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On the ratio maximal function for an ergodic flow

by

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Abstract. In this paper an integrability problem is investigated for the supremum of ergodic ratios defined by means of a conservative and ergodic measurable flow of measure preserving transformations on a σ-finite measure space. The results obtained below include, as a special case, the continuous parameter versions of Davis's recent results concerning the supremum of ergodic averages defined by means of an invertible and ergodic measure preserving transformation on a probability measure space.

1. Introduction. Let $(\Omega, \mathfrak{F}, \mu)$ be a σ-finite measure space and $\{T_t\}_{t\in R}$ a conservative and ergodic measurable flow of measure preserving transformations on $(\Omega, \mathfrak{F}, \mu)$. In what follows we shall assume that μ is nonatomic and complete. As is easily seen, this is done without loss of generality.

Fix any $0 < e \in L_1(\mu)$ such that $\int e\,d\mu = 1$. If $f \in L_1(\mu)$, the ratio maximal function $M_e(f)(\omega)$ with respect to e is defined by

$$M_e(f)(\omega) = \sup_{b>0} \Big|\int_0^b f(T_t\omega)\,dt \Big/ \int_0^b e(T_t\omega)\,dt\Big| \qquad (\omega\in\Omega).$$

Let $\hat{f}_e$ denote the decreasing function on the interval $[0, 1)$ which is equidistributed with f/e ($\in L_1(e\,d\mu)$) with respect to the measure $e\,d\mu$. Extending $\hat{f}_e$ to the real line R by $\hat{f}_e(t+1) = \hat{f}_e(t)$ for $t \in R$, we define

$$H_e(f) = \int_0^{1/2} \frac{1}{t}\Big|\int_{-t}^{t} \hat{f}_e(s)\,ds\Big|\,dt.$$

Clearly, $H_e(f) = 0$ if and only if f/e and $-f/e$ are equidistributed with respect to $e\,d\mu$. Further it is known (cf. [8]) that if $f \geqslant 0$ then $H_e(f) < \infty$ if and only if $\int f \log^+(f/e)\,d\mu < \infty$, where $\log^+ a = \log(\max\{a, 1\})$ for $a \geqslant 0$. This, together with Theorem 2 in [8], shows that if $f \geqslant 0$ then $H_e(f) < \infty$ if and only if $\int M_e(f)\cdot e\,d\mu < \infty$. However, if the nonnegativity of f is not assumed, then, as is easily seen by a simple example, $H_e(f) < \infty$ does not necessarily imply $\int M_e(f)\cdot e\,d\mu < \infty$. (It will be proved below that $\int M_e(f)\cdot e\,d\mu < \infty$ implies $H_e(f) < \infty$.) Therefore it would be of interest to know what condition on the ratio maximal function with respect to e is necessary (and sufficient) for the condition $H_e(f) < \infty$. This is the starting point for the study in this paper. We shall show that $H_e(f) < \infty$ if and only if there exists an $f' \in L_1(\mu)$ such that f'/e and f/e are equidistributed with respect to $e\,d\mu$ and $\int M_e(f')\cdot e\,d\mu < \infty$. When $(\Omega, \mathfrak{F}, \mu)$ is a probability space and $e = 1$ on Ω, this characterization reduces to the one which corresponds to the continuous parameter version of Davis's result [4] for an invertible and ergodic measure preserving transformation on a probability space.

2. Results. In this section we will prove our results using the ideas in [4] adapted to our situation.

THEOREM 1 (cf. [9]). *Let $(\Omega, \mathfrak{F}, \mu)$, $\{T_t\}_{t\in R}$ and $e \in L_1(\mu)$ be as in Introduction. There exists an absolute constant $c > 0$ such that*

$$\int M_e(f)\cdot e\,d\mu \geqslant cH_e(f) \quad \textit{for all } f \in L_1(\mu).$$

To prove Theorem 1 we need some lemmas. For convenience we will write $\mu_e = e\,d\mu$. The letters c and C will denote positive absolute constants; the same letters do not necessarily denote the same numbers.

LEMMA 1. *Let T be a conservative measure preserving transformation on $(\Omega, \mathfrak{F}, \mu)$ and $f \in L_1(\mu)$. Define, for $\omega \in \Omega$,*

$$M(T, e)f(\omega) = \sup_{n\geqslant 1}\Big|\sum_{i=0}^{n-1} f(T^i\omega)\Big|\Big/\sum_{i=0}^{n-1} e(T^i\omega).$$

Given an $\alpha > 0$, let $A = \{M(T, e)f \geqslant \alpha\}$. Then we have

$$\int_{[M(T,e)f<\alpha]\cap(A\cup T^{-1}A)} f\,d\mu \leqslant \alpha\mu_e([M(T, e)f < \alpha] \cap (A\cup T^{-1}A)),$$

where $[M(T, e)f < \alpha]$ denotes the smallest set in $\mathfrak{F}$ containing the set $\{M(T, e)f < \alpha\}$ and invariant under T.

Proof. Clearly, it suffices to consider the case where $\mu(\{M(T, e)f < \alpha\}) > 0$. Let us write $B = [M(T, e)f < \alpha]$. For simplicity, assume that $B = \Omega$. This is done without loss of generality. Since T is conservative, it then follows from [5] that

$$\int_A f\,d\mu = \sum_{n=1}^{\infty} \int_{(\Omega-A)\cap(\bigcap_{i=1}^{n} T^{-i}A)\cap T^{-(n+1)}(\Omega-A)} \Big(\sum_{i=1}^{n} T^i f\Big)d\mu,$$

where $T^i f(\omega) = f(T^i\omega)$. Since $T^{-1}A$ is the disjoint union of the sets $(\bigcap_{i=1}^{n} T^{-i}A)\cap T^{-(n+1)}(\Omega-A)$, $n \geqslant 1$, we have

$$\int_{A\cup T^{-1}A} f\,d\mu = \int_A f\,d\mu + \int_{(\Omega-A)\cap T^{-1}A} f\,d\mu$$
$$= \sum_{n=1}^{\infty} \int_{(\Omega-A)\cap(\bigcap_{i=1}^{n} T^{-i}A)\cap T^{-(n+1)}(\Omega-A)} \Big(\sum_{i=0}^{n} T^i f\Big)d\mu$$
$$\leqslant \sum_{n=1}^{\infty} \int_{(\Omega-A)\cap(\bigcap_{i=1}^{n} T^{-i}A)\cap T^{-(n+1)}(\Omega-A)} \alpha\Big(\sum_{i=0}^{n} T^i e\Big)d\mu$$
$$= \alpha \int_{A\cup T^{-1}A} e\,d\mu,$$

which completes the proof.

LEMMA 2. *Given an $\alpha > 0$, let $A = \{M_e(f) \geqslant \alpha\}$. If $\mu(\Omega - A) > 0$ then $\int_A f\,d\mu \leqslant \alpha\mu_e(A)$.*

Proof. By virtue of Lemma 1 this follows from an easy modification of the proof of Theorem 1 in [8]. We omit the details.

Now suppose that $f \in L_1(\mu)$ satisfies $\int f\,d\mu = 0$. If $\lambda > 0$, let $\theta_e(f, \lambda) = \sup\{t \in [0, 1)\colon \hat{f}_e(t) \geqslant \lambda\}$. Then $0 \leqslant \theta_e(f, \lambda) < 1$; and there exists a unique number $\varphi_e(f, \lambda)$ in $(-1, 0]$ such that

$$\int_{\varphi_e(f,\lambda)}^{\theta_e(f,\lambda)} \hat{f}_e(t)\,dt = \lambda[\theta_e(f, \lambda) - \varphi_e(f, \lambda)].$$

Define $\psi_e(f, \lambda) = \theta_e(f, \lambda) - \varphi_e(f, \lambda)$ for $\lambda > 0$.

LEMMA 3. *Suppose that $f \in L_1(\mu)$ satisfies $\int f\,d\mu = 0$. Then we have*

$$\mu_e(\{M_e(f) \geqslant \lambda\}) \geqslant \psi_e(f, \lambda) \quad (\lambda > 0).$$

Proof. Put $A = \{M_e(f) \geqslant \lambda\}$. Since $\psi_e(f, \lambda) \leqslant 1$, it suffices to consider the case where $\mu_e(A) < 1$. Then Lemma 2 implies

$$\int_A f\,d\mu \leqslant \lambda\mu_e(A).$$

On the other hand, since $f/e \leqslant M_e(f)$ on Ω by the local ergodic theorem (see e.g. [10]), it follows that $\{f/e \geqslant \lambda\} \subset A$, from which the lemma follows immediately.

For any $f \in L_1(\mu)$, define

$$A_e(f) = \sum_{n=0}^{\infty} \Big|\int_{-b_n}^{b_n} \hat{f}_e(t)\,dt\Big| \quad \text{where} \quad b_n = 2^{-(n+1)}.$$

LEMMA 4 (Davis [4]). *There exists an absolute constant $c > 0$ such that if $f \in L_1(\mu)$ satisfies $\int f\,d\mu = 0$, then*

$$\int_0^{\infty} [\psi_e(f, \lambda) + \psi_e(-f, \lambda)]\,d\lambda \geqslant cA_e(f).$$

Proof. See [4], pp. 157–158.

Proof of Theorem 1. Let $f\in L_1(\mu)$ and $\alpha=\int f\,d\mu$. Putting $g=f-\alpha e$, we have $f=g+\alpha e$ with $\int g\,d\mu=0$. Since for a.e. $\omega\in\Omega$

$$\alpha=\lim_{b\to\infty}\int_0^b f(T_t\omega)\,dt\Big/\int_0^b e(T_t\omega)\,dt$$

by the ratio ergodic theorem (see e.g. [7]), it follows that

$$|\alpha|\leqslant M_e(f)\quad\text{and}\quad M_e(g)\leqslant 2M_e(f)\quad\text{on } \Omega.$$

Since $\hat f_e=\hat g_e+\alpha$,

$$A_e(f)\leqslant A_e(g)+\sum_{n=0}^{\infty}\int_{-b_n}^{b_n}|\alpha|\,dt\leqslant C\left[A_e(g)+|\alpha|\right]\leqslant C\left[A_e(g)+\int M_e(f)\cdot e\,d\mu\right].$$

Here, $A_e(f)\geqslant cH_e(f)-C\int|f|\,d\mu$ by Lemma 2.1 in [3]. To estimate $A_e(g)$, we apply Lemma 3 and get

$$\int M_e(g)\cdot e\,d\mu=\int_0^\infty \mu_e(\{M_e(g)\geqslant\lambda\})\,d\lambda\geqslant\int_0^\infty\psi_e(g,\lambda)\,d\lambda.$$

Replacing g by $-g$, we also get

$$\int M_e(g)\cdot e\,d\mu\geqslant\int_0^\infty\psi_e(-g,\lambda)\,d\lambda.$$

Therefore

$$2\int M_e(f)\cdot e\,d\mu\geqslant\int M_e(g)\cdot e\,d\mu\geqslant\tfrac12\int_0^\infty[\psi_e(g,\lambda)+\psi_e(-g,\lambda)]\,d\lambda\geqslant cA_e(g),$$

and this completes the proof of Theorem 1.

For any f and g in $L_1(\mu)$, we will denote $f\overset{e}{\sim}g$ if $\hat f_e=\hat g_e$, i.e. if f/e and g/e are equidistributed with respect to the measure $e\,d\mu$. With this understanding we have the following

THEOREM 2. *Let $(\Omega,\mathfrak{F},\mu)$, $\{T_t\}_{t\in R}$ and $e\in L_1(\mu)$ be as in Introduction. There exists an absolute constant $C>0$ such that to each $f\in L_1(\mu)$ there corresponds an $f'\in L_1(\mu)$ such that $f'\overset{e}{\sim}f$ and*

$$\int M_e(f')\cdot e\,d\mu\leqslant C\left[H_e(f)+\int|f|\,d\mu\right].$$

To prove Theorem 2 we need a representation theorem for the conservative and ergodic flow.

Let $(X,\mathfrak{B},m)$ be a σ-finite complete measure space, T an invertible measure preserving transformation on $(X,\mathfrak{B},m)$, and h a positive real valued measurable function on $(X,\mathfrak{B},m)$ such that $h(x)\geqslant d$ for all $x\in X$, where $d>0$ is a constant. Let $\bar X=\{(x,u)\colon x\in X,\ 0\leqslant u<h(x)\}$. Considering the restriction of the completed product measure of μ and the Lebesgue measure on the real line to $\bar X$, we have a σ-finite complete measure space $(\bar X,\bar{\mathfrak{B}},\bar m)$. Define a family $\{S_t\}_{t\in R}$ of transformations from $\bar X$ onto itself by

$$\begin{aligned}S_t(x,u)&=(x,u+t),\qquad\text{if}\quad 0\leqslant u+t<h(x),\\&=\Big(T^nx,\ u+t-\sum_{i=0}^{n-1}h(T^ix)\Big),\\&\qquad\qquad\text{if}\quad\sum_{i=0}^{n-1}h(T^ix)\leqslant u+t<\sum_{i=0}^{n}h(T^ix),\quad n\geqslant1,\\&=\Big(T^{-n}x,\ u+t+\sum_{i=1}^{n}h(T^{-i}x)\Big),\\&\qquad\qquad\text{if}\quad-\sum_{i=1}^{n}h(T^{-i}x)\leqslant u+t<-\sum_{i=1}^{n-1}h(T^{-i}x),\quad n\geqslant1.\end{aligned}$$

It is easily seen that $\{S_t\}_{t\in R}$ is a measurable flow of measure preserving transformations on $(\bar X,\bar{\mathfrak{B}},\bar m)$. This flow is called the *flow built under the function h on the measure preserving transformation T* (cf. Ambrose [1]). $(X,\mathfrak{B},m)$ is a *base measure space*, T is a *base transformation*, and h is called a *ceiling function.*

THEOREM A. *Let $(\Omega,\mathfrak{F},\mu)$ and $\{T_t\}_{t\in R}$ be as in Introduction. There exists a σ-finite complete measure space $(X,\mathfrak{B},m)$, an invertible conservative and ergodic measure preserving transformation T on $(X,\mathfrak{B},m)$, and a positive real valued measurable function h on $(X,\mathfrak{B},m)$ satisfying $h(x)\geqslant d$ for all $x\in X$ and some constant $d>0$ such that $\{T_t\}_{t\in R}$ is isomorphic to the flow $\{S_t\}_{t\in R}$ on $(\bar X,\bar{\mathfrak{B}},\bar m)$ built under the function h on the measure preserving transformation T.*

Proof. If $\mu(X)<\infty$, this theorem reduces to Theorem 2 in [1]. Thus we consider the case where $\mu(X)=\infty$. Since the argument resembles Ambrose's in [1], we only sketch the proof.

Fix any $A\in\mathfrak{F}$ with $0<\mu(A)<\infty$. Denoting by 1_A the indicator function of A, it follows from the local ergodic theorem that

$$\lim_{\varepsilon\to+0}\frac1\varepsilon\int_0^\varepsilon 1_A(T_t\omega)\,dt=1_A(\omega)$$

for a.e. $\omega\in\Omega$; thus we can choose a positive number p so that if the function $P(\omega)$ on Ω is defined by

$$P(\omega)=\frac1p\int_0^p 1_A(T_t\omega)\,dt,$$

then for some $A_1, A_2 \in \mathfrak{F}$, with $0 < \mu(A_i) < \infty$ $(i = 1, 2)$, $A_1 \subset A$ and $A_2 \subset \Omega - A$, we get

$$A_1 \subset \{P > 3/4\} \quad \text{and} \quad A_2 \subset \{P < 1/4\}.$$

Let Q denote the rational numbers. Since $\{T_t\}_{t\in R}$ is conservative, each T_t is also conservative, and hence there exists a set B_1 in $\mathfrak{F}$ satisfying $B_1 \subset A_1$, $\mu(A_1 - B_1) = 0$ and

$$\sum_{n=0}^{\infty} 1_{A_1}(T_r^n \omega) = \infty$$

for every $\omega \in B_1$ and all $r \in Q$. Denote $\hat{B}_1 = \bigcup_{r\in Q} T_r B_1$. Since $T_r \hat{B}_1 = \hat{B}_1$ for all $r \in Q$, an easy approximation argument shows that $\mu(\hat{B}_1 \triangle T_t \hat{B}_1) = 0$ for all $t \in R$, where the symbol $\triangle$ denotes the symmetric difference. This implies $\mu(\Omega - \hat{B}_1) = 0$, because $\{T_t\}_{t\in R}$ is ergodic. Therefore to a.e. $\omega \in \Omega$ there corresponds an $r \in Q$, with $r \neq 0$, such that

$$\sum_{n=0}^{\infty} 1_{A_1}(T_r^n \omega) = \infty \quad \text{and} \quad \sum_{n=-\infty}^{0} 1_{A_1}(T_r^n \omega) = \infty.$$

Since a similar result holds for A_2, it follows that there exists an $E \in \mathfrak{F}$, with $\mu(E) = 0$ and $T_t E = E$ for all $t \in R$, such that if $\omega \notin E$ then the trajectory $T_t \omega$ has points in common with each of A_1 and A_2 for arbitrarily large and negatively arbitrarily large $t \in R$. In what follows we may assume that E is empty. We now apply the argument in Theorem 2 in [1] and observe that if we let

$$X = \{\omega \in \Omega\colon\ P(\omega) = \tfrac{1}{2} \text{ and } P(T_t \omega) > \tfrac{1}{2} \text{ for all } 0 < t \leqslant p/8\},$$

$$h(x) = \min\{t > 0\colon\ T_t x \in X\} \quad \text{for} \quad x \in X,$$

and

$$Tx = T_{h(x)} x \quad \text{for} \quad x \in X,$$

then $h(x) \geqslant p/8$ for all $x \in X$ and T is an invertible transformation from X onto itself. We may and will regard without loss of generality that

$$\Omega = \bar{X} = \{(x, u)\colon\ x \in X,\ 0 \leqslant u < h(x)\}$$

and

$$\{T_t\}_{t\in R} = \{S_t\}_{t\in R}$$

where $\{S_t\}_{t\in R}$ is defined by means of h and T as in a previous paragraph. Further we see that both the functions F and G on $\bar{X}$ defined by $F(x, u) = h(x)$ and $G(x, u) = u$ are measurable with respect to $\mathfrak{F}$.

Next, to finish the proof, we intend to apply the argument in Theorem 1 in [1]. To do this we must check the σ-finiteness condition of the measure space $(X, \mathfrak{B}, m)$, where $\mathfrak{B}$ is defined to be the σ-field of those sets A in X

with the property that the tube $A^* = \{(x, u)\colon\ x \in A,\ 0 \leqslant u < h(x)\}$ based on A belongs to $\mathfrak{F}$, and where $m(A)$, for $A \in \mathfrak{B}$, is defined by

$$m(A) = \frac{1}{d}\mu(A^*(0, d)),$$

where $d > 0$ is a constant satisfying $h(x) \geqslant d$ for all $x \in X$ and $A^*(0, d) = A^* \cap \{G < d\}$. Since the measure space $(\bar{X}, \mathfrak{F}, \mu)$ is σ-finite, we can take a finite equivalent measure ν on $\mathfrak{F}$. Let

$$\mathfrak{M} = \{A \in \mathfrak{B}\colon\ A = \bigcup_{i=1}^{\infty} A_i \text{ for some } A_i \in \mathfrak{B} \text{ with } m(A_i) < \infty\}$$

and

$$a = \sup\{\nu(A^*(0, d))\colon\ A \in \mathfrak{M}\}.$$

Clearly, there exists an $A \in \mathfrak{M}$ such that $a = \nu(A^*(0, d))$. It will be proved that $\mu(\bar{X} - A^*) = 0$, which, in turn, implies that $m(X - A) = 0$, and hence that $(X, \mathfrak{B}, m)$ is σ-finite. Suppose the contrary: $\mu(\bar{X} - A^*) > 0$. Let

$$\bar{H} = \{(x, u)\colon\ x \in X,\ 0 \leqslant u < d\}.$$

Then we can choose a set $\bar{M}$ in $\mathfrak{F}$ satisfying

$$\bar{M} \subset \bar{H} \cap (\bar{X} - A^*) \quad \text{and} \quad 0 < \mu(\bar{M}) < \infty.$$

Write

$${}^*\tilde{M} = \{(x, u, t)\colon\ S_{t-u}(x, u) \in \bar{M},\ 0 \leqslant u < d,\ 0 \leqslant t < d\}.$$

It follows (cf. [1], p. 731) that if $(\bar{X} \times R, \mathfrak{F} \otimes \mathfrak{L}, \tilde{\mu})$ denotes the completed product measure space of $(\bar{X}, \mathfrak{F}, \mu)$ and the Lebesgue measure space on the real line, then

$${}^*\tilde{M} \in \mathfrak{F} \otimes \mathfrak{L} \quad \text{and} \quad 0 < \tilde{\mu}({}^*\tilde{M}) < \infty;$$

further if we define, for $t \in R$,

$$\bar{M}_t = \{x \in X\colon\ (x, t) \in \bar{M}\},$$

$${}^*\tilde{M}^t = \{(x, u)\colon\ (x, u, t) \in {}^*\tilde{M}\},$$

then, for all $t \in [0, d)$,

$$\begin{aligned} {}^*\tilde{M}^t &= \bar{H} \cap \{x \in X\colon\ (x, t) \in \bar{M}\} \\ &= \{(x, u) \in \bar{X}\colon\ (x, t) \in \bar{M},\ 0 \leqslant u < d\}; \end{aligned}$$

since $\tilde{\mu}({}^*\tilde{M}) = \int_0^d \mu({}^*\tilde{M}^t)\,dt$ by Fubini's theorem, there exists $t \in [0, d)$ such that

$${}^*\tilde{M}^t \in \mathfrak{F} \quad \text{and} \quad 0 < \mu({}^*\tilde{M}^t) < \infty.$$

Then, immediately, $\bar{M}_t \in \mathfrak{B}$ and $0 < m(\bar{M}_t) < \infty$. But this is a contradiction, because if we set $B = \bar{M}_t \cup A$, then $B \in \mathfrak{M}$ and $\nu(B^*(0, d)) > a = \nu(A^*(0, d))$.

Since we have proved that $(X, \mathfrak{B}, m)$ is σ-finite, the proof of Theorem 1 in [1] can be applied directly to establish our Theorem A. We omit the details.

Proof of Theorem 2. First, by Theorem 3.1 in [3], any $f \in L_1(\mu)$ can be written $f = \bar{g} + \sum_{i=1}^{\infty} f_i$, where the f_i have disjoint supports and

(i) $\|\bar{g}/e\|_\infty \leqslant 3\int |f|\,d\mu$,

(ii) each f_i/e takes on only two nonzero values and $\int f_i\,d\mu = 0$,

(iii) $\sum_{i=1}^{\infty} [H_e(f_i) + \int |f_i|\,d\mu] \leqslant C[H_e(f) + \int |f|\,d\mu]$.

Suppose there exist functions g_i, $i \geqslant 1$, in $L_1(\mu)$ such that the sets $\{g_i \neq 0\}$ are disjoint, $g_i \overset{e}{\sim} f_i$, and

$$\int M_e(g_i)\cdot e\,d\mu \leqslant C[H_e(f_i) + \int |f_i|\,d\mu].$$

Then, letting h_i, for $i \geqslant 1$, be any function in $L_1(\mu)$ such that $\{h_i \neq 0\} \subset \{g_i \neq 0\}$,

$$h_i 1_{\{g_i > 0\}} \overset{e}{\sim} \bar{g} 1_{\{f_i > 0\}} \quad \text{and} \quad h_i 1_{\{g_i < 0\}} \overset{e}{\sim} \bar{g} 1_{\{f_i < 0\}},$$

and h_0 be any function in $L_1(\mu)$ such that $\{h_0 \neq 0\} \subset \{g_i = 0 \text{ for all } i \geqslant 1\}$ and

$$h_0 \overset{e}{\sim} \bar{g} 1_{\{g_i = 0 \text{ for all } i \geqslant 1\}},$$

it follows that $\sum_{i=0}^{\infty} h_i \overset{e}{\sim} \bar{g}$, and the function $f' = h_0 + \sum_{i=1}^{\infty} (g_i + h_i)$ satisfies $f' \overset{e}{\sim} f$ and

$$\int M_e(f')\cdot e\,d\mu \leqslant \sum_{i=1}^{\infty} \int M_e(g_i)\cdot e\,d\mu + \|\bar{g}/e\|_\infty$$
$$\leqslant C\sum_{i=1}^{\infty} [H_e(f_i) + \int |f_i|\,d\mu] + 3\int |f|\,d\mu.$$

Thus, to establish Theorem 2, it suffices to prove the existence of such functions g_i, $i \geqslant 1$. For this purpose we apply Theorem A and assume without loss of generality that $(\Omega, \mathfrak{F}, \mu) = (\bar{X}, \bar{\mathfrak{B}}, \bar{m})$ and $\{T_t\}_{t\in R} = \{S_t\}_{t\in R}$. Define

$$G(x) = \int_0^{h(x)} e(x, u)\,du \quad \text{for} \quad x \in X,$$

where h is the ceiling function on the base measure space $(X, \mathfrak{B}, m)$. It

follows from Fubini's theorem that for each $i \geqslant 1$ there exists a measurable function h_i on $(X, \mathfrak{B}, m)$ such that $0 < h_i < h$ on X and

$$\int_0^{h_i(x)} e(x, u)\,du = G(x)\,\mu_e(\{f_k \neq 0 \text{ for some } 1 \leqslant k \leqslant i\})$$

for $x \in X$. Let us write $A_i = \{(x, u)\colon x \in X,\ h_{i-1}(x) \leqslant u < h_i(x)\}$, where we let $h_0(x) = 0$ for $x \in X$.

Fix any $i \geqslant 1$. For an integer $N \geqslant 1$ then there exist measurable functions j_k, $0 \leqslant k \leqslant N$, on $(X, \mathfrak{B}, m)$ such that $h_{i-1} = j_0 < j_1 < \ldots < j_N = h_i$ on X and

$$\int_{j_{k-1}(x)}^{j_k(x)} e(x, u)\,du = \frac{1}{N} G(x)\,\mu_e(\{f_i \neq 0\}) \quad (x \in X);$$

further there exist measurable functions j_k', $1 \leqslant k \leqslant N$, on $(X, \mathfrak{B}, m)$ such that $j_{k-1} < j_k' < j_k$ on X and

$$\int_{j_{k-1}(x)}^{j_k'(x)} e(x, u)\,du = \frac{1}{N} G(x)\,\mu_e(\{f_i > 0\}) \quad (x \in X).$$

Denoting by α and β, with $\alpha > 0 > \beta$, the two nonzero values which f_i/e takes on, define a function g_i on $\Omega = \bar{X}$ by

$$g_i(x, u) = \begin{cases} \alpha e(x, u), & \text{if} \quad j_{k-1}(x) \leqslant u < j_k'(x),\ 1 \leqslant k \leqslant N, \\ \beta e(x, u), & \text{if} \quad j_k'(x) \leqslant u < j_k(x),\ 1 \leqslant k \leqslant N, \\ 0, & \text{otherwise}. \end{cases}$$

Then, clearly, $\{g_i \neq 0\} = A_i$ and $g_i \overset{e}{\sim} f_i$. It is easily seen that if N is sufficiently large then

$$\int_{\bar{X} - A_i} M_e(g_i)\cdot e\,d\mu < \int |f_i|\,d\mu.$$

Next, for any $f \in L_1(\mu)$, define

$$M_e^*(f)(\omega) = \sup_{b>0} \int_0^b f(T_t\omega)\,dt \Big/ \int_0^b e(T_t\omega)\,dt \quad \text{for} \quad \omega \in \Omega.$$

By a maximal ergodic theorem (see e.g. [6]) for flows, we have

$$\int_{\{M_e^*(g_i) \geqslant \lambda\}} (g_i - \lambda e)\,d\mu \geqslant 0 \quad (\lambda > 0).$$

Since $g_i = 0$ on $\bar{X} - A_i$, this gives

$$\int_{\{M_e^*(g_i) \geqslant \lambda\} \cap A_i} g_i\,d\mu \geqslant \lambda\mu_e(\{M_e^*(g_i) \geqslant \lambda\} \cap A_i) \quad (\lambda > 0).$$

Using the facts that $\{g_i/e \geqslant \lambda\} \subset \{M_e^*(g_i) \geqslant \lambda\}$ and g_i/e takes on only the two nonzero values α and β, we observe from the definition of $\psi_e(g_i, \lambda)$ that

$$\mu_e(\{M_e^*(g_i) \geqslant \lambda\} \cap A_i) \leqslant \psi_e(g_i, \lambda) \quad (\lambda > 0).$$

Similarly,

$$\mu_e(\{M_e^*(-g_i) \geqslant \lambda\} \cap A_i) \leqslant \psi_e(-g_i, \lambda) \quad (\lambda > 0).$$

Thus, by the fact that $M_e(g_i)(\omega) = \max\{M_e^*(g_i)(\omega), M_e^*(-g_i)(\omega)\}$, we obtain

$$\begin{aligned}\int_{A_i} M_e(g_i)\cdot e\,d\mu &= \int_0^\infty \mu_e(\{M_e(g_i) \geqslant \lambda\} \cap A_i)\,d\lambda \\ &\leqslant \int_0^\infty [\mu_e(\{M_e^*(g_i) \geqslant \lambda\} \cap A_i) + \mu_e(\{M_e^*(-g_i) \geqslant \lambda\} \cap A_i)]\,d\lambda \\ &\leqslant \int_0^\infty [\psi_e(g_i, \lambda) + \psi_e(-g_i, \lambda)]\,d\lambda \\ &\leqslant C[H_e(g_i) + \int |g_i|\,d\mu] \qquad \text{(by Lemma 3.2 in [4])} \\ &= C[H_e(f_i) + \int |f_i|\,d\mu],\end{aligned}$$

where the last equality is due to the fact that $f_i \overset{e}{\sim} g_i$. Hence, if N is chosen sufficiently large, then the function g_i satisfies

$$\int_\Omega M_e(g_i)\cdot e\,d\mu \leqslant C[H_e(f_i) + \int |f_i|\,d\mu].$$

Since $\{g_i \neq 0\} = A_i$ and $A_i \cap A_j = \emptyset$ if $i \neq j$, this completes the proof of Theorem 2.

THEOREM 3. *Let $(\Omega, \mathfrak{F}, \mu)$, $\{T_t\}_{t\in R}$ and $e \in L_1(\mu)$ be as in Introduction. Suppose that $f \in L_1(\mu)$ satisfies $\int |f| \log^+(|f|/e)\,d\mu = \infty$. Then there exists an $f' \in L_1(\mu)$ such that $f' \overset{e}{\sim} f$ and $\int M_e(f')\cdot e\,d\mu = \infty$.*

Proof. If necessary, considering $-f$ instead of f, we may assume that there exists a set A in $\mathfrak{F}$ such that $f/e > 1$ on A, $\int_A f\,d\mu < 1$ and $\int_A f \log(f/e)\,d\mu = \infty$; further we may assume by Theorem A that $(\Omega, \mathfrak{F}, \mu) = (\bar{X}, \mathfrak{B}, \bar{m})$ and $\{T_t\}_{t\in R} = \{S_t\}_{t\in R}$. Choose a positive number α so that if we let $\bar{A} = \{(x, u) \in \bar{X}\colon 0 \leqslant u < \alpha\}$ then $\mu_e(\bar{A}) = \mu_e(A)$. Next, take an increasing function $w(u)$ on the interval $[0, \alpha)$ so that if a function g on $\Omega = \bar{X}$ is defined by $g(x, u) = w(u)e(x, u)$ for $(x, u) \in \bar{A}$ and 0 otherwise, then $g \overset{e}{\sim} f 1_A$. By Theorem 2 in [8], we have

$$\int_{\bar{A}} M_e(g)\cdot e\,d\mu = \infty.$$

Thus, if f' is any function in $L_1(\mu)$ such that $f' = g$ on $\bar{A}$ and $f' \overset{e}{\sim} f$, then $\int M_e(f')\cdot e\,d\mu = \infty$. The proof is complete.

3. Remarks. (i) The proof of Theorem 1 shows that this theorem holds for any conservative and ergodic *semiflow* $\{T_t\}_{t\geqslant 0}$. (ii) It is not difficult to check that Ambrose and Kakutani's representation theorem [2] for conservative flows holds even if the underlying measure space is not a probability space but a σ-finite measure space; thus it follows that, except for a pathological case, Theorems 2 and 3 hold for any conservative flow $\{T_t\}_{t\in R}$.

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