Let $\xi_1(\omega) = \omega$, $\xi_2(\omega) = c \in R$ for $\omega \in \Omega$ and

$$\xi_3(\omega) = \begin{cases} \operatorname{tg} \omega & \text{for} \quad \omega \in \bigcup_{i \in Z} I_i, \\ 0 & \text{otherwise,} \end{cases}$$

where $I_i = (-\pi/2 + \pi i, \pi/2 + \pi i)$ for $i \in Z$. The Rényi probability distribution, distribution function and density function of ξ_1 are given in $\mathscr{R}$ by the following representatives: $\mu_1(E) = |E|$, $F_1(x) = x$, $f_1(x) = 1$, respectively.

For ξ_2 and ξ_3, the Rényi probability distributions do not exist, because (d_0) does not hold. But the Rényi distributions of random variables:

$$\xi_{2i}(\omega) = c \quad (\omega \in (-i, i)) \quad \text{for} \quad i \in N,$$

where $c \in (-i, i)$, and

$$\xi_{3i}(\omega) = \operatorname{tg} \omega \quad (\omega \in I_i) \quad \text{for} \quad i \in Z,$$

in the respective Rényi spaces (restrictions of $\mathscr{R}$) exist and are given by

$$\mu_{2i}(E) = \delta_c(E) \quad (E \subset (-i, i)),$$

$$\mu_{3i}(E) = \int_E \frac{1}{1+t^2} dt \quad (E \subset I_i),$$

where δ_c denotes the probability measure concentrated at c.

Now, we formulate sufficient conditions for a random variable ξ in order that the Rényi distribution function of ξ would exist in K.

THEOREM 1.5. (cf. [17], p. 251). *Let ξ be a random variable with values in K which satisfies condition* (d_1) *and*

$$(d_2) \qquad P(\xi \in K_n | \xi \in K_{n+1}) > 0 \quad (n \in N).$$

Then the Rényi distribution function of ξ exists in K.

Proof. First note that condition (d_2) implies that

$$P(\xi \in K_i | \xi \in K_j) > 0$$

for all $i, j \in N$, $i < j$. This is a consequence of the relation

$$\prod_{k=i}^{j-1} P(B_k | B_{k+1}) = P(B_i | B_j)$$

for $B_i \subset B_{i+1} \subset \ldots \subset B_j$; $B_k \in \mathscr{B}$, which easily follows from axioms (I)–(III) (see [12]).

Consider the case where $K \in \mathscr{M}_\xi$ (i.e., one can put $K_n = K$ in (d_1)). Let $x_0 = (x_1^0, \ldots, x_q^0)$ be a fixed and $x = (x_1, \ldots, x_q)$ an arbitrary points in K. Let I_x be the interval defined by (10)–(11) and let k be as in (12). The function

$$F(x) = (-1)^k P(\xi \in I_x | \xi \in K) \quad (x \in K)$$

fulfils (19), in view of additivity of the measure $P(\cdot | \xi \in K)$ and axiom (III).

If $K \notin \mathscr{M}_\xi$, then $K_n \neq K$ for $n \in N$ in condition (d_1). Let x_0 be a fixed point in K_1 and x be an arbitrary point in K, so $x \in K_n$ for some $n \in N$, by (d_1). We define

$$F(x) = (-1)^k \frac{P(\xi \in I_x | \xi \in K_n)}{P(\xi \in K_1 | \xi \in K_n)}, \tag{22}$$

where k and I_x have the same meaning as previously. In view of (III), definition (22) does not depend on the choice K_n such that $x \in K_n$. In a similar way to the preceding case, we can show that F fulfils (19) and this completes the proof.

Remark 1.2. Suppose that $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ is a given Rényi space fulfilling condition (IV) and let $\{\xi_\alpha\}$ be an arbitrary family of random variables with values in $K \subset R^q$, satisfying conditions (d_1)–(d_2). By Theorem 3.2 in [12], we can extend the family $\mathscr{B}$ to a family $\tilde{\mathscr{B}}$ such that $\xi_\alpha^{-1}(J) \in \tilde{\mathscr{B}}$ for each α and sufficiently large intervals $J \subset K$ (more exactly: for each α there is an interval $I_\alpha \subset K$ such that $\xi_\alpha^{-1}(J) \in \tilde{\mathscr{B}}$ for each interval J such that $I_\alpha \subset J \subset K$). If $\mathscr{R}$ satisfies axiom (IV′), then the family $\mathscr{B}$ itself has this property.

Let ξ be a random variable in a Rényi space $\mathscr{R}$, fulfilling (IV), with values in $K \subset R^q$ such that conditions (d_1)–(d_2) hold. Let $[F]$ be the Rényi distribution function of ξ. In view of Theorem 5.1 and Remark 2.1 in [12], the set $\xi^{-1}(K)$ belongs or can be joined to the family $\mathscr{B}$ iff

$$\prod_{i=1}^{\infty} P(\xi \in K_i | \xi \in K_{i+1}) = \lim_{n \to \infty} \Delta_{K_1} F / \Delta_{K_n} F > 0,$$

i.e., iff $[F]$ is a bounded distribution function on K.

In the sequel, we shall consider random variables satisfying conditions (d_1)–(d_2) for $K = R^q$.

One of fundamental tools in the classical probability theory is a concept of characteristic function, i.e., the Fourier transform of a given probability distribution. In the case of Rényi probability distributions which are not bounded, in general, the Fourier transform can be understood in the sense of the theory of distributions of L. Schwartz; the Fourier transform $\mathscr{F}(f)$ is defined for $f \in \mathscr{S}'$ by the formula:

$$\langle \mathscr{F}(f), \omega \rangle = \langle f, \mathscr{F}(\omega) \rangle \quad (\omega \in \mathscr{S}),$$

where

$$\mathcal{F}(\omega)(t) = \int_{R^q} e^{2\pi i t x}\,\omega(x)\,dx \in \mathcal{S},$$

so $\mathcal{F}(f) \in \mathcal{S}'$.

We say that a Rényi probability distribution $[\mu]$ in R^q is *tempered*, if

$$\int_{R^q} (1+|t|^2)^{-m}\,\mu(dt) < \infty$$

for some $m \in N \cup \{0\}$.

We shall always identify $[\mu]$ with the class $[\tilde{\mu}] = \{\alpha\tilde{\mu}\colon \alpha > 0\}$ of tempered distributions, where $\tilde{\mu}$ is defined by the formula

$$\langle \tilde{\mu}, \omega \rangle = \int_{R^q} \omega(t)\,\mu(dt) \quad (\omega \in \mathcal{S}).$$

In the sequel, the classes $[f] = \{\alpha f\colon \alpha > 0\}$, where $f \in \mathcal{D}'$, will be called *distributors*.

By the *characteristic distributor* of a tempered Rényi probability distribution $[\mu]$ (or of the respective Rényi distribution function), we mean the class $[\Phi]$, where $\Phi = \mathcal{F}(\mu)$.

Let us recall that a continuous complex-valued function φ in R^q is said to be *positive-definite* if

$$\sum_{i,j=1}^{n} \varphi(x_i - x_j)\, z_i \bar{z}_j \geqslant 0$$

for arbitrary systems $x_1, \ldots, x_n$ of points in R^q and $z_1, \ldots, z_n$ of complex numbers.

Let $h^*(x) = \overline{h(-x)}$ for any complex-valued distribution h. A complex-valued distributor $[\Phi]$ in R^q will be called *positive-definite* if

$$\langle \Phi, \omega * \omega^* \rangle \geqslant 0$$

for each complex-valued function $\omega \in \mathcal{D}$ or, equivalently, if the function $\Phi * h * h^*$ is positive-definite for all $h \in \mathcal{E}'$ or, equivalently, for all $h \in \mathcal{D}$ (e.g., see [21], 123–124).

Let us formulate the known Bochner–Schwartz characterization of positive-definite distributions in terms of distributors:

THEOREM 1.6 (e.g., see [21], p. 127). *Let $[\Phi]$ be a distributor in R^q. The following conditions are equivalent*:

(i) $[\Phi]$ *is positive-definite*;

(ii) $[\Phi]$ *is a characteristic distributor of some Rényi probability distribution*;

(iii) $\Phi = \alpha(1-\Delta)^m \varphi$ *for some* $\alpha > 0$ *and* $m \in N \cup \{0\}$, *where* Δ *is the Laplace operator and* φ *is a characteristic function of some Kolmogorov probability distribution.*

2. Convergences. In this section, we consider functions defined on R and measures on $B(R)$. For a given function $F\colon R \to R$ and an arbitrary interval $I = [a, b) \subset R$, we shall use the notation: $F(I) = F(b) - F(a)$ instead of $\Delta_I F$. Let us recall that a sequence of Kolmogorov distribution functions F_n converges to a Kolmogorov distribution function F ($F_n \xrightarrow{w} F$) if one of the following equivalent conditions is satisfied:

(a) $F_n(x) \to F(x)$ for each point $x \in R$ at which $F(x)$ is continuous;

(b) $F_n(x) \to F(x)$ on some dense set in R;

(c) $\int_{-\infty}^{\infty} \gamma(x)\,dF_n(x) \to \int_{-\infty}^{\infty} \gamma(x)\,dF(x)$ for each continuous and bounded function γ;

(d) $\int_{-\infty}^{\infty} \gamma(x)\,dF_n(x) \to \int_{-\infty}^{\infty} \gamma(x)\,dF(x)$ for each $\gamma \in \mathcal{D}$.

Condition (d), expressing the distributional convergence of F_n to F, is usually omitted in characterizations, given in literature, of the convergence of Kolmogorov distribution functions. The equivalence of this condition and the others is a simple consequence of the obvious implication (c) $\Rightarrow$ (d) and the compactness of an arbitrary sequence of Kolmogorov distribution functions (see also [15]).

For an arbitrary Rényi distribution function $[F]$, by $C(F)$ we mean the set of all points of R at which F is continuous and by $\bar{C}(F)$ the set of all intervals $I = [a, b) \subset R$ such that F is continuous at a and b and we define

$$F_I(x) = \begin{cases} 0 & \text{if} \quad x < a, \\ \dfrac{F(x)-F(a)}{F(b)-F(a)} & \text{if} \quad a \leqslant x < b, \\ 1 & \text{if} \quad x \geqslant b \end{cases}$$

for $I = [a, b) \in \bar{C}(F)$. Note that the definitions of $C(F)$, $\bar{C}(F)$, F_I do not depend on the choice of a representative of $[F]$.

THEOREM 2.1. *Let $[F_n]$, $[F]$ be Rényi distribution functions. The following conditions are equivalent*:

(i) $F_{n,I} \xrightarrow{w} F_I$ *for each* $I \in \bar{C}(F)$ (*for each interval I with ends belonging to some dense set*) *such that* $F_n(I) > 0$ *and* $F(I) > 0$;

(ii) $\dfrac{F_n(I_1)}{F_n(I_2)} \to \dfrac{F(I_1)}{F(I_2)}$ *for each* $I_1, I_2 \in \bar{C}(F)$ (*for each intervals I_1, I_2 with ends belonging to some dense set*) *such that* $F_n(I_2) > 0$ *and* $F(I_2) > 0$;

(iii) *there exist constants* $\alpha_n > 0$, $\beta_n \in R$ *such that*

(1) $$\alpha_n F_n(x) + \beta_n \to F(x)$$

for each $x \in C(F)$ (for each x belonging to some dense set);

(iv) *there exist positive constants α_n such that*

$$\alpha_n \int_{-\infty}^{\infty} \gamma(x)\, dF_n(x) \to \int_{-\infty}^{\infty} \gamma(x)\, dF(x) \tag{2}$$

for each continuous function γ of bounded support;

(v) *there exist positive constants α_n such that* (2) *holds for all $\gamma \in \mathcal{D}$.*

Remark 2.1. In conditions (i) and (ii), we assume that there exists an interval $I_0 \subset R$ such that $F_n(I_0) > 0$ for (almost) all $n \in N$. Since Rényi distribution functions are non-trivial, this assumption implies that there is an interval $I_1 \subset R$ for which $F_n(I_1) > 0$, $F(I_1) > 0$ $(n \in N)$.

Proof. First we shall prove the equivalence of conditions (i)–(iii) formulated for points of continuity of $[F]$. Suppose that (i) holds and let $I_1, I_2 \in \bar{C}(F)$, $F_n(I_2) > 0$, $F(I_2) > 0$. Let J be an interval from $\bar{C}(F)$ such that $J \supset I_1 \cup I_2$. Owing to (i), we have

$$\frac{F_n(I_i)}{F_n(J)} \to \frac{F(I_i)}{F(J)} \qquad (i = 1, 2),$$

which implies (ii).

Suppose that (ii) holds. Fix $I_0 \in C(F)$ such that $F_n(I_0) > 0$, $F(I_0) > 0$ and $a_0 \in C(F)$. Let

$$\alpha_n = \frac{F(I_0)}{F_n(I_0)} \quad \text{and} \quad \beta_n = F(a_0) - \alpha_n F_n(a_0).$$

We have

$$\alpha_n F_n(x) + \beta_n = F(I_0)\frac{F_n(x) - F_n(a_0)}{F_n(I_0)} + F(a_0),$$

so

$$\alpha_n F_n(x) + \beta_n \to F(x),$$

provided $x \in C(F)$, i.e., (iii) holds.

Now, assume (iii). Note that $F(I_0) > 0$ for sufficiently large interval $I_0 \in C(F)$ and, since $\alpha_n F_n(I_0) \to F(I_0)$, we have also $F_n(I_0) > 0$ for almost all $n \in N$. Let now $I = [a, b) \in \bar{C}(F)$, $F_n(I) > 0$, $F(I) > 0$ and let $x \in C(F) \cap I$. Then

$$F_{n,I}(x) = \frac{\bar{F}_n(x) - \bar{F}_n(a)}{\bar{F}_n(I)} \to \frac{F(x) - F(a)}{F(I)} = F_I(x),$$

where $\bar{F}_n(x) = \alpha_n F_n(x) + \beta_n$. Moreover, we have

$$F_{n,I}(x) = 0 = F_I(x) \quad \text{for} \quad x < a$$

and

$$F_{n,I}(x) = 1 = F_I(x) \quad \text{for} \quad x \geqslant b$$

and thus (i) holds.

The equivalence of conditions (i), (ii), (iii) formulated in the stronger version is proved. The proof in case of the other formulation is analogous.

Now, suppose that conditions (i)–(iii) hold in the weaker formulation and let γ be a continuous function with the support contained in an interval $J = [c, d)$, where c, d belong to the dense set involved in conditions (i), (ii), (iii), such that $F(J) > 0$. By (iii), we have

$$\alpha_n F_n(J) \to F(J). \tag{3}$$

On the other hand,

$$\int_{-\infty}^{\infty} \gamma(x)\, dF_{n,J}(x) \to \int_{-\infty}^{\infty} \gamma(x)\, dF_J(x),$$

in view of (i) and the characterization of the convergence of Kolmogorov's distribution functions. That means,

$$\frac{1}{F_n(J)} \int_J \gamma(x)\, dF_n(x) \to \frac{1}{F(J)} \int_J \gamma(x)\, dF(x).$$

This and (3) yield (2), i.e., conditions (i)–(iii) in the weaker version imply (iv).

The implication (iv) $\Rightarrow$ (v) is obvious.

Suppose (v) and define $\beta_n = F(a_0) - \alpha_n F_n(a_0)$, where a_0 is a fixed point belonging to $C(F)$. Let $x \in C(F) \cap (a_0, \infty)$ and let ε be an arbitrary positive number. There is a $\delta > 0$ such that $a_0 + 2\delta < x$ and

$$|F(t) - F(a_0)| < \varepsilon \quad \text{for} \quad |t - a_0| \leqslant \delta, \tag{4}$$

$$|F(t) - F(x)| < \varepsilon \quad \text{for} \quad |t - x| \leqslant \delta. \tag{5}$$

Let

$$g_1(t) = \begin{cases} 1 & \text{if} \quad a_0 + \delta/2 \leqslant t < x - \delta/2, \\ 0 & \text{if} \quad t < a_0 + \delta/2 \quad \text{or} \quad t \geqslant x - \delta/2 \end{cases}$$

and

$$g_2(t) = \begin{cases} 1 & \text{if} \quad a_0 - \delta/2 \leqslant t < x + \delta/2, \\ 0 & \text{if} \quad t_0 < a_0 - \delta/2 \quad \text{or} \quad t \geqslant x + \delta/2. \end{cases}$$

We can find a non-negative smooth function ω such that

$$\omega(t) = 0 \quad \text{for} \quad |t| > \delta/2$$

and

$$\int \omega(t)\, dt = 1.$$

The functions $\gamma_i = g_i * \omega$ $(i = 1, 2)$ are smooth and

$$0 \leqslant \gamma_i(t) \leqslant 1 \qquad (t \in R).$$

Moreover,

$$\gamma_1(t) = \begin{cases} 1 & \text{if} \quad a_0+\delta \leqslant t < x-\delta, \\ 0 & \text{if} \quad t < a_0 \quad \text{or} \quad t \geqslant x \end{cases} \tag{6}$$

and

$$\gamma_2(t) = \begin{cases} 1 & \text{if} \quad a_0 \leqslant t < x, \\ 0 & \text{if} \quad t < a_0-\delta \quad \text{or} \quad t \geqslant x+\delta. \end{cases} \tag{7}$$

In view of (4)–(7), we have

$$\int \gamma_1(t)\, dF(t) \geqslant F(x-\delta) - F(a_0+\delta) \geqslant F(x) - F(a_0) - 2\varepsilon, \tag{8}$$

$$\int \gamma_2(t)\, dF(t) \leqslant F(x+\delta) - F(a_0-\delta) \leqslant F(x) - F(a_0) + 2\varepsilon, \tag{9}$$

$$\int \gamma_1(t)\, dF_n(t) \leqslant F_n(x) - F_n(a_0), \tag{10}$$

$$\int \gamma_2(t)\, dF_n(t) \geqslant F_n(x) - F_n(a_0). \tag{11}$$

Applying (v), we deduce from (8)–(11) the inequalities:

$$F(x) - F(a_0) - 3\varepsilon \leqslant \alpha_n\big(F_n(x) - F_n(a_0)\big) \leqslant F(x) - F(x_0) + 3\varepsilon$$

for sufficiently large $n \in N$. Changing roles of x and a_0, we get similar inequalities for $x < a_0$, i.e., (1) holds for all $x \in C(F)$, $x \neq a_0$. Since

$$\alpha_n F_n(a_0) + \beta_n = F(a_0),$$

we have (1) for all $x \in C(F)$ and thus (v) implies conditions (i)–(iii) in the stronger form. In this way, the proof of the theorem is completed.

We say that a sequence of Rényi distribution functions $[F_n]$ *converges* to a Rényi distribution function $[F]$ and write $[F_n] \to [F]$ if one of conditions (i)–(v) in Theorem 2.1 holds.

Since Rényi distribution functions uniquely determine Rényi probability distributions, the above definition can be applied for Rényi probability distributions. Namely, let $[\mu_n]$ and $[\mu]$ be Rényi probability distributions defined by the formulae

$$\mu_n(I) = F_n(I), \qquad \mu(I) = F(I) \qquad (I \subset R).$$

Then we say that $[\mu_n]$ *converges* to $[\mu]$ and write $[\mu_n] \to [\mu]$ if one of conditions (iv)–(v) in Theorem 2.1 is satisfied.

Let us consider now the case where the Rényi distribution functions are absolutely continuous. In this case, we can say about the convergence of Rényi density functions.

A sequence of Rényi density functions $[f_n]$ is said to *converge* to a density function $[f]$ if one of the following conditions is fulfilled:

(iv) *there exist positive constants* α_n *such that*

$$\alpha_n \int_{-\infty}^{\infty} \gamma(x) f_n(x)\, dx \to \int_{-\infty}^{\infty} \gamma(x) f(x)\, dx \tag{12}$$

for each continuous function γ *of bounded support;*

(v) *there exist positive constants* α_n *such that* (12) *holds for each* $\gamma \in \mathcal{D}$.

In view of Theorem 2.1, the above conditions are equivalent.

Remark 2.2. Note that the above definitions of the convergences of Rényi distributions, probability distributions and density functions coincide with the quotient convergence of the respective classes.

For Rényi distributions, e.g., we have $[F_n] \to [F]$ iff $\tilde{F}_n \to \tilde{F}$ for some $\tilde{F}_n \in [F_n]$ and $\tilde{F} \in [F]$, i.e.,

$$\alpha_n F_n(x) + \beta_n \to \alpha F(x) + \beta$$

on a dense set in R for some constants $\alpha_n, \alpha > 0$ and $\beta_n, \beta \in R$. Moreover such a definition is consistent. In fact, if

$$\alpha_n F_n(x) + \beta_n \to F(x) \quad \text{as} \quad n \to \infty \tag{13}$$

and

$$\tilde{\alpha}_n F_n(x) + \tilde{\beta}_n \to G(x) \quad \text{as} \quad n \to \infty \tag{14}$$

on a dense set in R for some non-decreasing, left-continuous functions F, G and constants α_n, $\tilde{\alpha}_n > 0$ and β_n, $\tilde{\beta}_n \in R$, then for every (sufficiently large) interval $I = [a, b)$ with a, b belonging to some dense set in R, we get

$$\frac{F(x) - F(x_0)}{G(x) - G(x_0)} = \lim_{n \to \infty} \frac{\alpha_n}{\tilde{\alpha}_n} = \alpha \in R$$

for every $x \in R$ and a fixed $x_0 \in R$. But this means that F and G belong to the same class. It should be noted that if $F = G$ in (13)–(14), then $\alpha_n/\tilde{\alpha}_n \to 1$ as $n \to \infty$.

By S we denote the set of all *rapidly decreasing continuous functions* on R, i.e. such continuous functions G on R that the functions pG are bounded on R for every polynomial p.

Theorem 2.2. *Let* $[F_n]$, $[F]$ *be tempered Rényi distribution functions in* R. *The following conditions are equivalent:*

(i) *There exist* $m \in N \cup \{0\}$ *and constants* $\alpha_n > 0$, $\beta_n \in R$, $a > 0$, *such that* (1) *holds for each* $x \in C(F)$ (*for each* x *from some dense set*) *and*

$$\alpha_n \int_{-\infty}^{\infty} (1+t^2)^{-m}\, dF_n(t) < a; \tag{15}$$

(ii) *There exist* $k \in N \cup \{0\}$ *and positive constants* λ_n, λ *such that*

(16) $$F_n^0(x) = \lambda_n \int_{-\infty}^{x} (1+t^2)^{-k} dF_n(t), \quad F^0(x) = \lambda \int_{-\infty}^{x} (1+t^2)^{-k} dF(t)$$

are Kolmogorov distribution functions and

(17) $$F_n^0 \xrightarrow{w} F^0 \quad as \quad n \to \infty;$$

(iii) *There exist positive constants* α_n *such that* (2) *holds for each* $\gamma \in S$;

(iv) *There exist positive constants* α_n *such that* (2) *holds for each* $\gamma \in \mathscr{S}$.

Proof. Suppose that (i) holds. By (1) and (15), we can easily infer that

(18) $$\int_{-\infty}^{\infty} (1+t^2)^{-m-1} dF(t) < b < \infty,$$

so for $k = m+2$ and arbitrary $\varepsilon > 0$ we have

(19) $$\alpha_n \int_{|t|>x_0} (1+t^2)^{-k} dF_n(t) < \varepsilon, \quad \int_{|t|>x_0} (1+t^2)^{-k} dF(t) < \varepsilon$$

for sufficiently large $x_0 > 0$. We can assume that $x_0 \in C(F)$ and $-x_0 \in C(F)$. Moreover, since F is non-constant, we have

(20) $$\int_{-\infty}^{\infty} (1+t^2)^{-k} dF(t) > \eta > 0$$

and, owing to (1),

(21) $$\alpha_n \int_{-\infty}^{\infty} (1+t^2)^{-k} dF_n(t) > \eta' > 0 \quad (n \in N).$$

We adopt

(22) $$\lambda_n = \Big(\int_{-\infty}^{\infty} (1+t^2)^{-k} dF_n(t)\Big)^{-1}, \quad \lambda = \Big(\int_{-\infty}^{\infty} (1+t^2)^{-k} dF(t)\Big)^{-1}.$$

By (19), (20) and (21), the functions F_n^0, F^0 defined in (16) are Kolmogorov distribution functions. In view of (19) and Theorem 2.1, we obtain

$$\Big|\alpha_n \int_{-\infty}^{\infty} (1+t^2)^{-k} dF_n(t) - \int_{-\infty}^{\infty} (1+t^2)^{-k} dF(t)\Big|$$
$$\leqslant \Big|\alpha_n \int_{-x_0}^{x_0} (1+t^2)^{-k} dF_n(t) - \int_{-x_0}^{x_0} (1+t^2)^{-k} dF(t)\Big| +$$
$$+ \alpha_n \int_{|t|>x_0} (1+t^2)^{-k} dF_n(t) + \int_{|t|>x_0} (1+t^2)^{-k} dF(t) < 3\varepsilon$$

for sufficiently large n, i.e.,

(23) $$\alpha_n \lambda_n^{-1} \to \lambda^{-1} \quad \text{as} \quad n \to \infty.$$

Let M be a positive constant such that

(24) $$\alpha_n \lambda_n^{-1} > M^{-1} \quad (n \in N).$$

By Theorem 2.1, (iv), we have for $x \in C(F)$ and sufficiently large n

$$\Big|\alpha_n \int_{-x_0}^{x} (1+t^2)^{-k} dF_n(t) - \int_{-x_0}^{x} (1+t^2)^{-k} dF(t)\Big| < \varepsilon,$$

so

$$|F_n^0(x) - F^0(x)| \leqslant \lambda_n \alpha_n^{-1} \Big|\alpha_n \int_{-\infty}^{x} (1+t^2)^{-k} dF_n(t) - \int_{-\infty}^{x} (1+t^2)^{-k} dF(t)\Big| +$$
$$+ \lambda_n \alpha_n^{-1} (1 - \lambda \lambda_n^{-1} \alpha_n) \Big|\int_{-\infty}^{x} (1+t^2)^{-k} dF(t)\Big| < (3+b) M\varepsilon$$

for an arbitrary $x \in C(F)$ and for sufficiently large n, by virtue of (18), (19), (23), (24). Consequently, (17) holds and the implication (i) $\Rightarrow$ (ii) is proved.

Now, suppose (ii). We have

$$\lambda_n \int_{-\infty}^{\infty} \gamma(x)(1+x^2)^{-k} dF_n(x) = \int_{-\infty}^{\infty} \gamma(x) dF_n^0(x) \to \int_{-\infty}^{\infty} \gamma(x) dF^0(x)$$
$$= \lambda \int_{-\infty}^{\infty} \gamma(x)(1+x^2)^{-k} dF(x)$$

for every continuous bounded function γ on R. In particular, the above relation holds if $\gamma(x) = (1+x^2)^k G(x)$, where $G \in S$, and thus (iii) holds with $\alpha_n = \lambda_n \cdot \lambda^{-1}$.

The implication (iii) $\Rightarrow$ (iv) is trivial.

Since $\mathscr{D} \subset \mathscr{S}$, we can apply Theorem 2.1 to derive that (iv) implies (1) for $x \in C(F)$ and constants α_n in (1) can be taken the same as in (iv) (see the proof of the implication (v) $\Rightarrow$ (iii) in Theorem 2.1). It remains to prove that (15) holds for some $m \in N \cup \{0\}$ and $a > 0$.

Condition (iv) means that the sequence of the non-negative tempered measures $\alpha_n \mu_n$ determined by the equality

$$\mu_n(I) = F_n(I) \quad (I \subset R)$$

is weakly convergent in $\mathscr{S}'$. But this means that the sequence $\alpha_n \mu_n$ is convergent in the norm $\|\cdot\|_{-m}$ of some space $\mathscr{S}'_m$, $m \in N \cup \{0\}$ (e.g., see [21], p. 91). Hence

(25) $$|\alpha_n \langle \mu_n, \gamma \rangle| \leqslant \|\mu_n\|_{-m} \|\gamma\|_m < a < \infty,$$

for each $n \in N$, where $\|\cdot\|_{-m}$ is the norm in $\mathscr{S}'_m$ and $\|\cdot\|_m$ the norm in $\mathscr{S}_m$.

Let $\eta_k(t) = \eta(t/k)$ for $t \in R$, $k \in N$, where $\eta \in \mathscr{D}$, $\eta(t) > 0$ and $\eta(0) = 1$. By Fatou's lemma, we get from (23):

$$\alpha_n \int_{-\infty}^{\infty} (1+t^2)^{-m}\, dF_n(t) = \liminf_{k \to \infty} \alpha_n \int_{-\infty}^{\infty} \eta_k(t)(1+t^2)^{-m} \mu_n\, dt \leqslant a$$

for each $n \in N$, as desired. The proof is finished.

Remark 2.3. It can be derived from the proof of Theorem 2.2 that the following relation holds between constants λ_n, λ in condition (ii) and constants α_n in any of conditions (i), (iii), (iv): $\alpha_n \lambda_n^{-1} \to \lambda^{-1}$.

The convergence described in Theorem 2.2 will be called *tempered convergence of Rényi distribution functions* and denoted by $[F_n] \xrightarrow{t} [F]$. Similarly, we define the *tempered convergence of Rényi distributions* and *density functions*.

Remark 2.2 can be reformulated for the tempered convergence.

Now, we pass on to the convergence of characteristic distributors.

We say that a sequence $[\Phi_n]$ of characteristic (positive-definite) distributors *converges* to a characteristic (positive-definite) distributor $[\Phi]$ and write $[\Phi_n] \xrightarrow{t} [\Phi]$, if $\alpha_n \Phi_n \to \Phi$ in $\mathscr{S}'$ for some positive constants α_n.

The definition does not depend on the choice of representatives of the classes $[\Phi_n]$ and $[\Phi]$.

The following theorem is a consequence of the fact that the Fourier transform preserves the convergence in $\mathscr{S}'$:

THEOREM 2.3. *Let $[F_n]$ and $[F]$ be tempered Rényi distribution functions. Then $[F_n] \xrightarrow{t} [F]$ iff $[\Phi_n] \xrightarrow{t} [\Phi]$, where $[\Phi_n] = [\mathscr{F}(F_n)]$ and $[\Phi] = [\mathscr{F}(F)]$ are characteristic distributors of $[F_n]$ and $[F]$, respectively.*

As an immediate corollary from Theorems 2.2 and 2.3, Remark 2.2 and the known theorem on the convergence of usual characteristic functions, we obtain:

THEOREM 2.4. *Let $[\Phi_n]$ and $[\Phi]$ be characteristic distributors. The following conditions are equivalent:*

(i) $[\Phi_n] \xrightarrow{t} [\Phi]$;

(ii) *There exist $m \in N \cup \{0\}$ and positive constants λ_n, a such that*

$$\Phi_n = (1-D^2)^m \varphi_n, \qquad \Phi = (1-D^2)^m \varphi,$$

$$\lambda_n \varphi_n(0) < a \quad \text{for} \quad n \in N$$

and

$$\lambda_n \varphi_n(t) \to \varphi(t) \quad \text{as} \quad n \to \infty, \tag{26}$$

where $Dh(x) = \dfrac{d}{dx} h(x)$ for $h \in \mathscr{D}'$, φ_n and φ are continuous positive-definite functions and the convergence in (26) is pointwise or, equivalently, almost uniform in R.

3. Independence and convolution. Let $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ be a fixed Rényi space. We say that two random variables ξ_1 and ξ_2 with values in R^q are *independent* if

$$J_1 \in \mathscr{M}_{\xi_1},\ J_2 \in \mathscr{M}_{\xi_2} \text{ implies } J_1 \times J_2 \in \mathscr{M}_{\xi}, \tag{1}$$

where $\xi = (\xi_1, \xi_2)$, and

$$P(\xi \in I_1 \times I_2 \mid \xi \in J_1 \times J_2) = P(\xi_1 \in I_1 \mid \xi_1 \in J_1) \cdot P(\xi_2 \in I_2 \mid \xi_2 \in J_2) \tag{2}$$

for any intervals $I_1 \subset J_1$, $I_2 \subset J_2$ such that $J_1 \in \mathscr{M}_{\xi_1}$ and $J_2 \in \mathscr{M}_{\xi_2}$.

By axiom (III), it follows that if random variables ξ_1, ξ_2 are independent, then

$$P(\xi \in E_1 \times E_2 \mid \xi \in E_1' \times E_2') = P(\xi_1 \in E_1 \mid \xi_1 \in E_1') \cdot P(\xi_2 \in E_2 \mid \xi_2 \in E_2')$$

for any Borel sets $E_1 \subset E_1'$, $E_2 \subset E_2'$ such that $E_1' \subset J_1$, $E_2' \subset J_2$ and $P(E_1' \mid J_1) > 0$, $P(E_2' \mid J_2) > 0$ for some intervals $J_1 \in \mathscr{M}_{\xi_1}$ and $J_2 \in \mathscr{M}_{\xi_2}$.

THEOREM 3.1. *Suppose that ξ_1, ξ_2 are random variables in $\mathscr{R}$ with values in R^q, satisfying conditions* (d_1) *and* (d_2) *for $K = R^q$. If ξ_1 and ξ_2 are independent, then*

$$[F(x)] \cdot [G(y)] = [H(x, y)] \qquad (x, y \in R^q), \tag{3}$$

i.e.,

$$\tilde{F}(x) \cdot \tilde{G}(y) = \tilde{H}(x, y) \qquad (x, y \in R^q) \tag{4}$$

for some representatives $\tilde{F}$, $\tilde{G}$, $\tilde{H}$ of the Rényi distribution functions $[F]$, $[G]$, $[H]$ of ξ_1, ξ_2 and $\xi = (\xi_1, \xi_2)$, respectively.

Conversely, if relation (3) *holds and for each $J_1 \in \mathscr{M}_{\xi_1}$, $J_2 \in \mathscr{M}_{\xi_2}$ we have $\Delta_{J_1} F > 0$ and $\Delta_{J_2} G > 0$, then ξ_1 and ξ_2 are independent.*

Proof. First note that the random variable $\xi = (\xi_1, \xi_2)$ satisfies conditions (d_1)–(d_2) for $K = R^{2q}$, so the Rényi distribution functions of ξ_1, ξ_2 and ξ exist, by Theorem 1.4. Suppose that ξ_1 and ξ_2 are independent random variables. Since $[F]$, $[G]$ and $[H]$ are non-trivial, there exist intervals $Q_1 \in \mathscr{M}_{\xi_1}$ and $Q_2 \in \mathscr{M}_{\xi_2}$ such that $Q = Q_1 \times Q_2 \in \mathscr{M}_{\xi}$ and

$$\Delta_{Q_1} F > 0,\ \Delta_{Q_2} G > 0,\ \Delta_Q H > 0. \tag{5}$$

Fix intervals $Q_1, Q_2 \subset R^q$ and $Q \subset R^{2q}$, satisfying (5), and let P_1 and P_2 be arbitrary intervals in R^q. Now, let J_1 and J_2 be intervals in R^q such that $J_1 \in \mathscr{M}_{\xi_1}$, $J_2 \in \mathscr{M}_{\xi_2}$ and $P_1 \cup Q_1 \subset J_1$, $P_2 \cup Q_2 \subset J_2$. We have $P \cup Q \subset J$, where $P = P_1 \times P_2$, $J = J_1 \times J_2$ and

$$\Delta_{J_1} F > 0, \qquad \Delta_{J_2} G > 0, \qquad \Delta_J H > 0.$$

Applying (2) and formula (14) in Section 1 for $I_1 = P_1$, $I_2 = P_2$ and for $I_1 = Q_1$, $I_2 = Q_2$, we obtain

$$\Delta_{P_1} F \cdot \Delta_{P_2} G = \alpha \Delta_P H,$$

where $\alpha = (\Delta_{Q_1} F)\cdot(\Delta_{Q_2} G)\cdot(\Delta_Q H)^{-1}$. Hence (4) follows with $\tilde{F}(x) = \Delta_{ax} F$, $\tilde{G}(y) = \Delta_{ay}$ and $\tilde{H}(x, y) = \alpha\Delta_{(a,a)(x,y)} H$ for any fixed $a \in R^q$ (see (6) in Section 1).

It is easy to see that if (4) holds, then

$$\Delta_{Q_1}\tilde{F}\cdot\Delta_{Q_2}\tilde{G} = \Delta_Q\tilde{H} \tag{6}$$

for arbitrary intervals $Q_1, Q_2 \subset R^q$; $Q = Q_1 \times Q_2 \subset R^{2q}$. Owing to formula (19) in Section 1 and the additional assumption in the second part of the theorem, we infer from (6) that the random variables ξ_1 and ξ_2 are independent, which completes the proof.

As a corollary, we obtain:

THEOREM 3.2. *Suppose that ξ, η are random variables in $\mathscr{R}$ with values in R^q, satisfying conditions* (d$_1$) *and* (d$_2$) *for $K = R^q$ and let the Rényi distribution functions be absolutely continuous with the Rényi density functions* $[f]$ *and* $[g]$, *respectively. If ξ and η are independent, then the Rényi distribution function of the random variable $\zeta = (\xi, \eta)$ is absolutely continuous and*

$$[f(x)]\cdot[g(y)] = [h(x, y)] \tag{7}$$

i.e.,

$$f(x)\cdot g(y) = \alpha h(x, y)$$

for some $\alpha > 0$, where (h) is the Rényi density function of ζ.

Conversely, if (7) *holds and*

$$\int_{J_1} f(x)\,dx > 0, \qquad \int_{J_2} g(y)\,dy > 0$$

for each $J_1 \in \mathscr{M}_\xi$ and $J_2 \in \mathscr{M}_\eta$, then ξ and η are independent.

Remark 3.1. In [17] (p. 252), the independence of random variables is defined, in the case where their Rényi distribution functions exist, by formula (4).

The definition (2) of independence and Theorem 3.1 can be easily transferred for the case of an arbitrary finite number of random variables.

From now to the end of the section, we shall consider the one-dimensional case. For a given non-negative measure μ on $B(R)$ (finite on $B_0 R)$), a non-negative, locally integrable function $f\colon R \to R$ and a non-decreasing, left-continuous function $F\colon R \to R$, denote

$$\mu_a(E) = \mu(E \cap (-a, a)) \qquad (E \in B(R)),$$

$$f_a(x) = \begin{cases} 0 & \text{for} \quad |x| > a, \\ f(x) & \text{for} \quad |x| \leqslant a \end{cases}$$

and

$$F_a(x) = \begin{cases} F(-a) & \text{for} \quad x \leqslant -a, \\ F(x) & \text{for} \quad |x| \leqslant a, \\ F(a) & \text{for} \quad x \geqslant a, \end{cases}$$

respectively, where a is an arbitrary positive number.

Note that the convolutions $\mu_a * \mu_b$, $f_a(x) * g_b(x)$ and $(F_a(x)+\beta) * (G_b(x)+\beta')$ exist in the usual sense for arbitrary $a, b > 0$ and $\beta, \beta' \in R$ and represent a non-negative measure on $B(R)$ (finite on $B_0(R)$), a non-negative, locally integrable function on R and a nondecreasing, left-continuous function on R, respectively.

Suppose now that $\mathscr{R}$ fulfils axiom (IV′).

THEOREM 3.3. *Let ξ and η be independent random variables in $\mathscr{R}$ such that ξ and η satisfy conditions* (d$_1$)-(d$_2$) *for $K = R$. Let $[\mu]$ and $[\nu]$ be the Rényi probability distributions of ξ and η, respectively. If there exists a non-trivial non-negative measure λ on $B(R)$ (finite on $B_0(R)$) and numbers a_n, b_n, α_n ($a_n \to \infty$, $b_n \to \infty$, $\alpha_n > 0$) such that*

$$\lim \alpha_n(\mu_{a_n} * \nu_{b_n})(I) = \lambda(I) \tag{8}$$

for each interval I with ends belonging to some dense set in R, then $[\lambda]$ is the Rényi probability distribution of $\xi+\eta$.

Proof. Let $\zeta = (\xi, \eta)$ and $\chi = \xi+\eta$. We have to prove that

$$P(\chi \in I \mid \chi \in J) = \frac{\lambda(I)}{\lambda(J)} \tag{9}$$

for intervals I, J such that $I \subset J \in \mathscr{M}_\chi$ and $\lambda(J) > 0$. In view of axiom (IV′), we can assume that ends of the intervals I, J belong to a dense set, for which (8) holds.

To prove (9) for such intervals it suffices to show that

$$P(\chi \in I \mid \chi \in J) = \lim_n \frac{\mu_n * \nu_n(I)}{\mu_n * \nu_n(J)}, \tag{10}$$

where $\mu_n = \mu_{a_n}$ and $\nu_n = \nu_{b_n}$, owing to (8). Let

$$E = \{(x, y) \in R^2\colon\ x+y \in I\}, \qquad E' = \{(x, y) \in R^2\colon\ x+y \in J\}$$

and

$$E_n = \{(x, y) \in R^2\colon\ x \in I_n,\ y \in J_n\},$$

where $I_n = (-a_n, a_n)$, $J_n = (-b_n, b_n)$. Since

$$\bigcup_{n=1}^{\infty} E_n = R^2 \quad \text{and} \quad \chi^{-1}(J) = \zeta^{-1}(E')$$

we have

$$\lim_n P(\zeta \in E' \cap E_n \mid \zeta \in E') = 1.$$

By axiom (IV′), we get

$$\zeta^{-1}(E' \cap E_n) \in \mathscr{B}$$

for sufficiently large n. Hence

$$P(\chi \in I \mid \chi \in J) = \lim_n P(\zeta \in E \cap E_n \mid \zeta \in E' \cap E_n), \tag{11}$$

by Theorem 2.1 from [12]. Because of the independence of ξ and η and Theorem 3.1 relation (11) implies (10) and the proof is completed.

In the above theorem, the convolution of two Rényi probability distributions is given by formula (8). Now, we shall formulate a little stronger definition of the the convolution of Rényi probability distributions. Simultaneously, we shall give definitions of the convolutions of Rényi density functions and Rényi distribution functions.

Let

(a) $[\mu]$, $[\nu]$, $[\lambda]$ be Rényi probability distributions,
(b) $[f]$, $[g]$, $[h]$ be Rényi density functions,
(c) $[F]$, $[G]$, $[H]$ be Rényi distribution functions.

Then we write

(a) $[\lambda] = [\mu] * [\nu]$,
(b) $[h] = [f] * [g]$,
(c) $[H] = [F] * [G]$

if for arbitrary sequences $a_n \to \infty$ and $b_n \to \infty$:

(a) $[\mu_{a_n} * \nu_{b_n}] \to [\lambda]$,
(b) $[f_{a_n} * g_{b_n}] \to [h]$,
(c) there are sequences β_n and β_n' of real numbers such that

$$[(F_{a_n} + \beta_n) * (G_{b_n} + \beta_n')] \to [H].$$

The above definitions can be easily transferred to the q-dimensional case. In view of Remark 2.1, the definitions are consistent. The following theorem is obvious:

THEOREM 3.4. *If $[F]$ and $[G]$ are Rényi distribution functions corresponding to Rényi probability distributions $[\mu]$ and $[\nu]$, then $[F] * [G]$ corresponds to $[\mu] * [\nu]$, provided one of the convolution exists. If $[F]$ and $[G]$ are absolutely continuous, then $[F] * [G]$ is absolutely continuous and*

$$[[F] * [G]]' = [f] * [g],$$

where $[f] = [F']$ and $[g] = [G']$ are the Rényi density functions corresponding to $[F]$ and $[G]$.

Let us give some examples of convolutions.

EXAMPLE 3.1. Let f and g be two non-negative polynomials:

$$f(x) = \sum_{k=0}^{n} c_k x^k, \quad g(x) = \sum_{l=0}^{m} a_l x^l \quad (x \in R).$$

Of course, the assumption $f \geq 0$, $g \geq 0$ implies that $n = 2r$, $m = 2s$ and $c_{2r} \geq 0$, $d_{2s} \geq 0$. The classes $[f]$ and $[g]$ are Rényi density functions.

We shall prove that

$$[f] * [g] = [1]. \tag{12}$$

and the convolution is tempered. Let a_n and b_n be arbitrary positive numbers such that $a_n \to \infty$, $b_n \to \infty$. For an arbitrary $x \in R$, we have $x + a_n > 0$, $x + b_n > 0$, $x - a_n < 0$ and $x - b_n < 0$ for sufficiently large n. Denote

$$\varkappa_n(x) = \max(x - a_n, -b_n), \quad K_n(x) = \min(x + a_n, b_n),$$
$$A = \{i \in N:\ a_i \geq b_i\}, \quad B = \{i \in N:\ a_i < b_i\},$$
$$a(x) = \{i \in N:\ x + a_i \geq b_i\}, \quad b(x) = \{i \in N:\ x + a_i < b_i\}$$

and

$$A(x) = \{i \in N:\ a_i \geq x + b_i\}, \quad B(x) = \{i \in N:\ a_i < x + b_i\}.$$

Note that

$$A \cap B = a(x) \cap b(x) = A(x) \cap B(x) = \emptyset$$

and

$$A \cup B = a(x) \cup b(x) = A(x) \cup B(x) = N.$$

By the Leibniz formula, we have

$$\begin{aligned} f_{a_n} * g_{b_n}(x) &= \sum_{k=0}^{2r} \sum_{l=0}^{2s} c_k d_l \int_{\varkappa_n(x)}^{K_n(x)} (x-t)^k t^l \, dt \\ &= \sum_{k=0}^{2r} \sum_{l=0}^{2s} \sum_{i=0}^{k} \theta(k, l, i) x^i \int_{\varkappa_n(x)}^{K_n(x)} t^{k+l-i} \, dt \\ &= \sum_{k=0}^{2r} \sum_{l=0}^{2s} \sum_{i=0}^{k} \tilde{\theta}(k, l, i) x^i k_n(x, k, l, i), \end{aligned} \tag{13}$$

where

$$\theta(k, l, i) = (-1)^{k-i} \binom{k}{i} c_k d_l, \ \tilde{\theta}(k, l, i) = \theta(k, l, i)(k+l+1-i)^{-1}$$

and

$$k_n(x, k, l, i) = (K_n(x))^{k+l+1-i} - (\varkappa_n(x))^{k+l+1-i}.$$

Let $\beta_n = \min(a_n, b_n)$ and $\alpha_n = \beta_n^{-2r-2s-1}$. Of course, $\beta_n \to \infty$ and $\alpha_n \to 0$.

Note that if any of the cases: 1° $n \in A \cap a(x)$, 2° $n \in B \cap b(x)$, 3° $n \in A \cap b(x)$, 4° $n \in B \cap a(x)$ holds for infinitely many n, then, given $\varepsilon > 0$, we have

$$|\beta_n^{-1} K_n(x) - 1| < \varepsilon \tag{14}$$

for sufficiently large n.

In fact, in case 1°, we have $\beta_n^{-1} K(x) = 1$ and, in case 2°,

$$\beta_n^{-1} K_n(x) = \frac{x}{a_n} + 1,$$

so (14) is obvious in these cases.

In case 3°, inequality (14) follows from the relations:

$$\beta_n^{-1} K_n(x) = \frac{x}{b_n} + \frac{a_n}{b_n}$$

and

$$x + b_n \leqslant x + a_n < b_n.$$

Finally, in case 4°, we have

$$\beta_n^{-1} K_n(x) = \frac{b_n}{a_n}$$

and (14) holds, because 4° yields

$$a_n < b_n \leqslant x + a_n.$$

Since cases 1°–4° exhaust all possibilities, we have proved that

$$\beta_n^{-1} K_n(x) \to 1 \quad \text{as} \quad n \to \infty. \tag{15}$$

Similarly, considering the sets $A \cap A(x)$, $B \cap B(x)$, $A \cap B(x)$ and $B \cap A(x)$, one can show that

$$\beta_n^{-1} \varkappa_n(x) \to -1 \quad \text{as} \quad n \to \infty. \tag{16}$$

By (15) nad (16), we have

$$\alpha_n k_n(x, 2r, 2s, 0) \to 2$$

and

$$\alpha_n k_n(x, k, l, i) \to 0$$

for any k, l, i, such that $k + l - i < 2r + 2s$. Hence, because of (13),

$$\alpha_n f_{a_n} * g_{b_n}(x) \to \frac{2c_{2r} d_{2s}}{2r + 2s + 1} > 0 \tag{17}$$

as $n \to \infty$ for each $x \in R$, i.e., (12) holds. Moreover, it is easy to see that

$$\beta_n^{-1} |K_n(x)| < c(1 + |x|) \quad (x \in R)$$

and

$$\beta_n^{-1} |\varkappa_n(x)| < c(1 + |x|) \quad (x \in R)$$

for some $c > 0$, i.e.,

$$\alpha_n |f_{a_n} * g_{b_n}(x)| < d(1 + |x|)^{2r + 2s + 1} \tag{18}$$

for some $d > 0$. Relations (17) and (18) imply that

$$\alpha_n \int_{-\infty}^{\infty} \gamma(x)(f_{a_n} * g_{b_n})(x)\, dx \to \int_{-\infty}^{\infty} \gamma(x)\, dx$$

for each $\gamma \in \mathscr{S}$, i.e., the convolution $[f] * [g] = [1]$ exists in $[\mathscr{S}']$ (see Section 6).

EXAMPLE 3.2. Let

$$\mu(E) = \sum_{i=-\infty}^{\infty} |i|^k \delta_i(E)$$

and

$$\nu(E) = \sum_{j=-\infty}^{\infty} |j|^l \delta_j(E)$$

for $E \in B(R)$, where δ_i is the normed measure concentrated at the point i. Using similar arguments as in the preceding example one can show that the convolution $[\mu] * [\nu]$ exists in $[\mathscr{S}']$ (see Section 6) and

$$[\mu] * [\nu] = \Big[\sum_{i=-\infty}^{\infty} \delta_i\Big].$$

EXAMPLE 3.3. It can be shown, analogously as in Example 3.1 that the convolution $[e^x] * [e^x]$ exists in $[\mathscr{D}']$ (see Section 6) and

$$[e^x] * [e^x] = [1].$$

However, this convolution does not exist in $[\mathscr{S}']$.

It is worth noting that Examples 3.1–3.3 have an interesting probabilistic interpretation. The Rényi density function (1) represents the so-called uniform probability distribution on the whole real line and the Rényi probability distribution $[\sum_{i=-\infty}^{\infty} \delta_i]$ can be called the uniform probability distribution on $Z = \{0, \pm 1, \pm 2, \ldots\}$. Examples 3.1–3.3 show that the sum $\xi + \eta$ of independent random variables ξ and η with polynomial density functions on R (with the exponential density functions on R; with Rényi probability distributions concentrated on Z of polynomial growth) has the uniform probability distribution on R (on Z), provided $\xi + \eta$ satisfies conditions (d_1) and (d_2) for $K = R$. In particular, the sum of independent random variables with the uniform distribution on R (on Z) has again the uniform distribution on R (on Z), under the mentioned conditions.

4. A particular case of convolution. Now, consider the particular case of the convolution of Rényi probability distributions $[\mu]$ and $[\nu]$ in R^q, where the limit in definition (a′) in Section 3 exists with the constants $\alpha_n = 1$, i.e.,

$$\lim_n \mu_{a_n} * \nu_{b_n} = \lambda \tag{1}$$

for each a_n, $b_n > 0$ such that $a_n \to \infty$ and $b_n \to \infty$. Relation (1) can be adopted as one of possible definitions of the convolution of measures (not necessarily bounded) μ, ν on $B(R^q)$. Of course, if the convolution $\mu * \nu$ exists, then the convolution of Rényi probability distributions $[\mu]$ and $[\nu]$ exists and

$$[\mu] * [\nu] = [\mu * \nu].$$

Non-negative measures on $B(R^q)$ can be treated as distributions of L. Schwartz (cf. formula (17) in Section 1).

In [11], the convolution $f * g$ in $\mathscr{D}'$ (in $\mathscr{S}'$) for $f, g \in \mathscr{D}'(R^q)$ ($f, g \in \mathscr{S}'(R^q)$) is defined in one of the three equivalent ways:

$$f * g = \lim_n (\eta_n f) * g, \tag{2}$$

$$f * g = \lim_n f * (\eta_n g) \tag{3}$$

$$f * g = \lim_n (\eta_n f) * (\tilde{\eta}_n g), \tag{4}$$

where the limits are supposed to exist in $\mathscr{D}'(R^q)$ (in $\mathscr{S}'(R^q)$) for all sequences of smooth functions $\{\eta_n\}$ and, in the last equality, $\{\tilde{\eta}_n\}$ belonging to one of the classes E, $\bar{E}$ (one of the classes E, $\bar{E}$, E^s, $\bar{E}^s$) of so-called unit-sequences.

Let us recall that the sequence $\{\eta_n\}$ belongs to $\bar{E}$ if

(i) for every interval $I \subset R^q$ there is an index n_0 such that $\eta_n(x) = 1$ for $x \in I$ and $n > n_0$,

and

(ii) $\sup\{|\eta_n^{(k)}(x)|:\ x \in R^q,\ n \in N\} < M_k < \infty$.

The product in $\mathscr{D}'$ (in $\mathscr{S}'$) of Schwartz (tempered) distributions can be defined by one of the formulae:

$$f \cdot g = \lim_{n\to\infty} (f * \delta_n) \cdot g, \tag{5}$$

$$f \cdot g = \lim_{n\to\infty} f \cdot (g * \delta_n), \tag{6}$$

$$f \cdot g = \lim_{n\to\infty} (f * \delta_n) \cdot (g * \tilde{\delta}_n), \tag{7}$$

where the limits exist in $\mathscr{D}'(R^q)$ (in $\mathscr{S}'(R^q)$) for all delta-sequences $\{\delta_n\}$ and $\{\tilde{\delta}_n\}$, belonging to one of the classes Δ, Δ_m (Δ, Δ^s, Δ_m or Δ_m^s) — cf. [11].

As a consequence of the theorem on the exchange formulae in [10], p. 122 (see also [5], Corollary 3), we have

THEOREM 4.1. *Let f, g be tempered measures in R^q. If one of the convolutions (2)–(4) exists in $\mathscr{S}'$ then the products of $\mathscr{F}(f)$ and $\mathscr{F}(g)$ in the sense (5)–(7) exist in $\mathscr{S}'$ and are equal. Moreover, using the usual notation for the convolutions and products, we have*

$$\mathscr{F}(f * g) = \mathscr{F}(f) \cdot \mathscr{F}(g).$$

5. Compatibility and polynomial compatibility. In this section, we shall mean the convolution and the product of distributions in the sense of (4) and (10) (Section 4), respectively.

There are known criteria for the convolution (of functions, measures, distributions) to exist, in terms of supports. Namely, the existence of the convolution of two arbitrary distributions in R^q is guaranteed if their supports A, B fulfil one of the following equivalent conditions:

(i) for every interval $I \subset R^q$, there exists an interval $J \subset R^q$ such that $x \in I$ implies $\sigma_{AB}(x) \subset J$, where

$$\sigma_{AB}(x) = \{y \in R^q:\ x - y \in A,\ y \in B\};$$

(ii) for every interval $I \subset R^q$, there exists an interval $J \subset R^q$ such that $x \in I$ implies $\sigma_{BA}(x) \subset J$;

(iii) for every interval $K \subset R^q$ the set $(A \times B) \cap K^\Delta$, where

$$K^\Delta = \{(x, y) \in R^{2q}:\ x + y \in K\},$$

is bounded;

(iv) if $x_n \in A$, $y_n \in B$ and $|x_n| + |y_n| \to \infty$, then $|x_n + y_n| \to \infty$.

Conditions (i)–(iii) are usually formulated in terms of compact sets. Condition (iv) is formulated in [1], p. 125. The sets $A, B \subset R^q$ satisfying one of conditions (i)–(iv) will be called, as in [1], *compatible*.

Let us formulate the following theorem, which shows that the compatibility of supports of distributions is an apt concept:

THEOREM 5.1. *Let $A, B \subset R^q$. If the convolution $f * g$ exists for all non-negative (tempered) measures f, g whose supports are contained in A, B, respectively, then the sets A, B are compatible.*

Proof. Suppose that A, B are not compatible, i.e., there exist $x_n \in A$, $y_n \in B$ such that

$$|x_n| \to \infty, \quad |y_n| \to \infty$$

and

$$x_n + y_n \to z \in R^q.$$

We can assume that

(1) $$|x_{n+1}|-|x_n| > 1, \quad |y_{n+1}|-|y_n| > 1 \quad (n \in N)$$

and that

(2) $$|x_n+y_n-z| < 1 \quad (n \in N).$$

Let

$$f(t) = \sum_{i=1}^{\infty} \delta(t-x_i) \quad (t \in R^q)$$

and

$$g(t) = \sum_{i=1}^{\infty} \delta(t-y_i) \quad (t \in R^q).$$

Note that f and g are well defined non-negative measures on R^q, with supports in A and B, respectively, and they are tempered.

Let η be a non-negative function of the class $\mathcal{S}$ such that

(3) $$\eta(t) = 1 \quad \text{for} \quad |x| \leqslant 1$$

and let

(4) $$\eta_n(t) = \eta(t/n) \quad (t \in R^q,\ n \in N).$$

Further, let $\omega \in \mathcal{S}$, $\omega(x) \geqslant 0$ for $x \in R^q$ and

(5) $$\omega(t) = 1 \quad \text{for} \quad |t-z| \leqslant 1.$$

Then, by (2)–(5),

$$\langle (\eta_n f)*(\eta_n g)(t),\ \omega(t)\rangle = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \eta_n(x_i)\eta_n(y_j)\,\omega(x_i+y_j) \geqslant p_n,$$

where p_n is the number of the indices i, for which $|x_i| \leqslant n$ and $|y_i| \leqslant n$. Since $p_n \to \infty$, we conclude that the convolution $f*g$ does not exist, which contradicts the assumption, so the assertion is proved.

In [19], the following problem was posed. Suppose that the convolution $f*g$ of two tempered distributions exists in $\mathcal{D}'$. Has the distribution $f*g$ to be a tempered distribution?

The negative answer was given in the doctoral dissertation of the author (which was completed in 1974 and edited in the form of a preprint [10]); the result without a proof was published in [5]. Independently, the problem was solved by P. Dierolf and J. Voigt in [3].

In [10], p. 77 (see also [5]), the answer is given in a stronger form. Namely, given a non-negative function $g(x)$ on R^q (which can increase as $|x| \to \infty$ as rapidly as we wish) there exists a smooth function $f(x)$ on R^q whose support is compatible with itself such that $f(x) \to 0$ as $|x| \to \infty$ and

$$f*f(x) \geqslant g(x)$$

for each $x \in R^q$.

The following definition, given in [10], p. 87 (see also [5]), is a modification of the concept of compatibility.

The sets $A,\ B \subset R^q$ are said to be *polynomially compatible* if there exists a polynomial p in R such that

$$|x|+|y| \leqslant p(|x+y|)$$

for each $x \in A$ and $y \in B$.

It is not difficult to show that the following conditions are equivalent for arbitrary $A,\ B \subset R^q$ (cf. [10], p. 91):

(i) A, B are polynomially compatible;

(ii) There exists $k \in N$ and $c > 0$ such that

$$|x|+|y| \leqslant c(1+|x+y|)^k$$

for each $x \in A$ and $y \in B$;

(iii) There exists a polynomial p such that if $x-y \in A$ and $y \in B$, then $|y| < p(|x|)$.

It is clear that polynomially compatible sets in R^q are compatible. The converse is not true (see [10], p. 91).

The following theorem holds:

THEOREM 5.2. *If tempered distributions f, g have polynomially compatible supports, then $f*g \in \mathcal{S}'$.*

Conversely, let $A,\ B \subset R^q$ be sets such that $f*g \in \mathcal{S}'$ for any tempered distributions f, g with supports contained in A and B, respectively. Then A and B are polynomially compatible.

Proof. Suppose that $f_1, f_2 \in \mathcal{S}'$ and that the supports A^1 and A^2 of f_1 and f_2, respectively, are polynomially compatible. We have $f_1 = F_1^{(k_1)}$ and $f_2 = F_2^{(k_2)}$ on R^q, where $k_1,\ k_2 \in P^q$ and F_1, F_2 are bounded by polynomials. Given a set $K \subset R^q$ and $\varepsilon > 0$, let

$$K_\varepsilon = \{y \in R^q\colon |y-x| < \varepsilon \text{ for some } x \in K\}.$$

Clearly, K is an open set. Let $\omega_1,\ \omega_2 \in \mathcal{D}$ and

$$\omega_m(x) = \begin{cases} 1 & \text{for} \quad x \in A_\varepsilon^m, \\ 0 & \text{for} \quad x \notin A_{2\varepsilon}^m \end{cases}$$

for $m = 1, 2$. We have

(6) $$f_m = f_m\,\omega_m = \sum_{0 \leqslant i \leqslant k_m} (F_m\,\omega_m^{(i)})^{(k_m-i)} \quad (m = 1, 2)$$

and

(7) $$|F_1\,\omega_1^{(i)}| * |F_2\,\omega_2^{(j)}|(x) \leqslant C_1\,C_2 \int_{\sigma(x)} |F_1(x-y)\,F_2(y)|\,dy,$$

where

$$C_m = \max\{\omega_m^{(i)}(t):\ t\in R^q,\ 0\leqslant i\leqslant k_m\} \qquad (m=1,\,2)$$

and

$$\sigma(x) = \{y\in R^q:\ x-y\in A_{2\varepsilon}^1,\ y\in A_{2\varepsilon}^2\}.$$

Since the sets $A_{2\varepsilon}^1$ and $A_{2\varepsilon}^2$ are also polynomially compatible, we have

$$\sigma(x) \subset [-p(|x|),\,p(|x|)] \subset R^q$$

for some non-negative polynomial p (see condition (iii) of the definition of polynomial compatibility). This and (7) imply that the convolution on the left-hand side of (7) exists and is bounded by a polynomial. To prove that $f_1 * f_2 \in \mathscr{S}'$, it remains to use formula (6) and the known properties of the convolution. Thus the first part of the theorem is proved.

Now, suppose that $A,\ B\subset R^q$ have the property formulated in the second part of the theorem. In view of Theorem 5.1, the sets A, B are compatible. Assume that they are not polynomially compatible, i.e., there are points $x_i\in A$ and $y_i\in B$ such that

(8) $$|x_i|+|y_i| > 2^i(1+|z_i|)^i$$

for each $i\in N$, where $z_i = x_i+y_i$. Hence $|x_i|+|y_i|\to\infty$.

There are three possibilities:

(a) $|x_i|\to\infty$ and $|y_i|\to\infty$,
(b) $|x_i|\to\infty$ and $|y_i|\not\to\infty$,
(c) $|x_i|\not\to\infty$ and $|y_i|\to\infty$.

First consider case (a). Since the sets A and B are compatible, we infer that $|z_i|\to\infty$. Of course, we can choose subsequences $\bar{x}_i = x_{k_i}$, $\bar{y}_i = y_{k_i}$, and $\bar{z}_i = z_{k_i}$ such that $|\bar{x}_{i+1}|-|\bar{x}_i|>1$, $|\bar{y}_{i+1}|-|\bar{y}_i|>1$ and $|\bar{z}_{i+1}|-|\bar{z}_i|>1$ for all $i\in N$. Let

$$f(t) = \sum_{i=1}^{\infty}(1+|\bar{x}_i|)\,\delta(t-\bar{x}_i)$$

and

$$g(t) = \sum_{i=1}^{\infty}(1+|\bar{y}_i|)\,\delta(t-\bar{y}_i).$$

Clearly, f and g are tempered measures and $\operatorname{supp} f\subset A$, $\operatorname{supp} g\subset B$. Moreover, the convolution $f*g$ exists and is a measure, namely

$$f*g(t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(1+|\bar{x}_i|)(1+|\bar{y}_j|)\cdot\delta(t-\bar{x}_i-\bar{y}_j).$$

But this measure is not a tempered distribution, because

$$f*g(t) \geqslant \sum_{i=1}^{\infty}(|\bar{x}_i|+|\bar{y}_i|)\,\delta(t-\bar{z}_i) \geqslant \sum_{i=1}^{\infty}2^{k_i}(1+|\bar{z}_i|)^{k_i}\,\delta(t-\bar{z}_i),$$

in view of (8). But this contradicts the assumption.

Now, consider case (b). In this case there is a subsequence $\{y_{m_i}\}$ of $\{y_i\}$ such that

$$y_{m_i}\to y\in R^q$$

as $i\to\infty$. Of course,

$$|x_{m_i}+y_{m_i}|\to\infty.$$

We can assume that $|x_{m_{i+1}}|-|x_{m_i}|>1$ and $|z_{m_{i+1}}|-|z_{m_i}|>1$ for $i\in N$, where $z_j = x_j+y_j$. Let

$$f(t) = \sum_{i=1}^{\infty}(1+|\bar{x}_i|)\,\delta(t-\bar{x}_i)$$

and

$$g(t) = \sum_{i=1}^{\infty}2^{-i}(1+|\bar{y}_i|)\,\delta(t-\bar{y}_i),$$

where $\bar{x}_i = x_{m_i}$ and $\bar{y}_i = y_{m_i}$. Note that f and g are tempered measures such that $\operatorname{supp} f\subset A$, $\operatorname{supp} g\subset B$ and the convolution exists and is a measure given by the formula:

$$f*g(t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}2^{-j}(1+|\bar{x}_i|)(1+|\bar{y}_j|)\,\delta(t-\bar{x}_i-\bar{y}_j).$$

However, this measure is not tempered, because

$$f*g(t) \geqslant \sum_{i=1}^{\infty}2^{-i}(|\bar{x}_i|+|\bar{y}_i|)\,\delta(t-\bar{z}_i),$$

where $\bar{z}_i = \bar{x}_i+\bar{y}_i$, and thus

$$f*g(t) \geqslant \sum_{i=1}^{\infty}2^{m_i-i}(1+|\bar{z}_i|)^{m_i}\,\delta(t-z_i).$$

The obtained contradiction proves the assertion in case (b).

Case (c) is completely symmetric to (b), so the theorem is proved.

Remark 5.1. The proof of the first part of Theorem 5.2 is given in [10], p. 97 (see also [5]). The second part of Theorem 5.2 and Theorem 5.1 are answers to the problems of S. Ju. Priščepionok, communicated to me orally.

From the first part of the proof of Theorem 5.2, it follows that tempered distributions f, g with compatible supports can be represented in the form

$$f = \sum_{i=1}^{r} F_i^{(i)}, \quad g = \sum_{j=1}^{s} G_j^{(i)},$$

where the convolution $|F_i| * |G_j|$ exists and is bounded by a polynomial. Hence, in view of Theorem in [6] (see also [10], p. 122 and [5]), we have

THEOREM 5.3. *If tempered distributions f, g have polynomially compatible supports, then the product $\mathcal{F}(f)\cdot\mathcal{F}(g)$ exists in $\mathcal{S}'$ and*

$$\mathcal{F}(f*g) = \mathcal{F}(f)\cdot\mathcal{F}(g).$$

COROLLARY. *If tempered Rényi probability distributions $[\mu]$ and $[\nu]$ are concentrated on polynomially compatible sets (i.e., there are representatives $\tilde{\mu}$, $\tilde{\nu}$ of $[\mu]$, $[\nu]$, respectively, with polynomially compatible supports), then*

$$[\mu]*[\nu] = [\Phi]\cdot[\Psi],$$

where $[\Phi]$ and $[\Psi]$ are the characteristic distributors of $[\mu]$ and $[\nu]$ and the product $[\Phi]\cdot[\Psi]$ exists in the sense $[\Phi]\cdot[\Psi] = [\Phi\cdot\Psi]$.

6. Distributors. The method of identifying non-negative measures (their Fourier transforms), which differ from each other by a constant factor, can be used for arbitrary functions and Schwartz distributions (see [8]).

Consider the space $\mathcal{D}'$ ($\mathcal{S}'$) of all Schwartz distributions (tempered distributions) in R^q and define the relation $\sim$ in $\mathcal{D}'$ ($\mathcal{S}'$): $f_1 \sim f_2$ if $f_1 = \alpha f_2$ for some $\alpha > 0$. This is an equivalence relation. The equivalent classes with respect to this relation will be called *distributors* (*tempered distributors*) and be denoted by $[f]$, $[g]$, ..., where $f, g, \ldots \in \mathcal{D}'$ ($f, g, \ldots \in \mathcal{S}'$). The set of all distributors (tempered distributors) will be denoted by $[\mathcal{D}']$ (by $[\mathcal{S}']$).

We introduce the convergence in $[\mathcal{D}']$ (in $[\mathcal{S}']$) as follows: a *sequence $[f_n]$ of distributors (tempered distributors) is convergent in $[\mathcal{D}']$ (in $[\mathcal{S}']$)* if, for some representatives $\tilde{f}_n$ of $[f_n]$, the sequence $\tilde{f}_n$ is convergent in $\mathcal{D}'$ (in $\mathcal{S}'$), i.e., if the sequence $\alpha_n f_n$ is convergent in $\mathcal{D}'$ (in $\mathcal{S}'$) for some constants $\alpha_n > 0$.

Note that convergent sequences have not unique limits, in general. Namely, if $[f_n] \to [f]$, then always $[f_n] \to [0]$ in $[\mathcal{D}']$ (or $[\mathcal{S}']$). However, if $\alpha_n f_n \to f$ and $\tilde{\alpha}_n f_n \to g$ in $\mathcal{D}'$ (or in $\mathcal{S}'$) for $f \neq 0 \neq g$ and constants $\alpha_n, \tilde{\alpha}_n > 0$, then $f = \alpha g$ for some $\alpha > 0$. In fact, the assumption $f \neq 0 \neq g$ implies $\langle f, \psi\rangle \neq 0 \neq \langle g, \psi\rangle$ for some $\psi \in \mathcal{D}$ (or $\psi \in \mathcal{S}$) and thus

$$\frac{\alpha_n}{\tilde{\alpha}_n} \to \frac{\langle f, \psi\rangle}{\langle g, \psi\rangle} > 0,$$

i.e.,

$$f = \lim_{n\to\infty} \alpha_n f_n = \lim_{n\to\infty} \frac{\alpha_n}{\tilde{\alpha}_n}\tilde{\alpha}_n f_n = \alpha g,$$

where $\alpha = \langle f, \psi\rangle\cdot\langle g, \psi\rangle^{-1}$. This means, the definition of the convergence is

consistent. Evidently, this is the quotient convergence, with respect to the relation $\sim$, generated by the weak or, equivalently, strong topology in $\mathcal{D}'$ ($\mathcal{S}'$).

To avoid non-uniqueness we adopt the following convention: $[0]$ is the limit of $[f_n]$ iff there exist constants $\alpha_n > 0$ such that $\alpha_n f_n \to 0$ and if $\beta_n f_n \to f$ for some $\beta_n > 0$, then $f = 0$. This means that if $\alpha_n f_n \to f \neq 0$ for some constants $\alpha_n > 0$, then $[f]$ is the only limit of $[f_n]$. Of course, if $f_n \to f \neq 0$ in $\mathcal{D}'$ (in $\mathcal{S}'$), then $[f_n] \to [f]$ in $[\mathcal{D}']$ (in $[\mathcal{S}']$).

EXAMPLE 6.1. One can check that the sequence $[x^n]$ is convergent in $[\mathcal{S}']$ to $[0]$ and the sequence $[\delta^{(n)}]$ is not convergent in $[\mathcal{S}']$.

We are going to present a general scheme of defining operations on distributors. Before that let us modify the Mikusiński method of irregular operations on Schwartz distributions, proposed in [14] and developed in [1].

In [14] and [1], the starting point of the method is the assumption that a given operation is defined for smooth functions. Then, by using a fixed class of delta-sequences, the operation is extended to those distributions, for which the respective distributional limit of smooth functions exists. Only particular operations (regular operations) can be extended for all distributions.

Some of operations, e.g. the convolution, cannot be defined for all smooth functions and the procedure is more complicated for them (cf. [1], p. 153; 130–131). On the other hand, all natural operations are defined for the functions of the class $\mathcal{S}$ and the construction below will be based on this assumption.

We can consider operations of an arbitrary finite number of arguments. For simplicity, suppose that a given operation A is of two arguments, i.e., to each pair of functions $\varphi, \psi \in \mathcal{S}$ a smooth function $A(\varphi, \psi)$ is assigned.

Let $\bar{E}_m$ and Δ_m be the classes of unit-sequences and delta-sequences defined as in [7] and [11], i.e., of the form:

(1) $$\eta_n(x) = \eta\left(\frac{x}{\tau_n}\right) \quad (x \in R^q)$$

and

(2) $$\delta_n(x) = \tau_n^q \sigma(\tau_n x) \quad (x \in R^q),$$

where $\tau_n \to \infty$, $\eta, \sigma \in \mathcal{S}$, $\eta(x) = 1$ for x belonging to some neighbourhood of 0 and $\int \sigma = 1$. Additionally, we shall assume that η, σ are real-valued even functions. For fixed $f, g \in \mathcal{S}'$, we adopt

(3) $$A(f, g) = \lim_n \Phi_{nn},$$

where

(4) $$\Phi_{ij} = A((\eta_i f)*\delta_j, (\eta_i g)*\delta_j) \quad (i, j \in N),$$

provided the limit exists in $\mathscr{D}'$ for each $\{\eta_n\} \in \bar{E}_m$ and $\{\delta_n\} \in \Delta_m$ and does not depend on the choice of $\{\eta_n\}$ and $\{\delta_n\}$. The above definition can be simplified for some operations.

We say that a given operation A is *local* if for each interval $I \subset R^q$ there exist intervals $I_1, I_2 \subset R^q$ such that

$$A(\varphi_1, \varphi_2) = A(\psi_1, \psi_2) \quad \text{on} \quad I$$

for each $\varphi_1, \varphi_2, \psi_1, \psi_2 \in \mathscr{D}$ such that $\varphi_1 = \psi_1$ on I_1 and $\varphi_2 = \psi_2$ on I_2.

Clearly, local operations can be defined for all smooth functions φ, ψ by the formula

$$A(\varphi, \psi) = \lim_n A(\eta_n \varphi, \eta_n \psi),$$

where $\{\eta_n\}$ is an arbitrary unit-sequence from $\bar{E}_m$. The operation of multiplication $A(\varphi, \psi) = \varphi \cdot \psi$ $(\varphi, \psi \in \mathscr{D})$ is local.

The following theorem is obvious:

THEOREM 6.1. *Let A be a local operation and $f, g \in \mathscr{D}'$. Then $A(f, g)$ exists in $\mathscr{D}'$ in the sense of* (3)–(4) *iff*

$$A(f, g) = \lim_{n\to\infty} A(f * \delta_n, g * \delta_n) \tag{5}$$

in $\mathscr{D}'$ for each $\{\delta_n\} \in \Delta_m$.

We say that a given operation A is *locally regular* if for each interval $I \subset R^q$ and arbitrary fundamental (i.e., distributionally convergent) sequences $\{\varphi_n\}$ and $\{\psi_n\}$ of smooth functions such that supp $\varphi_n \subset I$ and supp $\psi_n \subset I$ $(n \in N)$, the sequence $\{A(\varphi_n, \psi_n)\}$ is fundamental.

Note that locally regular operations can be defined for all distributions of the class $\mathscr{E}'$ by the formula:

$$A(f, g) = \lim_{j\to\infty} A(f * \delta_j, g * \delta_j),$$

where $\{\delta_j\}$ is an arbitrary delta-sequence of the class Δ_m.

We say that an operation A *commutes* with delta-sequences if for each $\{\delta_n\} \in \Delta_m$ there exists $\{\tilde{\delta}_n\} \in \Delta_m$ such that

$$A(\varphi * \delta_n, \psi * \delta_n) = A(\varphi, \psi) * \tilde{\delta}_n$$

for each $\varphi \in \mathscr{D}$.

The operation of convolution $A(\varphi, \psi) = \varphi * \psi$ $(\varphi, \psi \in \mathscr{D})$ is locally regular and commutes with delta-sequences.

Analogously to the first part of the proof Theorem in [9], we obtain

THEOREM 6.2. *Let A be a locally regular operation which commutes with delta-sequences and let $f, g \in \mathscr{D}'$. Then $A(f, g)$ exists in the sense of* (3)–(4) *iff*

$$\lim_{n\to\infty} A(\eta_n f, \eta_n g) = A(f, g) \tag{6}$$

in $\mathscr{D}'$ for each $\{\eta_n\} \in \bar{E}_m$.

Formulae (5) and (6) can be adopted as the definitions of local and locally regular operations for distributions. Similarly, we define local and locally regular operations in $\mathscr{S}'$. Namely, we assume that for given $f, g \in \mathscr{S}'$, (5) and (6) hold in $\mathscr{S}'$ for all $\{\delta_n\} \in \Delta_m^s$ and $\{\eta_n\} \in E_m^s$, i.e., for all delta-sequences and unit-sequences of the form (2) and (1), respectively, where σ and η are real-valued even functions of the class $\mathscr{S}$ such that $\int \sigma = 1$ and $\eta(0) = 1$.

Now, define the relation $\sim$ in the space $\mathscr{D}$ ($\mathscr{S}$) and denote by $[\varphi]$ the equivalence class containing $\varphi \in \mathscr{D}$ $(\varphi \in \mathscr{S})$.

Suppose that a given operation A fulfils the condition:

(m) for any $a_1, a_2 > 0$ there exists $a > 0$ such that

$$A(a_1 \varphi_1, a_2 \varphi_2) = aA(\varphi_1, \varphi_2)$$

for each $\varphi_1, \varphi_2 \in \mathscr{D}$ $(\varphi_1, \varphi_2 \in \mathscr{S})$.

Then A can be defined on the classes $[\varphi_1]$, $[\varphi_2]$ as follows:

$$A([\varphi_1], [\varphi_2]) = [A(\varphi_1, \varphi_2)]$$

and the definition is consistent. Note that the operations of multiplication and convolution satisfy condition (m) and the operations of addition and subtraction do not.

Denote

$$[\varphi] * \delta_n = [\varphi * \delta_n] \quad \text{and} \quad \eta_n \cdot [\varphi] = [\eta_n \varphi]$$

for $\varphi \in \mathscr{D}$ and $\{\delta_n\} \in \Delta_m$, $\{\eta_n\} \in \bar{E}_m$ or for $\varphi \in \mathscr{S}$ and $\{\delta_n\} \in \Delta_m^s$, $\{\eta_n\} \in E_m^s$, respectively. For given distributors $[f]$, $[g]$ and an operation A fulfilling condition (m), let

$$A([f], [g]) = \lim_n \Phi_{nn} \tag{7}$$

where Φ_{ij} is given by formula (4), provided the limit exists in (in ($\mathscr{S}'$)) for all $\{\eta_n\} \in E_m$ and $\{\delta_n\} \in \Delta_m$($\{\eta_n\} \in E_m^s$ and $\{\delta_n\} \in \Delta_m^s$) and does not depend on the choice of $\{\eta_n\}$ and $\{\delta_n\}$.

Similarly, for a given local or locally regular operation A fulfilling condition (m), we adopt

$$A([f], [g]) = \lim_{n\to\infty} [A(f * \delta_n, g * \delta_n)] \tag{8}$$

or

$$A([f], [g]) = \lim_{n\to\infty} [A(\eta_n f, \eta_n g)] \tag{9}$$

under the assumption that the respective limits exist for all sequences from the respective classes. It is evident that

THEOREM 6.3. *Let A be an operation satisfying condition* (m) *and let f, g be fixed distributions (tempered distributions) such that $A(f, g)$ exists in the sense of* (3), (5) *or* (6) *and $A(f, g) \neq 0$. Then $A([f], [g])$ exists in the sense of* (7), (8) *or* (9), *respectively, and*

$$A([f], [g]) = [A(f, g)].$$

THEOREM 6.4. *Let A be an arbitrary (local) operation fulfilling condition* (m). *Then $A([f], [g])$ exists in $[\mathscr{D}']$ in the sense of* (7) *(in the sense of* (8)) *for any distributors $[f]$, $[g]$.*

Proof. Putting

$$\alpha_n = \left(n \cdot \sup\{|\Phi_{nn}(t)|:\ |t| \leqslant n\}\right)^{-1},$$

we have

$$|\alpha_n \langle \Phi_{nn}, \varphi\rangle| \leqslant \frac{1}{n}\int |\varphi| \to 0$$

for each $\varphi \in \mathscr{D}$, i.e., $A([f], [g])$ exists in $[\mathscr{D}']$ in the sense of (7). Similarly, if A is local, then $A([f], [g])$ exists in the sense of (8) in $[\mathscr{D}']$.

Locally regular operations need not exist in the sense of (9) for all distributors $[f]$, $[g]$. Namely, it can be shown that the convolution $[f]*[g]$, where $f(x) = \sum_{i=0}^{\infty} \delta^{(i)}(x-i)$ and $g(x) = \sum_{i=0}^{\infty} \delta^{(i)}(x+i)$, does not exist in the sense of (9). However, the convolution of arbitrary tempered distributors exists in $[\mathscr{S}']$ in the sense of (9). The same is true for the operation of product. Namely, we have

THEOREM 6.5. *The product $[f]\cdot[g]$ and the convolution $[f]*[g]$ exist in $[\mathscr{S}']$ for any distributors $[f], [g] \in [\mathscr{S}']$ in the sense of* (8) *and* (9), *respectively.*

Proof. We have $f = F^{(k)}$ and $g = G^{(l)}$ on R^q, where $k, l \in P^q$ and F, G are continuous functions, bounded by polynomials. Let $\{\delta_n\}$ be an arbitrary delta-sequence of the form (2) with $\sigma \in \mathscr{S}$. It is evident that

$$\tau_n^{-k-l-2q} P^{-1}(F * \delta_n^{(k)}) \cdot (G * \delta_n^{(l)}) \rightrightarrows 0 \quad \text{in} \quad R^q$$

for some polynomial $P > 0$, so

$$\alpha_n (f * \delta_n) \cdot (g * \delta_n) \to 0 \quad \text{in} \quad \mathscr{S}'$$

for $\alpha_n = \tau_n^{-k-l-2q}$.

Now, let $\{\eta_n\} \in E_m^s$. The functions $H_n^{ij} = (\eta_n^{(i)} F) * (\eta_n^{(j)} G)$ are continuous rapidly decreasing for $0 \leqslant i \leqslant k$, $0 \leqslant j \leqslant l$ and $n \in N$. For an arbitrary $\varphi \in \mathscr{S}$, we have

$$|\alpha_n \langle (\eta_n f) * (\eta_n g), \varphi\rangle| \leqslant \frac{1}{n} \sum_{0 \leqslant i \leqslant k} \sum_{0 \leqslant j \leqslant l} c_{ij} \int |\phi^{(k+l-i-j)}(t)|\, dt \to 0$$

as $n \to \infty$, where $c_{ij} = \binom{k}{i} \cdot \binom{l}{j}$ and $\alpha_n = n^{-1}(\sup_{i,j,t} |H_n^{ij}(t)|)^{-1}$. The proof is completed.

The convolution and the product of distributors $[f]$, $[g]$ below will be meant in $[\mathscr{S}']$ in the sense of (9) and (8), respectively.

Now, as a corollary from Theorems 1, 2 in [7], we obtain:

THEOREM 6.6. *The convolution and the product of arbitrary distributors $[f], [g] \in [\mathscr{S}']$ exist in $[\mathscr{S}']$. Moreover,*

$$\mathscr{F}([f]*[g]) = [\mathscr{F}(f)] \cdot [\mathscr{F}(g)] \quad \text{and} \quad \mathscr{F}([f]\cdot[g]) = [\mathscr{F}(f)] * [\mathscr{F}(g)].$$

Finally, we shall give some non-trivial examples of the product and the convolution of distributors in the cases where the operations are not feasible for the respective distributions or the result equals 0.

EXAMPLE 6.2. Suppose that $k, l \in N \cup \{0\}$, $k \geqslant l$. It can be shown that, for an arbitrary function $\sigma \in \mathscr{S}(R)$,

$$\int_{-\infty}^{\infty} \sigma^{(k)}(t)\sigma^{(l)}(t)\, dt = (-1)^{(k-l)/2} \int_{-\infty}^{\infty} (\sigma^{(m)}(t))^2\, dt \tag{10}$$

with $m = (k+l)/2$, provided $k+l$ is even, and

$$\sigma^{(k)} \cdot \sigma^{(l)} = \chi', \tag{11}$$

where χ is a function such that

$$\int_{-\infty}^{\infty} \chi(t)\, dt = (-1)^{(k-l-1)/2} \alpha \int_{-\infty}^{\infty} (\sigma^{(m')}(t))^2\, dt \tag{12}$$

for some $\alpha > 0$ and $m' = (k+l-1)/2$, provided $k+l$ is odd. For $f = \delta^{(k)}$ and $g = \delta^{(l)}$ and $\{\delta_n\}$ of form (2), we have

$$(f * \delta_n)(x)(g * \delta_n)(x) = \tau_n^{k+l+2} \varrho(\tau_n x),$$

where $\varrho = \sigma^{(k)} \cdot \sigma^{(l)} \in \mathscr{S}$. Let $k+l$ be even. Then, by (10),

$$\alpha_n (f * \delta_n)(g * \delta_n) \to (-1)^{(k-l)/2} \delta \quad \text{in} \quad \mathscr{S}' \tag{13}$$

for $\alpha_n = \tau_n^{-k-l-1}$, since $\int (\sigma^{(m)}(t))^2\, dt > 0$. Now, let $k+l$ be odd. By (11) and (12), we have

$$\tilde{\alpha}_n (f * \delta_n) \cdot (g * \delta_n) \to (-1)^{(k-l-1)/2} \delta' \quad \text{in} \quad \mathscr{S}' \tag{14}$$

for $\tilde{\alpha}_n = \tau_n^{-k-l}$ since $\varrho = \chi'$ and $\int_{-\infty}^{\infty} (\sigma^{(m')}(t))^2\, dt > 0$. Relations (13) and (14) give the formula:

$$[\delta^{(k)}] \cdot [\delta^{(l)}] = \begin{cases} (-1)^{(k-l)/2} \cdot [\delta] & \text{if } k+l \text{ is even,} \\ (-1)^{(|k-l|-1)/2} \cdot [\delta'] & \text{if } k+l \text{ is odd} \end{cases}$$

for any $k, l \in N \cup \{0\}$, where the product of distributors exists in $[\mathscr{S}']$.

EXAMPLE 6.3. Let k, $l \in N \cup \{0\}$. In view of Theorem 6.6, we get, after simple calculations, the formula:

$$[x^k] * [x^l] = \begin{cases} (-1)^m \cdot [1] & \text{if} \quad k+l \text{ is even}, \\ (-1)^m [x] & \text{if} \quad k+l \text{ is odd}, \end{cases}$$

where $m = \min(k, l)$ and the convolution exists in $[\mathcal{S}']$. It is known that $x^k \delta^{(m)} = 0$ if $k > m$ and

$$x^k \delta^{(m)} = \frac{(-1)^k \cdot m!}{(m-k)!} \delta^{(m-k)} \quad \text{if} \quad k \leqslant m$$

in the sense of distributions.

EXAMPLE 6.4. One can prove that

$$[x^k] \cdot [\delta^{(m)}] = \begin{cases} [\delta] & \text{if} \quad k > m,\ k+l \text{ is even}, \\ [\delta'] & \text{if} \quad k > m,\ k+l \text{ is odd}, \end{cases}$$

i.e., the product of distributions can differ from the product of the respective distributions if the latter equals to 0.

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