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On Riemann integration of functions with values in topological linear spaces

by

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Abstract. The definition of Riemann integral with respect to arbitrary non-atomic measure of functions with values in linear topological spaces is introduced.

Introduction. Let X be a linear metric space. Let $f\colon [0, 1] \to X$ be a Lebesgue measurable function defined on the unit interval $[0, 1]$ with values in X. If the space X is not locally convex then, in general, it is impossible to define a Lebesgue integral for f (see S. Rolewicz [9]). This follows from the fact that it is easy to construct a sequence of simple functions x_n with

$$x_n := \sum_{m=1}^{n} a_{m,n} \chi(E_m)$$

such that $\sup_m \|a_{m,n}\| \to 0$ for $n \to \infty$ and such that $\sum_m a_{m,n} \lambda(E_m)$ does not tend to zero, where $\lambda(E_m)$ denotes the Lebesgue measure of the set E_m.

Since it is not possible to define the Lebesgue integral, D. Przeworska-Rolewicz and S. Rolewicz [7] and independently B. Gramsch [2], [3], [4] have introduced a Riemann integral. The definition is more or less like the classical one. However, there are continuous functions which are not Riemann integrable. Even more, if each continuous function is Riemann integrable, then the space X is locally convex. This shows that the class of Riemann integrable functions is relatively small for a suitable space X.

In the papers mentioned above ([7], [2], [3], [4]), the definition was extended to analytic manifolds. The use of these integrals gives us the possibility to define analytic functions in locally bounded algebras. Consequently, it yields an extension of Levy's theorem for exponents p with $0 < p < 1$.

Then the natural question arises: Can we define a Riemann integral of functions with values in a linear metric space defined on a compact space K

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which is endowed with a Radon measure μ? This note gives a positive answer to this question. Even more, such an integral will be defined for functions with values in a topological linear space.

There are also some recent studies of Riemann's integrability. C. S. Hönig [5] found several examples of Riemann integrable, Hilbert space valued functions defined on the unit interval $[0, 1]$ which are not measurable with respect to the complete Lebesgue measure. G. C. da Rocha Filho [8] analysed definitions of abstract Riemann integrals. A. Pełczyński gave a definition of Riemann integrable functions defined on an arbitrary measure space with values in a Banach space.

1. Preliminary results. Throughout this paper we will consider a compact (Hausdorff-) space K endowed with a non-negative Radon measure μ such that $\operatorname{supp}(\mu) = K$. Here $\operatorname{supp}(\mu) := \{x \in K \mid \mu(U) > 0$ for every open neighbourhood U of $x\}$ denotes the *support* of μ. If $B \subset K$ then $\bar{B}$ denotes the closure of B in K; instead of $\bar{B}$ we sometimes use the notation $\operatorname{cl}(B)$. $B \subset K$ is *regular closed* if the closure of the interior of B coincides with B, i.e., if $\operatorname{clint}(B) = B$. The (topological) *boundary* of B is $\partial B := \bar{B} \cap \overline{K \setminus B}$. $\mathscr{B}(K)$ denotes the Borel algebra of K, i.e., the smallest σ-field on K containing all compact subsets of K. An element $B \in \mathscr{B}(K)$ is called a *μ-continuity set* if $\mu(\partial B) = 0$. The class of all μ-continuity sets is a field, but generally not a σ-field. $\mathrm{RC}(K, \mu)$ denotes the class of all regular closed μ-continuity subsets of K. $Z(\mu) \subset \mathscr{B}(K)$ is the Boolean σ-ideal of μ-zero Borel subsets of K. The following lemma is valid especially for K and μ:

LEMMA 1. *Let X be a compact space and let ν be a non-negative Radon measure on X. Then the class of all reaular closed ν-continuity sets having a non-void interior is a neighbourhood basis system of X.*

Proof. Let D be an open neighbourhood of $x \in X$. Let $\varepsilon > 0$. Since ν is regular, there is a closed set $B \subset D$ with $\nu(D \setminus B) < \varepsilon$. We may assume $x \in B$, for otherwise we consider $B \cup \{x\}$. There is a continuous Urysohn function $g\colon X \to [0,1]$ with $g|_B \equiv 1$ and $g|_{X \setminus D} \equiv 0$. For $0 \leqslant \alpha \leqslant 1$ put $G_\alpha := \{x \in X \mid g(x) = \alpha\}$. Then $X = \bigcup_{0 \leqslant \alpha \leqslant 1} G_\alpha$. Since ν is bounded, there are at most countably many G_α with $\nu(G_\alpha) > 0$. Therefore there is a β, $0 < \beta < 1$, with $\nu(G_\beta) = 0$. We put $C := \{x \in X \mid g(x) \geqslant \beta\}$ and $CC := \operatorname{clint}(C)$. One verifies that CC is a regular closed set having a non-void interior. Moreover, $x \in \operatorname{int}(CC)$ and $CC \subset D$. Since $\partial CC \subset \partial C \subset \{x \in X \mid g(x) = \beta\}$, the set CC is a ν-continuity set. ∎

An *$\mathrm{RC}(K, \mu)$-partition* of K is a finite class $\mathscr{P} := \{P_i\}_{i=1}^n \subset \mathrm{RC}(K, \mu) \setminus \{\emptyset\}$ such that $\bigcup_{i=1}^n P_i = K$ and such that $P_i \cap P_j \subset Z(\mu)$ for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$. An $\mathrm{RC}(K, \mu)$-partition $\mathscr{P}' := \{P'_j\}_{j=1}^m$ of K is called a *refinement* of $\mathscr{P}$, or is *finer* than $\mathscr{P}$, if for every P_i there is a subset $p(i) \subset \{1, \ldots, m\}$ with $P_i = \bigcup_{j \in p(i)} P'_j$.

LEMMA 2. *Let $\mathscr{P} := \{P_i\}_{i=1}^n$ and $\mathscr{P}' := \{P'_j\}_{j=1}^m$ be two $\mathrm{RC}(K, \mu)$-partitions of K. Then $\mathscr{P} \cap \mathscr{P}' := \{\operatorname{clint}(P_i \cap P'_j) \mid 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m\} \setminus \{\emptyset\}$ is a $\mathrm{RC}(K, \mu)$-partition of K which is finer than both $\mathscr{P}$ and $\mathscr{P}'$.*

Proof. Due to its definition $\mathscr{P} \cap \mathscr{P}'$ consists of regular closed sets. Since $\partial(\operatorname{clint}(P_i \cap P'_j)) \subset \partial(P_i \cap P'_j)$, it follows that $\mathscr{P} \cap \mathscr{P}'$ consists of μ-continuity sets. Let $1 \leqslant i_1, i_2 \leqslant n$, $1 \leqslant j_1, j_2 \leqslant m$ and $(i_1, j_1) \neq (i_2, j_2)$. Then $(P_{i_1} \cap P'_{j_1}) \cap (P_{i_2} \cap P'_{j_2}) \subset Z(\mu)$. Therefore $(\operatorname{clint}(P_{i_1} \cap P'_{j_1})) \cap (\operatorname{clint}(P_{i_2} \cap P'_{j_2})) \subset Z(\mu)$. Let $x \in K$ and suppose that x is not contained in an element of $\mathscr{P} \cap \mathscr{P}'$. Then there is a closed neighbourhood $U(x)$ of x such that $P_i \cap P'_j$ is nowhere dense in $U(x)$ for every pair (i, j). This is a contradiction to the Baire theorem. Hence $\mathscr{P} \cap \mathscr{P}'$ is a $\mathrm{RC}(K, \mu)$-partition of K.

Let $1 \leqslant i \leqslant n$. We consider the elements of $\mathscr{P} \cap \mathscr{P}'$ of the following form: $\operatorname{clint}(P_i \cap P'_j)$ with $1 \leqslant j \leqslant m$. These elements cover P_i, as again follows from Baire's theorem. Hence $\mathscr{P} \cap \mathscr{P}'$ is a refinement of $\mathscr{P}$ and analogously of $\mathscr{P}'$. ∎

LEMMA 3. *Suppose that $\{A_i\}_{i=1}^n \subset \mathrm{RC}(K, \mu)$ is a finite covering of K. There is a $\mathrm{RC}(K, \mu)$-partition $\mathscr{P} := \{P_j\}_{j=1}^m$ of K such that for every $j \in \{1, \ldots, m\}$ there is an $i(j) \in \{1, \ldots, n\}$ with $P_j \subset A_{i(j)}$.*

Proof. We construct $\mathscr{P}$ by induction. Let $B_1 := A_1 \setminus (\bigcup_{i=2}^n A_i)$. If $\operatorname{int}(B_1) = \emptyset$ then $A_1 \subset \bigcup_{i=2}^n A_i$ for $K \setminus (\bigcup_{i=2}^n A_i)$ is open. Therefore, if $\operatorname{int}(B_1) = \emptyset$, then we may consider the covering $\{A_i\}_{i=2}^n$ of K and start again. Let $\operatorname{int}(B_1) \neq \emptyset$. Then $\operatorname{clint}(B_1)$ is a regular closed, non-empty μ-continuity set, because $\partial(\operatorname{int}(B_1)) \subset \bigcup_{i=1}^n \partial A_i$. Moreover, $\operatorname{clint}(B_1) \subset A_1$ and we can choose $P_1 := \operatorname{cl}\operatorname{int}(B_1)$. In the next step we consider $K_1 := K \setminus \operatorname{int}(B_1)$. If $\operatorname{int}(K_1) = \emptyset$ then $K \subset P_1$ and we are finished. If $\operatorname{int}(K_1) \neq \emptyset$ then $K_1 \in \mathrm{RC}(K, \mu)$ is compact. $\{A_i\}_{i=2}^n \subset \mathrm{RC}(K, \mu)$ is a covering of K_1. Hence we can apply the first step again which yields P_2. Repeating this procedure at most n times, we obtain $\mathscr{P}$. ∎

We denote by $S\mathscr{P}$ the set of all $\mathrm{RC}(K, \mu)$-partitions of K. An *order* $\geqslant$ on $S\mathscr{P}$ is defined as follows: $\mathscr{P}' \geqslant \mathscr{P}$ iff $\mathscr{P}'$ is finer than $\mathscr{P}$. Recall the meaning of an order: (i) $\mathscr{P} \geqslant \mathscr{P}$, (ii) $\mathscr{P}'' \geqslant \mathscr{P}'$ and $\mathscr{P}' \geqslant \mathscr{P} \Rightarrow \mathscr{P}'' \geqslant \mathscr{P}$, (iii) $\mathscr{P}' \geqslant \mathscr{P}$ and $\mathscr{P} \geqslant \mathscr{P}' \Rightarrow \mathscr{P}' = \mathscr{P}$. Due to Lemma 2 the ordered set $(S\mathscr{P}, \geqslant)$ is *reticulated*, i.e., if $\mathscr{P}_1, \ldots, \mathscr{P}_n \in S\mathscr{P}$ then there is a $\mathscr{P}' \in S\mathscr{P}$ with $\mathscr{P}' \geqslant \mathscr{P}_i$ for $i = 1, \ldots, n$.

2. The definition of a Riemann integral. In the following let L be a topological linear (Hausdorff) space over the field of real numbers $\boldsymbol{R}$ or over the field $\boldsymbol{C}$ of complex numbers. $\hat{L}$ denotes the completion of L with respect to the translation-invariant uniform structure of L. A subset $B \subset L$ is called *bounded* if for every open neighbourhood U of the zero element in L there is a constant $b(U)$ with $B \subset b(U)U$. A map $f\colon K \to L$ from the compact space K into L is called *bounded* if $f(K)$ is a bounded subset of L.

Let $f\colon K \to L$ be a bounded map. If $\mathscr{P} := \{P_i\}_{i=1}^n$ is an $\mathrm{RC}(K, \mu)$-partition of K and if $x_i \in f(P_i)$ for $i = 1, \ldots, n$ then $x := \sum_{i=1}^n \mu(P_i) x_i$ is called a *$\mathscr{P}$-sum* of f. We denote $S(f, \mathscr{P}) := \{x \in L \mid x \text{ is a } \mathscr{P}\text{-sum of } f\}$. $S(f, \mathscr{P}) \subset L$. If the indexed class $\{S(f, \mathscr{P}) \mid \mathscr{P} \in (S\mathscr{P}, \geqslant)\}$ is converging in $\hat{L}$, i.e., is finer than a Cauchy filter in L, then f is called *$\mathrm{RC}(K, \mu)$-Riemann integrable* and

$$\overset{R}{\underset{K}{\int}} f\,d\mu := \lim_{\mathscr{P} \in (S\mathscr{P}, \geqslant)} S(f, \mathscr{P})$$

is called the $\mathrm{RC}(K, \mu)$ *Riemann integral* of f over K, i.e. $y \in \hat{L}$ is the $\mathrm{RC}(K, \mu)$ Riemann integral of f over K if for every open neighbourhood U of y there is a $\mathscr{P} \in (S\mathscr{P}, \geqslant)$ such that $S(f, \mathscr{P}') \subset U$ for all $\mathscr{P}' \geqslant \mathscr{P}$. One recognizes that the above definition is a generalization of Riemann's original definition.

Since the $\mathrm{RC}(K, \mu)$ Riemann integral is a limit, all operations compatible with limits are compatible with the above integral. If $A \subset K$ then $\chi(A)$ denotes the *characteristic function* of A, i.e., $\chi(A)(x) = 1$ if $x \in A$ and $\chi(A)(x) = 0$ if $x \notin A$. Suppose that $f\colon K \to L$ is $\mathrm{RC}(K, \mu)$ Riemann integrable; then $f\chi(A)$ is $\mathrm{RC}(K, \mu)$ Riemann integrable for every $A \in \mathrm{RC}(K, \mu)$. Conversely, if $\mathscr{P} \in S\mathscr{P}$, $\mathscr{P} := \{P_i\}_{i=1}^n$ and if $f\chi(P_i)$ is $\mathrm{RC}(K, \mu)$ Riemann integrable for every i then f is $\mathrm{RC}(K, \mu)$ Riemann integrable and

$$\overset{R}{\underset{K}{\int}} f\,du = \sum_{i=1}^n \overset{R}{\underset{K}{\int}} f\chi(P_i)\,d\mu =: \sum_{i=1}^n \overset{R}{\underset{P_i}{\int}} f\,d\mu.$$

A constant function $g\colon K \to L$ defined by $x \mapsto g_0 \in L$ is $\mathrm{RC}(K, \mu)$ Riemann integrable and its $\mathrm{RC}(K, \mu)$ Riemann integral is equal to $\mu(K) g_0$. The classical result states that if X is finite-dimensional, then a bounded function f is Riemann integrable if and only if it is continuous almost everywhere. In infinite-dimensional spaces the theorem does not hold as follows from

Example 1 (C. S. Hönig). Let $A := \{a_i \mid i \in N\}$ be a dense set in the unit interval $[0, 1]$. Let $\{e_i \mid i \in N\}$ be an orthonormal system in the Hilbert space l^2. Define $f\colon [0, 1] \to l^2$ as follows: $f(a_i) = e_i$ for $i \in N$ and $f(x) = 0$ if $x \notin A$. Then f is $\mathrm{RC}([0, 1], \lambda)$ Riemann integrable but nowhere continuous. Here λ denotes the Lebesgue measure. Using a non-separable Hilbert space and a non-Lebesgue measurable set in $[0, 1]$ one obtains analogously an $\mathrm{RC}([0, 1], \lambda)$ Riemann integrable function which is not measurable with respect to the complete Lebesgue measure.

Obviously, the class of $\mathrm{RC}(K, \mu)$ Riemann integrable functions depends on the geometry of the topological linear space considered. We study in the following the translation of the classical results.

3. Darboux integrability in Banach spaces. We consider a Banach space B with the norm $\| \;\|$ over the field $\boldsymbol{R}$ or $\boldsymbol{C}$. If $M \subset B$ then $\mathrm{dia}(M) := \sup\{\|x-y\| \mid x, y \in M\}$ denotes the *diameter* of M. M is bounded iff $\mathrm{dia}(M) < \infty$. Suppose that $f\colon K \to B$ is a bounded map, i.e., that $\mathrm{dia}(f(K)) < \infty$. Let $\mathscr{P} := \{P_i\}_{i=1}^n$ an $\mathrm{RC}(K, \mu)$-partition of K. Then $\mathrm{dia}(S(f, \mathscr{P}))$ is called the *μ-diameter* of f with respect to $\mathscr{P}$.

If $\mathscr{P}'$ is a $\mathrm{RC}(K, \mu)$-partition of K finer than $\mathscr{P}$, then $\mathrm{dia}(S(f, \mathscr{P})) \geqslant \mathrm{dia}(S(f, \mathscr{P}'))$. Indeed, for $0 \leqslant \lambda \leqslant 1$ we obtain $\|(\lambda x + (1-\lambda) y) - (\lambda v + (1-\lambda) w)\| \leqslant \max\{\|x - v\|, \|y - w\|\}$. We claim that f is $\mathrm{RC}(K, \mu)$ Riemann integrable iff there is a sequence $\{\mathscr{P}_i\}_{i \in N}$ of $\mathrm{RC}(K, \mu)$-partitions with $\mathrm{dia}(S(f, \mathscr{P}_i)) \to 0$ for $i \to \infty$. Here, the 'if' part follows, because $\mathrm{dia}(\text{convex hull of } S(f, \mathscr{P})) = \mathrm{dia}(S(f, \mathscr{P}))$ and because $S(f, \mathscr{P}') \subset \text{convex hull of } S(f, \mathscr{P})$ if $\mathscr{P}' \geqslant \mathscr{P}$. Remark that the inequality $\mathscr{P}_{i+1} \geqslant \mathscr{P}_i$ can be assumed to hold without loss of generality for all i.

We call $\mathrm{dis}(f, \mathscr{P}) := \sum_{i=1}^n \mathrm{dia}(f(P_i))\mu(P_i)$ the *μ-distance sum* of f with respect to $\mathscr{P}$. We obtain $0 \leqslant \mathrm{dis}(f, \mathscr{P}') \leqslant \mathrm{dis}(f, \mathscr{P}) < \infty$ with $\mathscr{P}' \geqslant \mathscr{P}$. f is called $\mathrm{RC}(K, \mu)$ *Darboux integrable* if $\inf\{\mathrm{dis}(f, \mathscr{P}) \mid \mathscr{P} \in S\mathscr{P}\} = 0$.

The $\mathrm{RC}(K, \mu)$ Darboux integrability implies the $\mathrm{RC}(K, \mu)$ Riemann integrability. If B is finite-dimensional then the converse is true due to Riemann. It follows from Example 1 that in the infinite-dimensional case the inverse implication need not be true. The definition of the $\mathrm{RC}(K, \mu)$ Darboux integrability is suitable for the translation of classical results:

Proposition 4. *Let K be a compact space and let μ be a non-negative Radon measure with $\mathrm{supp}(\mu) = K$. Let B be a Banach space and let $f\colon K \to B$ be a bounded map.*

(a) *If f is $\mathrm{RC}(K, \mu)$ Darboux integrable then f is continuous at μ-almost every point of K (i.e., the set of discontinuity points of f is contained in a μ-zero set).*

(b) *If f is continuous then f is $\mathrm{RC}(K, \mu)$ Darboux integrable.*

(c) *If K is a metrizable and if f is continuous at μ-almost every point of K then f is $\mathrm{RC}(K, \mu)$ Darboux integrable.*

(d) *If f is $\mathrm{RC}(K, \mu)$ Darboux integrable, then f is $\bar{\mu}$ Bochner integrable and the two integrals coincide. Here $\bar{\mu}$ denotes the completion of μ.*

Proof. (a) Suppose that f is $\mathrm{RC}(K, \mu)$ Darboux integrable. Then for each positive integer n we can choose an $\mathrm{RC}(K, \mu)$-partition $\mathscr{P}_n$ such that $\mathrm{dis}(f, \mathscr{P}_n) < 1/n$. Replacing these partitions, due to Lemma 2, with finer

RC(K, μ)-partitions if necessary we can assume that for each n the RC(K, μ)-partition $\mathscr{P}_{n+1}$ is a refinement of the RC(K, μ)-partition $\mathscr{P}_n$. Let $\mathscr{P}_n := \{P_i(n) \mid 1 \leqslant i \leqslant p(n)\}$ and let $\mathscr{P}_{n+1} := \{P_i(n+1) \mid 1 \leqslant i \leqslant p(n+1)\}$ where $p(n)$ and $p(n+1)$ are positive integers. We can suppose that there are indices $\{p(n, i) \mid 1 \leqslant i \leqslant p(n)\}$ with $p(n, i) < p(n, i+1)$ for all $i \in \{1, \ldots \ldots, p(n)-1\}$ such that $P_i(n) = \bigcup \{P_j(n+1) \mid p(n, i) \leqslant j < p(n, i+1)\}$ for all $i \in \{1, \ldots, p(n)\}$. Starting the above construction of a numeration with $n = 1$, we can suppose that the equations are valid for each n.

For each n we define a real valued function $d_n\colon K \to \boldsymbol{R}$ as follows: If $1 \leqslant i \leqslant p(n)$ and $x \in P_i(n) \setminus \bigcup_{1 \leqslant j < i} P_j(n)$ then $d_n(x) := \operatorname{dia}\big(f(P_i(n))\big)$. Then d_n is a non-increasing sequence of non-negative simple Borel functions. Moreover, $\int_K d_n \, d\mu = \operatorname{dis}(f, \mathscr{P}_n)$.

Since $\{d_n\}_{n \in N}$ is a non-increasing sequence of non-negative real valued functions, we can define $d := \lim_n d_n$ pointwise. d is Borel measurable. Lebesgue's bounded convergence theorem implies that $\int_K d \, d\mu = \lim_n \operatorname{dis}(f, \mathscr{P}_n) = 0$. Since $d \geqslant 0$, it follows that there is a μ-zero set E with

$$(+) \qquad d(x) = 0 \quad \text{for all } x \in K \setminus E.$$

Let $\partial \mathscr{P}_n := \bigcup \{\partial P_i(n) \mid 1 \leqslant i \leqslant p(n)\}$ the union of the boundaries of the $\mathscr{P}_n$ sets. Since $\partial \mathscr{P}_n$ is a μ-zero set, it follows that $\bigcup_{n \in N} \partial \mathscr{P}_n$ is a μ-zero set. Let $x \in K$ and $x \notin E \cup \big(\bigcup_{n \in N} \partial \mathscr{P}_n\big)$. We recognize that f is continuous at x due to the definition of d_n. Indeed, if $\varepsilon > 0$ then there is an $n(\varepsilon) \in N$ such that $d_n(x) < \varepsilon$ for $n \geqslant n(\varepsilon)$ because of $(+)$. Since x is not a boundary point of $\mathscr{P}_{n(\varepsilon)}$, there is a $P_i(n(\varepsilon))$ with $x \in \operatorname{int}\big(P_i(n(\varepsilon))\big)$. As $\operatorname{dia}\big(f(P_i)\big) = d_{n(\varepsilon)}(x)$ it follows that $\|f(y) - f(z)\| < \varepsilon$ for all $y, z \in \operatorname{int}\big(P_i(n(\varepsilon))\big)$. Hence f is continuous at x and part (a) is proved.

We prove (b). f is supposed to be continuous on K. This implies that f is uniformly continuous on K. Let $\mathfrak{U} := \{U \mid U \in \mathfrak{U}\}$ be a neighbourhood basis of the uniformity of K. If $x \in K$ then there is a neighbourhood basis $\mathfrak{N}(x) := \{N(x, U) \mid U \in \mathfrak{U}\}$ of x fulfilling $N(x, U) \times N(x, U) \subset U$. On the analogy of Lemma 1 we can suppose that $\mathfrak{N}(x) \subset \mathrm{RC}(K, \mu)$. Since K is compact there are finitely many elements $\{x_i \mid 1 \leqslant i \leqslant n(U)\} \subset K$ such that $\{N(x_i, U) \mid 1 \leqslant i \leqslant n(U)\}$ is a covering of K. It follows from Lemma 3 that there is a RC(K, μ)-partition $\mathscr{P}_U := \{P_i(U) \mid 1 \leqslant i \leqslant m(U)\}$ such that each $P_i(U)$ fulfils $P_i(U) \times P_i(U) \subset U$.

If $n \in N$ then there is a $U(n) \in \mathfrak{U}$ with $\operatorname{dia}\Big(f\big(N(x, U(n))\big)\Big) < 1/n$. On the analogy of part (a), the system of RC(K, μ)-partitions $\{\mathscr{P}_{U(n)} \mid n \in N\}$ yields a system of RC(K, μ)-partitions $\{\mathscr{P}'_{U(n)} \mid n \in N\}$ with the following properties: (i) $\mathscr{P}'_{U(n)}$ is finer than $\mathscr{P}_{U(n)}$ and (ii) $\mathscr{P}'_{U(n+1)}$ is a refinement of $\mathscr{P}'_{U(n)}$ for every $n \in N$. Due to our construction we obtain $\operatorname{dis}(f, \mathscr{P}'_{U(n)}) \leqslant \mu(K)/n$. Hence f is RC(K, μ) Darboux integrable and (b) is proved.

We prove part (c). K is a compact metric space. Therefore there is a countable basis $\mathfrak{U} := \{U(n) \mid n \in N\}$ of the uniformity of K. As above we obtain a sequence $\mathscr{P}_n := \{P_i(n) \mid 1 \leqslant i \leqslant p(n)\}$ of RC(K, μ)-partitions of K such that (i) $\mathscr{P}_{n+1}$ is a refinement of $\mathscr{P}_n$ for all n and (ii) $P_i(n) \times P_i(n) \subset U(n)$ for all n and i. Let f be continuous at μ-almost every point of K. Define $d_n\colon K \to \boldsymbol{R}$ just as before. If f is continuous at $x \in K \setminus \bigcup_{n \in N} \partial \mathscr{P}_n$ then $\lim_n d_n(x) = 0$. Hence $\lim_n d_n = 0$ holds at μ-almost every point of K. Since $\int_K d_n \, d\mu = \operatorname{dis}(f, \mathscr{P}_n)$, Lebesgue's bounded convergence theorem implies that $\lim_n \operatorname{dis}(f, \mathscr{P}_n) = \lim_n \int_K d_n \, d\mu = 0$. It follows that f is RC(K, μ) Darboux integrable.

We prove (d). Since f is RC(K, μ) Darboux integrable, we can define d_n, d and $\mathscr{P}_n$ as in part (a). If $n \in N$ we define a function $f_n\colon K \to B$ as follows: Let $1 \leqslant i \leqslant p(n)$ and $y_i \in f\big(P_i(n) \setminus \bigcup_{1 \leqslant j < i} P_j(n)\big)$. If $x \in P_i(n) \setminus \bigcup_{1 \leqslant j < i} P_j(n)$ then $f_n(x) := y_i$. f_n is a simple Borel function. Hence f_n is totally measurable (cf. N. Dunford and J. T. Schwartz [1] for the definitions). Since $d_n \to 0$ μ-almost everywhere, we obtain $f_n \to f$ μ-almost everywhere. Therefore f is $\bar{\mu}$-measurable (cf. [1] Corollary III 6.14).

As f is bounded, there is a constant g with $\|f_n(x)\| \leqslant g$ for all $x \in K$. Due to the bounded convergence theorem, f is Bochner integrable and $\int_K f \, d\bar{\mu} = \lim_n \int_K f_n \, d\bar{\mu}$. Since $\int_K f_n \, d\bar{\mu} \in S(f, \mathscr{P}_n)$, it follows that $\int_K f \, d\bar{\mu} = \overset{R}{\int_K} f \, d\mu$. ∎

We mention some points: (1) It is well known that a function $f\colon [0, 1] \to \boldsymbol{R}$ which is Riemann integrable in the classical sense need not be Lebesgue–Borel measurable. (2) We proved part (c) for the metric case only, because the bounded convergence theorem is valid for Moore–Smith sequences in a slightly different version (cf. N. Dunford and J. T. Schwartz [1], III 3.7).

We finish this section with two canonical generalizations of the above proposition: (3) If F is a locally convex Fréchet space then the results remain true. Indeed, since we can consider the associated semi-normed spaces and since the countable union of μ-zero sets is a μ-zero set, the proof can be applied too. (4) Let σK be a locally compact space and let $\sigma\mu$ be a non-negative Radon measure defined on σK with supp $(\sigma\mu) = \sigma K$. If $\mathfrak{N}(\infty)$ is the neighbourhood basis of infinity consisting of all regular open $\sigma\mu$-continuity sets, then the classical definition of the Riemann integral on $\boldsymbol{R}$ can be applied analogously with respect to $\mathfrak{N}(\infty)$.

4. An application to uniformly distributed sequences. The classical Riemann integral is suitable for the approximation by average means with respect to uniformly distributed sequences (cf. L. Kuipers and H. Niederreiter [6] Chapter 1, Corollary 1.1). We want to state an analogous result for the generalized situation. For this purpose we consider in this part 4 a non-negative, *normalized* Radon measure μ_1 defined on the compact space K with $\operatorname{supp}(\mu_1) = K$. This means that we make the additional assumption that $\mu_1(K) = 1$.

A sequence $\{x_i\}_{i\in N} \subset K$ is called *μ_1-uniformly distributed* in K if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i) = \int_K f\,d\mu_1 \tag{4.1}$$

holds for every continuous real valued function f on K. It is well known that $\{x_i\}_{i\in N}$ is μ_1-uniformly distributed in K iff (4.1) holds for every characteristic function $\chi(A)$ where A is a μ_1-continuity set (cf. [6] Chapter 3, Theorem 1.2). If K is metrizable, then a μ_1-uniformly distributed sequence exists (cf. [6], Chapter 3, Lemma 2.1).

PROPOSITION 5. *Let K be a compact space and let μ_1 be a non-negative and normalized Radon measure on K with $\operatorname{supp}(\mu_1) = K$. Suppose that B is a Banach space with $\dim B \geq 1$. A sequence $\{x_i\}_{i\in N} \subset K$ is μ_1-uniformly distributed iff for every* $\mathrm{RC}(K, \mu_1)$ *Darboux integrable function f: $K \to B$ the following holds*:

$$\lim_n (1/n) \sum_{i=1}^{n} f(x_i) = \overset{R}{\int_K} f\,d\mu_1. \tag{4.1'}$$

Proof. Since every continuous function f is $\mathrm{RC}(K, \mu_1)$ Darboux integrable, the 'if' part follows for $\dim B \geq 1$.

We suppose now that $\{x_i\}_{i\in N}$ is μ_1-uniformly distributed in K and that f is $\mathrm{RC}(K, \mu_1)$ Darboux integrable. We need some preparations before we can prove that (4.1′) holds for f. Observe that $\inf\{\operatorname{dis}(f, \mathscr{P}) \mid \mathscr{P} \in S\mathscr{P}\} = 0$. If $A \subset K$ then $\operatorname{dia}(f(A)) = \operatorname{dia}(\text{convex hull of } f(A))$. Hence $\inf\{\sum_{i=1}^{n} \operatorname{dia}(\text{convex hull of } f(P_i))\, \mu_1(P_i) \mid \mathscr{P} = \{P_i\}_{i=1}^{n} \in S\mathscr{P}\} = 0$. If $\mathscr{P} = \{P_i\}_{i=1}^{n} \in S\mathscr{P}$, then we write

$$\operatorname{Scon}(f, \mathscr{P}) := \{\sum_{i=1}^{n} x_i \mu_1(P_i) \mid x_i \in \text{convex hull of } f(P_i)\}.$$

Hence

$$\overset{R}{\int_K} f\,d\mu_1 = \lim_{\mathscr{P}\in(S\mathscr{P},\geq)} \operatorname{Scon}(f, \mathscr{P}).$$

Moreover, if $x \in \operatorname{Scon}(f, \mathscr{P})$ and if $\operatorname{dis}(f, \mathscr{P}) < \varepsilon/2$, it follows that

$$\Big\|\overset{R}{\int_K} f\,d\mu_1 - x\Big\| < \varepsilon/2. \tag{4.2}$$

Lastly, since f is bounded, there is a constant $c(f) > 0$ with $\|f(x)\| < c(f)$ for all $x \in K$.

Let $\varepsilon \in \boldsymbol{R}$, $\varepsilon > 0$. Then there is an $\mathrm{RC}(K, \mu_1)$-partition $\mathscr{P}(\varepsilon) := \{P_i\}_{i=1}^{n}$ of K with $\operatorname{dis}(f, \mathscr{P}') < \varepsilon/2$ for all $\mathscr{P}' \geq \mathscr{P}(\varepsilon)$. We define $\mathscr{P}^*(\varepsilon) := \{P_i^*\}_{i=1}^{n}$ as follows:

$$P_i^* := P_i \setminus \Big(\bigcup_{k=1}^{i-1} P_k\Big).$$

$\mathscr{P}^*(\varepsilon)$ is a disjoint decomposition of K consisting of μ_1-continuity sets. Hence equation (4.1) holds for the characteristic functions $\chi(P_i^*)$ with $i = 1, \ldots, n$. Therefore we can choose an $m(\varepsilon) \in N$ such that

$$\Big|1 - m^{-1} \mu_1(P_i^*)^{-1} \sum_{j=1}^{m} \chi(P_i^*)(x_j)\Big| < \varepsilon/(2c(f)) \tag{4.3}$$

for $m \geq m(\varepsilon)$ and $i = 1, \ldots, n$.

Due to (4.3) there are elements $f(i, m)$ from the convex hull of $f(P_i^*)$ with

$$\Big\|f(i, m) - m^{-1} \mu_1(P_i^*)^{-1} \sum_{j=1}^{m} f(x_j)\chi(P_i^*)(x_j)\Big\| < c(f)\,\varepsilon/(2c(f)) = \varepsilon/2 \tag{4.4}$$

where $m \geq m(\varepsilon)$ and $i \in \{1, \ldots, n\}$. Since $\operatorname{dis}(f, \mathscr{P}(\varepsilon)) < \varepsilon/2$, it follows that

$$\begin{aligned}\Big\|\overset{R}{\int_K} f\,d\mu_1 - \frac{1}{m}\sum_{j=1}^{m} f(x_j)\Big\| &\leq \Big\|\overset{R}{\int_K} f\,d\mu_1 - \sum_{i=1}^{n} f(i, m)\,\mu_1(P_i)\Big\| + \\ &+ \Big\|\sum_{i=1}^{n} f(i, m)\,\mu_1(P_i^*) - \frac{1}{m}\sum_{j=1}^{m} f(x_j)\Big\| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon\end{aligned} \tag{4.5}$$

where $m \geq m(\varepsilon)$. Consequently

$$\Big\|\overset{R}{\int_K} f\,d\mu_1 - \lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} f(x_i)\Big\| \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrarily chosen, the proposition follows. ∎

On the analogy of Section 3, Proposition 5 can be formulated for metric locally convex spaces.

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Closed subgroups of nuclear spaces are weakly closed

by

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Abstract. A proof is given that a closed additive subgroup of a nuclear space is weakly closed. This generalizes the result obtained in [1].

It has been proved in [1] that if K is a discrete additive subgroup of a nuclear space E, then the quotient group E/K admits sufficiently many continuous characters, which means precisely that K is weakly closed in E. It appears, however, that it suffices to assume K to be closed. This result admits two equivalent formulations.

THEOREM A. *A closed additive subgroup of a nuclear space is weakly closed.*

THEOREM B. *If K is a closed additive subgroup of a nuclear space E, then the quotient group E/K admits sufficiently many continuous characters.*

We shall prove Theorem A. For the equivalence of A and B see Lemma 8 below. These theorems provide another illustration of the fact that nuclear spaces are more closely related to finite dimensional spaces than normed spaces are, since, as it has been proved in [2], they do not hold in any infinite dimensional normed space (see also Corollary 3 below). In fact, these theorems characterize nuclear spaces; more precisely, if they hold in a B_0^*-space E, then E is nuclear. The proof will be given elsewhere.

Let A be a subset of a topological vector space E. The symbols $\bar{A}$, $\bar{A}^w$, $\operatorname{span} A$ and $\operatorname{int} A$ will denote respectively the closure, the weak closure, the linear span and the interior of A. If E is a metric space, then $\operatorname{diam} A$ will denote the diameter of A, and $d(u, A)$ the distance of a point $u \in E$ to A. By $\operatorname{gp} A$ we shall denote the additive subgroup of E generated by A. Speaking of subgroups of vector spaces we shall omit the word "additive".

If E is a unitary space, then the scalar product of vectors $u, w \in E$ will be denoted by (u, w). By an *ellipsoid* in E we shall always mean an ellipsoid which is closed and convex. If T is a linear operator acting between normed spaces, then $d_n(T)$, $n = 1, 2, \ldots$, will denote the nth Kolmogorov number of T.

We shall obtain Theorem A as an easy consequence of the following proposition.