STUDIA MATHEMATICA, T. LXXX. (1984)

Hardy and Lipschitz spaces on subsets of R^n

by

ALF JONSSON (Umeå), PETER SJÖGREN (Göteborg), and HANS WALLIN (Umeå)

Abstract. Given a suitable closed set F in R^n, Lipschitz spaces on F can be defined by restriction or directly. We want to construct Hardy spaces H^p and local Hardy spaces h^p on F with these Lipschitz spaces as duals. Using a fixed nonnegative measure μ whose support is F, we define H^p and h^p by means of atoms on F. Polynomials are used in the moment condition for these atoms, and we assume that polynomials satisfy a version of Markov's inequality on F. Then H^p and h^p can (almost) be characterized in terms of a suitable maximal function. The desired duality property follows if we also assume that F is of constant dimension in the sense that μ behaves like a Hausdorff measure. Finally, the theory is applied to obtain a property of the atomic decomposition of $H^p(R^n)$.

0. Introduction

0.1. Hardy and Lipschitz spaces in R^n. The *Lipschitz spaces* in R^n are the duals of the Hardy spaces in R^n. More exactly: Let $H^p(R^n)$, $0 < p \leqslant 1$, be the Hardy spaces in R^n and $\dot{\Lambda}_\alpha(R^n)$, $\alpha > 0$, the homogeneous Lipschitz spaces in R^n defined in the following way: $\dot{\Lambda}_\alpha(R^n)$, $k < \alpha \leqslant k+1$, k a non-negative integer, consists of all $\varphi \in C^k(R^n)$ such that $\|\Delta_h D^\beta \varphi\|_\infty \leqslant c_1 |h|^{\alpha-k}$ if $\alpha \neq k+1$ and $\|\Delta_h^2 D^\beta \varphi\|_\infty \leqslant c_1 |h|$ if $\alpha = k+1$, for all $h \in R^n$ and all multi-indices β of length $|\beta| = k$; here Δ_h denotes the first and Δ_h^2 the second difference with step h, $\|\ \|_\infty$ denotes the $L^\infty(R^n)$-norm and $c_1 = c_1(\varphi)$ is a constant; $\inf c_1$ gives a semi-norm on $\dot{\Lambda}_\alpha(R^n)$. Then the dual $\big(H^p(R^n)\big)'$ of $H^p(R^n)$, $0 < p < 1$, is $\dot{\Lambda}_\alpha(R^n)$, where $\alpha = n(1/p-1)$; see for instance [2], p. 575. Let $\Lambda_\alpha(R^n)$ be the space of those $\varphi \in \dot{\Lambda}_\alpha(R^n)$ such that $\|D^\beta \varphi\|_\infty \leqslant c_2 < \infty$, for $|\beta| \leqslant k$, normed with the larger of $\inf c_1$ and $\inf c_2$. If $h^p(R^n)$ denote the local Hardy spaces (see § 2 or [5]), then $\big(h^p(R^n)\big)' = \Lambda_\alpha(R^n)$, if $\alpha = n(1/p-1)$, $0 < p < 1$ ([5], Th. 5).

The maximal function characterization of $H^p(R^n)$ was given in [3]. The characterization by means of atoms (§ 2) has been given by Coifman and Latter [10]; see [5] for $h^p(R^n)$.

0.2. Markov's inequality and d-sets. The Hardy and Lipschitz spaces in R^n have been generalized. In particular, the Lipschitz spaces have been defined on closed subsets F of R^n (see § 0.3). A natural question is: Can we define Hardy spaces of functions on F having these Lipschitz spaces as duals. It turns out that the answer is *yes* if F has two basic properties. The first one is that F will be a *d-set*, $0 < d \leqslant n$, in the sense that there exists a measure μ with support F (a *d-measure on* F), which behaves like a d-dimensional Hausdorff measure (see § 1.1 for the exact definition; d-sets were introduced and studied in [6] and [8]). The second property was introduced in [8], § 3 and [9] and has to do with Markov's inequality which gives an estimate of the derivatives of polynomials on a ball B by means of the maximum on B of the polynomials. The property is discussed in § 1 and means that Markov's inequality holds on F in the sense that derivatives of a polynomial on $B \cap F$ may be estimated by means of the maximum on $B \cap F$ of the polynomial (*F has the Markov property*).

0.3. Lipschitz spaces on $F \subset R^n$. The Lipschitz spaces $\Lambda_\alpha(R^n)$ have been generalized to Lipschitz spaces $\Lambda_\alpha(F)$ of functions defined on an arbitrary closed subspace F of R^n, so that $\Lambda_\alpha(F)$ consists of the restrictions to F of the functions in $\Lambda_\alpha(R^n)$. The space $\Lambda_\alpha(F)$ was introduced and studied in [7], [8], and [9]; among other things the restriction property was proved. When F has the Markov property the characterization of $\Lambda_\alpha(F)$ is particularly simple; see § 4.1 concerning this and the properties of $\Lambda_\alpha(F)$ which we need. Analogously, $\dot{\Lambda}_\alpha(R^n)$ may be generalized to $\dot{\Lambda}_\alpha(F)$ (see § 5.2).

0.4. Hardy spaces on F. $H^p(R^n)$ has for some p been generalized to Hardy spaces of functions defined on homogeneous spaces [2]. These Hardy spaces are defined by means of atomic sums ([2], p. 592) and a theory of the maximal function is given in [12]. If we interpret these Hardy spaces as a subset of $(\dot{\Lambda}_\alpha(F))'$, this theory gives a predual of $\dot{\Lambda}_\alpha(F)$ for $0 < \alpha < 1$ for global d-sets (§ 5.1) since these are homogeneous spaces. In this paper we construct (global) *Hardy spaces* $H^p(F)$, $0 < p \leqslant 1$, of distributions on F and *local Hardy spaces* $h^p(F)$, so that $(H^p(F))' = \dot{\Lambda}_\alpha(F)$ and $(h^p(F))' = \Lambda_\alpha(F)$ for all $\alpha > 0$, if $\alpha = d(1/p-1)$, $0 < p < 1$, and F is a d-set having the Markov property; we also cover the limit case $\alpha = 0$, $p = 1$. The space $h^p(F)$, and the more general spaces $h^{p,q}(F)$, $1 \leqslant q \leqslant \infty$, and $h^{p,q,s} = h^{p,q,s}(F, \mu)$, are introduced in § 2 (Definition 2.2) by means of *local atoms on F* (Definition 2.1) which, in turn, are introduced using a measure μ with support F satisfying the doubling condition (§ 1, inequality (1.1)). On homogeneous spaces only polynomials of degree zero are considered in the moment condition in the definition of atoms (§ 2, Equation (2.2)). Since we assume that F has the Markov property in relevant parts of §§ 3–5, we can handle moment conditions of higher degree and consequently get a more general theory for these F.

In § 3 we characterize $h^{p,q}(F)$ and $h^{p,q,s}$ by means of maximal functions if F has the Markov property and μ satisfies the doubling condition (Theorems 3.1 and 3.2). This leads to (Corollary 3.1) the conclusion that $h^{p,q,s}$ and $h^{p,q}(F)$ depend only on p and F for such F. This is true also for Hardy spaces on homogeneous spaces but not on general sets F (§ 4.4, Example 3). The duality is proved in § 4 (Theorem 4.2). The global Hardy spaces $H^p(F)$ and $H^{p,q}(F)$ are introduced and studied in § 5. In § 5.3 we apply our results to obtain a property of ordinary Hardy spaces in R^n. We prove that if the support of $f \in H^p(R^n)$ is contained in a bounded convex body F, then f can be written as an atomic sum with atoms having support contained in F.

In note [15] Strömberg and Torchinsky announce some results on weighted Hardy spaces which are related to this study.

1. Markov's inequality and d-sets

Throughout the paper, F denotes a non-empty closed subset of the n-dimensional Euclidean space R^n and μ is a positive Borel measure with support F, $\operatorname{supp}\mu = F$, such that μ is finite on compact sets. The closed ball with center $x \in R^n$ and radius $r > 0$ is denoted by $B(x, r)$.

1.1. The concept of d-set. We say that μ satisfies the *doubling condition* if, for some constant c_0,

$$\mu(B(x, 2r)) \leqslant c_0 \mu(B(x, r)) \quad \text{for} \quad x \in F,\ 0 < r \leqslant 1. \tag{1.1}$$

We call μ a *d-measure on* F ($0 < d \leqslant n$), and F a *d-set*, if, for some constants $c', c'' > 0$,

$$c' r^d \leqslant \mu(B(x, r)) \leqslant c'' r^d \quad \text{for} \quad x \in F,\ 0 < r \leqslant 1. \tag{1.2}$$

Clearly, (1.2) implies (1.1) and we may assume that the right-hand inequality in (1.2) holds for all $x \in R^n$. If μ_1 and μ_2 are d-measures on F, μ_1 and μ_2 are equivalent in the sense that there are positive constants c_1 and c_2 so that $c_1 \mu_1 \leqslant \mu_2 \leqslant c_2 \mu_1$ ([6], Proposition 1.1). If F is a d-set, then F has Hausdorff dimension d and the restriction to F of the d-dimensional Hausdorff measure in R^n is a d-measure on F ([6], Proposition 1.1); for $d = n$ this gives the n-dimensional Lebesgue measure on F. Examples of d-sets are given in [6], Section 2.

1.2. The Markov property. We say that F *has the Markov property* if I in Theorem 1.1 below holds for all positive integers N. In Theorem 1.3 we prove that every d-set $F \subset R^n$ with $d > n-1$ has the Markov property; if $F = R^n$, then I is Markov's inequality which is usually stated for cubes instead of balls B.

THEOREM 1.1. *Consider a fixed closed subset F of R^n, all balls*

$B = B(x_0, r)$, and all polynomials $P(x) = \sum_{|\beta| \leqslant N} a_\beta (x-x_0)^\beta$ of degree $\leqslant N$, where N is fixed, $x_0 \in F$ and $0 < r \leqslant 1$. Then the following two conditions I *and* II *are equivalent*:

I. $\max_{F \cap B} |\operatorname{grad} P| \leqslant c r^{-1} \max_{F \cap B} |P|$ *for all $P(x)$ and B.*

Here and from now on in § 1 *$c, c_1, c_2, \ldots$ denote positive constants depending only on F and N.*

II. $\max_{F \cap B} |P| \sim \sum |a_\beta| r^{|\beta|}$ *for all $P(x)$ and B, where $a \sim b$ means that $c_1 \leqslant a/b \leqslant c_2$.*

Furthermore, I *and* II *imply, for $1 \leqslant q \leqslant \infty$, if μ satisfies* (1.1),

III. $\left\{\frac{1}{\mu(B)} \int_B |P(x)|^q \, d\mu(x)\right\}^{1/q} \sim \max_{F \cap B} |P|$ *for all $P(x)$ and B,*

but not conversely.

Proof. *Step* 1, I $\Rightarrow$ II: Repeated application of I gives

$$|a_\beta| = C|D^\beta P(x_0)| \leqslant c_1 r^{-|\beta|} \max_{F \cap B} |P|.$$

Hence, $\sum |a_\beta| r^{|\beta|} \leqslant c_2 \max_{F \cap B} |P|$, which gives the non-trivial direction of II.

Step 2, II $\Rightarrow$ I: By using II we get, for $x \in B \cap F$,

$$|\operatorname{grad} P(x)| \leqslant c_1 r^{-1} \sum |a_\beta| r^{|\beta|} \leqslant c_2 r^{-1} \max_{F \cap B} |P|.$$

Step 3, II $\Rightarrow$ III: We note that the left member of III is less than or equal to the right-hand member. In order to prove the converse inequality we put $A = \sum |a_\beta| r^{|\beta|}$ and note that according to II there is a point $y \in F \cap B(x_0, r/2)$ where $P(y) \sim A$. Since $|\operatorname{grad} P| \leqslant c_1 r^{-1} A$ in B, this means that $|P| \sim A$ in a ball $B(y, cr)$, where $1/2 > c = c(F, N) > 0$. Consequently, the left-hand member of III is $\geqslant c_2 A \{\mu(B(y, cr))/\mu(B)\}^{1/q}$. According to the doubling condition (1.1) and II this is $\sim A \sim \max_{F \cap B} |P|$, which gives the converse inequality.

Step 4, III $\not\Rightarrow$ I, II: As an example we may choose a line segment $F \subset \boldsymbol{R}^2$ defined by $F = \{x = (x_1, x_2) \in \boldsymbol{R}^2\colon x_2 = 0, 0 \leqslant x_1 \leqslant 1\}$. By choosing $P(x) = x_2$, we see that I does not hold. On the other hand, if we consider F as a subset of $\boldsymbol{R}^1$, then the Markov property I is valid on F (see for instance Theorem 1.3 below). From Steps 1 and 3 in this proof we get that III holds.

Remark. From I and II in Theorem 1.1 it follows that the Markov property I is equivalent to the corresponding property for B changed to cubes with center in F and side of length $\leqslant 1$.

In order to verify the Markov property it is, according to the following theorem, enough to consider polynomials of degree 1.

Theorem 1.2. *If* I *in Theorem* 1.1 *holds for all polynomials $P(x)$ of degree* 1, *then F has the Markov property, i.e.* I *holds for all polynomials $P(x)$ of degree $\leqslant N$ for every N, with a constant depending on N and F.*

Proof. We use induction on N and suppose that I holds for $N-1$. Take $B = B(x_0, r)$, $x_0 \in F$, $0 < r \leqslant 1$, $P(x) = \sum_{|\beta| \leqslant N} a_\beta (x-x_0)^\beta$ and introduce the "coefficient norm" $A = \sum |a_\beta| r^{|\beta|}$ of P. It is enough to prove II in Theorem 1.1, i.e. that $|P| > c_1 A$ somewhere in $F \cap B$. We may assume that $|P(x_0)| = |a_0| \leqslant \sum_{\beta \neq 0} |a_\beta| r^{|\beta|}$, because otherwise II holds. This means that the "coefficient norm" of some component of $\operatorname{grad} P$ is $\sim A/r$. By the induction assumption we then get, if $B' = B(x_0, r/2)$,

$$\sup_{F \cap B'} |\operatorname{grad} P| \sim A/r.$$

Take a point $y \in F \cap B'$ such that $|\operatorname{grad} P(y)| \sim A/r$. Introduce the ball $B_1 = B(y, \varepsilon r)$, $0 < \varepsilon < 1/2$, and the polynomial $Q(x) = (x-y) \cdot \operatorname{grad} P(y)$. Since $Q(x)$ has degree 1 and $|\operatorname{grad} Q| = |\operatorname{grad} P(y)| \sim A/r$, the assumption in the theorem means that $|Q(x)| > cA\varepsilon$ for some $x \in F \cap B_1$. We shall estimate $P(x)$ by means of the Taylor expansion $P(x) = P(y) + Q(x) + R(x)$, where the remainder term $R(x)$ satisfies $|R(x)| \leqslant c_2 A \varepsilon^2$ for $x \in B_1$ since the absolute values of the second derivatives of P in B_1 are $\leqslant c_3 A/r^2$. By choosing $\varepsilon = \varepsilon(F, N)$ small we may assume that $|R(x)| \leqslant cA\varepsilon/4$ for $x \in B_1$ and we may also assume that $|P(y)| \leqslant cA\varepsilon/4$ (since otherwise there is nothing to prove). The estimates which we have for $|Q(x)|$, $|R(x)|$, and $|P(y)|$ give that $|P(x)| = |P(y) + Q(x) + R(x)| > cA\varepsilon/2$ for some $x \in (F \cap B_1) \subset (F \cap B)$, which is what we wanted to prove.

We now give a geometric criterion for the Markov property.

Theorem 1.3. *F has the Markov property if and only if the following condition holds: There exists an $\varepsilon > 0$ so that none of the sets $F \cap B(x_0, r)$, $x_0 \in F$, $0 < r \leqslant 1$, is contained in any band of type $\{x\colon |b \cdot (x-x_0)| < \varepsilon r\}$ where $|b| = 1$.*

Proof. (1) Suppose that the condition does not hold. Then, for any $\varepsilon > 0$, we can find x_0, r and b as in the theorem such that $F \cap B(x_0, r) \subset \{x\colon |b \cdot (x-x_0)| < \varepsilon r\}$. If we check I in Theorem 1.1 on the polynomial $P(x) = b \cdot (x-x_0)$, we see that I does not hold.

(2) Suppose that the condition holds for a certain $\varepsilon > 0$ and that P is a first degree polynomial, $P(x) = \sum a_\beta (x-x_0)^\beta = b \cdot (x-x_0) + a$, where $|b| = 1$. If $x_0 \in F$, and $0 < r \leqslant 1$, there exists an $x_1 \in F \cap B(x_0, r)$ with $|b \cdot (x_1 - x_0)| \geqslant \varepsilon r$. Hence, $|P(x_1) - P(x_0)| \geqslant \varepsilon r$, so at x_0 or x_1 we have $|P| \geqslant \varepsilon r/2 \sim \sum |a_\beta| r^{|\beta|}$ if $|a| \leqslant r$. If $|a| > r$, then $|P(x_0)| = |a| \sim \sum |a_\beta| r^{|\beta|}$. By II in Theorem 1.1 and Theorem 1.2, F has the Markov property.

1.3. The Markov property and d-sets.

THEOREM 1.4. *If $F \subset \mathbf{R}^n$ is a d-set with $d > n-1$, then F has the Markov property.*

Proof. We verify that the condition in Theorem 1.3 holds. Take $B = B(x_0, r)$, $x_0 \in F$, $0 < r \leqslant 1$, and a band $D = B \cap \{x: |b\cdot(x-x_0)| < \varepsilon r\}$ where $|b| = 1$ and $1 > \varepsilon > 0$. We can cover D by cubes with side εr so that we need at most $c_1 r^{n-1} \varepsilon r/(\varepsilon r)^n = c_1 \varepsilon^{1-n}$ cubes. Each cube intersecting F is contained in a ball with radius $c_2 \varepsilon r$ and center in F. This gives, by (1.2),

$$\mu(D) \leqslant c_3 \varepsilon^{1-n} (\varepsilon r)^d.$$

But, by (1.2), $\mu(B) \geqslant c' r^d$, i.e. $B \setminus D$ intersects F if $\varepsilon = \varepsilon(F)$ is so small that $c_3 \varepsilon^{d-(n-1)} < c'$.

2. Local Hardy spaces on $F \subset \mathbf{R}^n$

We assume that the exponents p and q are *admissible* in the sense that

$$0 < p \leqslant 1, \quad 1 \leqslant q \leqslant \infty, \quad \text{and} \quad p < q,$$

and that μ is a fixed measure, $\operatorname{supp} \mu = F \subset \mathbf{R}^n$, satisfying the doubling condition (1.1); s is a non-negative integer. If $B = B(x_0, r)$ we use, for the sake of simplicity, the notation $2B$ for the ball $B(x_0, 2r)$.

DEFINITION 2.1. A *local* (p, q, s)*-atom* (with respect to μ) is a function $a \in L^q(\mu)$ with compact support such that for some ball $B = B(x_0, r)$, $x_0 \in F$, satisfying $\operatorname{supp} a \subset B$, we have

$$\left\{\frac{1}{\mu(B)} \int |a|^q \, d\mu\right\}^{1/q} \leqslant \mu(2B)^{-1/p}, \tag{2.1}$$

and

$$\int a P d\mu = 0 \quad \text{if} \quad r \leqslant 1, \tag{2.2}$$

for all polynomials P of degree $\leqslant s$.

We call B a *supporting ball of* a. (2.1) gives that the $L^p(\mu)$-norm of a is $\leqslant 1$. We remark that the results of this paper remain true if in Definition 2.1 we change $\mu(B)$ to $\mu(2B)$ in the left-hand member of (2.1).

When F is a d-set, μ a d-measure on F, and $s = [d(1/p-1)]$, where [ ] denotes the integer part, i.e. the moment condition (2.2) holds for all polynomials P of degree $\leqslant [d(1/p-1)]$, a is called a *local* (p, q)*-atom on* F; a *local* p*-atom on* F is a local (p, ∞)-atom on F.

We need the following simple lemma.

LEMMA 2.1. *Let μ be a measure with support F satisfying the doubling condition* (1.1). *For any given $\varrho > 0$ there exist constants $c_1 = c_1(\mu, \varrho) > 0$ and $\beta = \beta(\mu) > 0$ such that*

$$\mu(B(x, r)) \geqslant c_1 r^\beta \quad \text{for all } x \in F \cap B(0, \varrho), \ 0 < r \leqslant 1.$$

Proof. We consider a maximal number of disjoint balls with center in $F \cap B(0, \varrho)$ and radius 1/3. Then any ball $B(x, 1)$, $x \in F \cap B(0, \varrho)$, must contain one of these finitely many disjoint balls and the desired inequality follows for $r = 1$. For $x \in F \cap B(0, \varrho)$, $0 < r < 1$ we take an integer j such that $1/2 \leqslant 2^j r < 1$ and obtain by repeated application of (1.1) with $c_0 = c_0(\mu) \geqslant 1$ that

$$\mu(B(x, r)) \geqslant c_0^{-j} \mu(B(x, 2^j r)) \geqslant r^\beta \mu(B(x, 1))/c_0,$$

for some $\beta = \beta(\mu) > 0$. This gives the lemma.

Using Lemma 2.1 we now prove the following basic lemma.

LEMMA 2.2. *Let μ be a measure with support F satisfying the doubling condition* (1.1), *and let p and q be admissible exponents. Then there exist a positive integer $s_0(\mu, p)$ and, for each $\varphi \in C_0^\infty(\mathbf{R}^n)$, a constant $c(\varphi) = c(\varphi, \mu)$ so that*

$$\left|\int a\varphi d\mu\right| \leqslant c(\varphi),$$

for all local (p, q, s)-atoms a with $s \geqslant s_0(\mu, p)$.

Proof. Let $\varphi \in C_0^\infty(\mathbf{R}^n)$ and let a be a local (p, q, s)-atom with a supporting ball $B = B(y, r)$. For $0 < r \leqslant 1$ we get by (2.2), Hölder's inequality, and (2.1), if $P_y(x-y)$ is the Taylor polynomial of degree s of φ around y,

$$\left|\int_B a\varphi \, d\mu\right| = \left|\int_B a(x)[\varphi(x) - P_y(x-y)] \, d\mu(x)\right|$$

$$\leqslant \mu(2B)^{-1/p} \cdot \mu(B)^{1/q} c_2 r^{s+1} \mu(B)^{1/q'},$$

$1/q + 1/q' = 1$, $c_2 = c_2(\varphi)$. We may assume that $B \cap \operatorname{supp} \varphi \neq \emptyset$ and use Lemma 2.1 with $c_1 = c_1(\mu, \varphi)$ to conclude that this is less than

$$\mu(B)^{1-1/p} c_2 r^{s+1} \leqslant (c_1 r^\beta)^{1-1/p} c_2 r^{s+1} \leqslant c, \quad \beta = \beta(\mu),$$

if $\beta(1-1/p) + s + 1 \geqslant 0$, i.e. if $s \geqslant s_0(\mu, p) = \beta(1/p - 1) - 1$. This is the desired estimate for $r \leqslant 1$. For $r > 1$ we get by Hölder's inequality and (2.1)

$$\left|\int_B a\varphi d\mu\right| \leqslant \mu(2B)^{-1/p} \mu(B)^{1/q} c_3 \mu(B)^{1/q'} \leqslant c_3 \mu(2B)^{1-1/p}, \quad c_3 = c_3(\varphi).$$

We may assume that there exists a point $x_0 \in B \cap \operatorname{supp} \mu \cap \operatorname{supp} \varphi$. Then $\mu(2B) \geqslant \mu(B(x_0, 1))$ and if we apply Lemma 2.1 with a ϱ such that $B(0, \varrho) \supset \operatorname{supp} \varphi$ we see that $\mu(B(x_0, 1)) \geqslant c_1$, $c_1 = c_1(\mu, \varphi)$, which gives the desired estimate for $r \geqslant 1$.

We identify the local (p, q, s)-atom a with the distribution (element in

$\mathcal{D}'$) whose value for $\varphi \in C_0^\infty(\mathbf{R}^n)$ is $\langle a, \varphi\rangle = \int a\varphi\, d\mu$. Now, suppose that a_j, $j = 1, 2, \ldots$, are local (p, q, s)-atoms and that λ_j are complex numbers satisfying $\sum |\lambda_j|^p < \infty$. Then $\sum |\lambda_j| < \infty$ and according to Lemma 2.2 $\sum \lambda_j a_j$ converges in $\mathcal{D}'$, if $s \geqslant s_0(\mu, p) = \beta(1/p-1)-1$, to a distribution f whose value for $\varphi \in C_0^\infty(\mathbf{R}^n)$ is

$$\langle f, \varphi\rangle = \sum \lambda_j \int a_j \varphi\, d\mu. \tag{2.3}$$

Definition 2.2. The *local Hardy space* $h^{p,q,s} = h^{p,q,s}(F, \mu)$ *on* F *with measure* μ, $s \geqslant s_0(\mu, p)$, is the space of all distributions f having a representation $f = \sum \lambda_j a_j$, where a_j are local (p, q, s)-atoms and $\sum |\lambda_j|^p < \infty$. We introduce

$$\|f\|_{h^{p,q,s}} = \inf \left(\sum |\lambda_j|^p\right)^{1/p},$$

where the infimum is taken over all such representations. This is a quasi-norm, i.e. all properties of a norm are valid except that the right-hand member of the triangle inequality must be multiplied by a fixed constant.

We note that if the doubling condition (1.1) holds for all $r > 0$, then we get the same local Hardy spaces if we replace $2B$ by B in the right-hand member of (2.1). For $p = 1$ we have $\int |a|\, d\mu \leqslant 1$ for all local (p, q, s)-atoms, i.e. $\sum \lambda_j a_j$ converges in $L^1(\mu)$ if $\sum |\lambda_j| < \infty$. Hence, we can *identify* $h^{1,q,s}$ *with a subset of* $L^1(\mu)$ and (2.3) holds for all $\varphi \in L^\infty(\mu)$.

Now let F be a d-set, μ a d-measure on F, and $s = [d(1/p-1)]$. In this case we may choose β in Lemma 2.1 equal to d, and hence $s \geqslant s_0(\mu, p) = \beta(1/p-1)-1$. Thus $h^{p,q,s}$ is defined. It is denoted by $h^{p,q} = h^{p,q}(F, \mu)$ and we speak about the *local Hardy space* $h^{p,q}(F, \mu)$ *on* F; we put $h^{p,\infty}(F, \mu) = h^p(F, \mu) = h^p$. We notice that it is easy to see that $h^p(F, \mu)$ for $F = \mathbf{R}^n$, μ Lebesgue measure, give the local Hardy spaces in $\mathbf{R}^n$ introduced by Goldberg ([5], pp. 36–37).

From Hölder's inequality it follows that the left-hand member of (2.1) is an increasing function of q. This means that $h^{p,q_1,s} \subset h^{p,q_2,s}$ with continuous imbedding if $q_1 > q_2$. In Section 3, Corollary 3.1, we prove that if F has the Markov property, then we have a converse imbedding, i.e. the spaces $h^{p,q,s}$ are the same, with equivalent quasi-norms, for fixed p and μ, for all q and all s that are sufficiently large; also, $h^{p,q}(F, \mu)$ is independent of q and μ, and denoted by $h^p(F)$ if F is a d-set having the Markov property, but this is not true for general F (see § 4.4, Example 3).

3. Maximal functions and local Hardy spaces

3.1. Maximal functions of $h^{p,q,s}$ distributions. We now define a maximal function adapted to $h^{p,q,s}$ by means of a class of smooth functions supported in the unit ball. In this subsection, we show that the maximal function of any $h^{p,q,s}$ distribution is in L^p, and § 3.2 contains a result in the converse sense.

From now on, we denote by c many different constants. The norm in $L^p(\mu)$ is denoted by $\|\cdot\|_p$.

Definition 3.1. Let

$$S_\sigma = \{\varphi \in C_0^\infty(\mathbf{R}^n)\colon\ \operatorname{supp}\varphi \subset B(0, 1),\ |D^\beta \varphi| \leqslant 1 \text{ for } |\beta| \leqslant \sigma\}$$

for $\sigma \in N$. If f is a distribution with support in F, we set for $x \in F$

$$m_\sigma f(x) = \sup_{\varphi \in S_\sigma} \sup_{0 < t \leqslant 1} \frac{1}{\mu(B(x, t))} \left\langle f, \varphi\left(\frac{x-\cdot}{t}\right)\right\rangle.$$

Theorem 3.1. *Let* μ *satisfy the doubling condition* (1.1). *If* $f \in h^{p,q,s}$ *for some admissible* p *and* q *and a sufficiently large* s (*depending on* μ *and* p), *then* $m_\sigma f \in L^p(\mu)$ *for* $\sigma > s$. *Further*,

$$\|m_\sigma f\|_p \leqslant c\, \|f\|_{h^{p,q,s}}, \qquad c = c(F, \mu, p, q, s, \sigma).$$

When μ *is a d-measure*, $s = [d(1/p-1)]$ *is large enough here, so* $h^{p,q,s}$ *can be replaced by* $h^{p,q}$.

This follows immediately from the next lemma.

Lemma 3.1. *If* μ, p, q, s, σ *are as in Theorem* 3.1, *any local* (p, q, s)-*atom* a *satisfies* $m_\sigma a \in L^p(\mu)$ *and* $\|m_\sigma a\|_p \leqslant c$.

Proof. For $p < 1$ it is enough to consider $q = 1$ since a is a $(p, 1, s)$-atom. We can assume that $B = B(0, r)$ is a supporting ball of a, so that $0 \in F$.

Case 1: $r \leqslant 1/2$. Define the ordinary maximal function by

$$mg(x) = \sup_{0 < \varrho \leqslant 1} \frac{1}{\mu(B(x; \varrho))} \int_{B(x,\varrho)} |g|\, d\mu, \qquad x \in F.$$

The operator m is of weak type (1,1), see e.g. [4], § 6.1. Clearly, $m_\sigma a \leqslant m|a|$, so if $_*$ denotes decreasing rearrangement we get $(m_\sigma a)_*(t) \leqslant c\, \|a\|_1\, t^{-1}$ and

$$\int_{|x| \leqslant 2r} (m_\sigma a)^p\, d\mu \leqslant \int_0^{\mu(2B)} (m_\sigma a)_*^p(t)\, dt$$

$$\leqslant c\, \|a\|_1^p \int_0^{\mu(2B)} t^{-p}\, dt \leqslant c\mu(2B)^{p-1+1-p} = c.$$

Here we used the simple fact that, because of (2.1),

$$\|a\|_1 \leqslant \mu(2B)^{1-1/p}. \tag{3.1}$$

Let $\varphi \in S_\sigma$. To estimate $m_\sigma a$ off $2B$, we use the vanishing moments of a, and denote by P_z the Taylor polynomial of φ at z of degree s. If $|t| \leqslant |x|/2$, the expression

$$\mu(B(x, t))^{-1} \int a(y)\, \varphi((x-y)/t)\, d\mu(y)$$

will vanish since $|x-y| \geqslant |x|/2$ and so $|x-y|/t \geqslant 1$ when $|x| > 2r$. And if $t > |x|/2$ and $|x| > 2r$, the same expression is at most

$$\mu(B(x, t))^{-1}\left|\int_{|y|\leqslant r} a(y)\left(\varphi\left(\frac{x-y}{t}\right)-P_{x/t}\left(-\frac{y}{t}\right)\right)d\mu(y)\right|$$
$$\leqslant c\mu(B(x, t))^{-1}\int |a(y)|(|y|/t)^{s+1}\,d\mu(y)$$
$$\leqslant c\mu(B(x, t))^{-1}\,\mu(B)^{1-1/p}(r/|x|)^{s+1},$$

because of (3.1). We can further assume $|x| < 3/2$ since we get 0 otherwise. Then $B = B(0, r) \subset B(x, \min(2|x|, 2))$, and the doubling condition implies $\mu(B) \leqslant c\mu(B(x, t))$. Then

$$\int_{|x|>2r} (m_\sigma a)^p\,d\mu \leqslant c\mu(B)^{-1}\int_{2r<|x|<3/2} (r/|x|)^{p(s+1)}\,d\mu(x).$$

Considering dyadic rings and using (1.1), we can estimate this last expression by

$$c\mu(B)^{-1}\sum_{2r<2^j\leqslant 2}\mu(B(0, 2^j))(r2^{-j})^{p(s+1)} \leqslant c\sum_{j>j_0} c_0^{j-j_0}2^{-p(s+1)(j-j_0)} \leqslant c$$

if s is large, where we define j_0 by $2^{j_0} \sim r$. The d-measure case with $s = [d(1/p-1)]$ requires some minor modifications at this point but is otherwise similar, see e.g. [13], proof of Theorem 3.1.

Case 2: $r > 1/2$. Clearly, $\operatorname{supp} m_\sigma a \subset B(0, r+1)$. Using m as before, we obtain

$$\int (m_\sigma a)^p\,d\mu \leqslant c\|a\|_1^p \int_0^{\mu(B(0,r+1))} t^{-p}\,dt \leqslant c\big(\mu(B(0, r+1))/\mu(B(0, 2r))\big)^{1-p} < c,$$

where we used (1.1) to get the last inequality, in case $r < 1$.

This proves the lemma when $p < 1$. In the opposite case, $q > 1$ and m is bounded on L^q. Then it is enough to replace weak L^1 by L^q in the preceding, and use Hölder's inequality instead of decreasing rearrangements.

Lemma 3.1 and Theorem 3.1 are proved.

3.2. A converse result.

THEOREM 3.2. *Let μ satisfy* (1.1) *and assume that F has the Markov property. If $f \in L^1_{\mathrm{loc}}(\mu)$ satisfies $m_\sigma f \in L^p(\mu)$ for some σ, where $0 < p \leqslant 1$, then $f \in h^{p,\infty,s}$ for any $s > 0$, and*

$$\|f\|_{h^{p,\infty,s}} \leqslant c\|m_\sigma f\|_p, \qquad c = c(F, \mu, p, s, \sigma).$$

Notice that f is identified with the distribution $f d\mu$ here. Similarly, equality or convergence in the distribution sense for functions in $L^1_{\mathrm{loc}}(\mu)$ will in this section refer to the distributions obtained when the functions are multiplied by $d\mu$.

COROLLARY 3.1. *Let μ and F be as in Theorem 3.2. The space $h^{p,q,s}$ is independent of q and s, with equivalence of norms, provided p, q are admissible and s is large enough. When F is also a d-set, $h^{p,q}$ is similarly independent of q and of the d-measure μ.*

We may thus write $h^p = h^p(F, \mu)$, and in the case of a d-set $h^p = h^p(F)$. The corollary follows if we apply the theorem and Lemma 3.1 to local (p, q, s)- and (p, q)-atoms. Notice here that $L^p(\mu)$ is independent of the d-measure μ in the d-set case. The authors do not know whether Theorem 3.2 holds for arbitrary distributions f supported on F.

Proof of Theorem 3.2. Our method is taken from Latter–Uchiyama [11] Theorem 1, see also [13], Theorem 3.5. We start by introducing Whitney balls and partitions of unity.

LEMMA 3.2. *If Ω is a relatively open proper subset of F, then there exist balls $B_i = B(x_i, r_i)$, $x_i \in F$, $i = 1, 2, \ldots$ such that $\Omega = \bigcup (F \cap B_i)$, the doubled balls $2B_i = B(x_i, 2r_i)$ have bounded overlap, i.e. no x can belong to more than c balls $2B_i$, and*

$$\frac{\operatorname{dist}(B_i, F\setminus\Omega)}{cM} < r_i < \frac{\operatorname{dist}(B_i, F\setminus\Omega)}{M} \tag{3.2}$$

where $M > 4$ is a preassigned constant. Further, there exist non-negative functions $\varphi_i \in C_0^\infty(R^n)$ such that $\sum \varphi_i = 1$ on Ω, $\varphi_i \sim 1$ on B_i, $\operatorname{supp}\varphi_i \subset 2B_i$, and

$$|D^\beta \varphi_i| \leqslant c_\beta r_i^{-|\beta|}, \quad |\beta| > 0. \tag{3.3}$$

To find the B_i, one covers $R^n\setminus(F\setminus\Omega)$ by Whitney cubes and replaces those cubes which intersect F by suitable balls centered on F. We omit the rest of the proof.

Now let $f \in L^1_{\mathrm{loc}}(\mu)$ with $m_\sigma f \in L^p(\mu)$. We apply Lemma 3.2 to those sets $\Omega_k = \{x \in F\colon m_\sigma f > 2^k\}$, $k \in Z$, which are not equal to F, obtaining $B_i^k = B(x_i^k, r_i^k)$ and φ_i^k, $i = 1, 2, \ldots$ Call the doubled ball $2B_i^k$ of *type I* if it is contained in a unit ball $B(y_i^k, 1)$, for some $y_i^k \in F\setminus\Omega_k$. Otherwise, $2B_i^k$ is of *type II*.

Fixing a ball $2B_i^k$ of type I, we let V_i^k denote the subspace of $L^2(\varphi_i^k\,d\mu)$ given by polynomials of degree at most s. Let $\pi_1, \ldots, \pi_L$ be an orthonormal basis in V_i^k, and define the projection of f into V_i^k by

$$P_i^k = \sum_l \pi_l \int f\pi_l\varphi_i^k\,d\mu.$$

LEMMA 3.3. *We have $|P_i^k| \leqslant c2^k$ in $2B_i^k$.*

Proof. Since $\varphi_i^k \sim 1$ in B_i^k, the L^2 norm of each π_l with respect to the normalized measure $d\mu/\mu(B_i^k)$ restricted to B_i^k is at most $c\mu(B_i^k)^{-1/2}$. By

Theorem 1.1, this implies

$$|D^\beta \pi_l| \leqslant c\mu(B_i^k)^{-1/2}(r_i^k)^{-|\beta|}, \quad |\beta| \geqslant 0, \tag{3.4}$$

in $2B_i^k$. Because of (3.3), we get similar bounds for the derivatives of $\pi_l \varphi_i^k$. We can find $y_i^k \in F \setminus \Omega_k$ such that

$$\operatorname{supp} \pi_l \varphi_i^k \subset B(y_i^k, cr_i^k)$$

because of (3.2). And $cr_i^k \leqslant 1$ since $2B_i^k$ is of type I. It follows that we may write

$$\pi_l(x)\varphi_i^k(x) = \mu(B_i^k)^{-1/2}\psi\left(\frac{y_i^k - x}{cr_i^k}\right)$$

with $\psi \in cS_\sigma$. Since $\mu(B(y_i^k, cr_i^k)) \leqslant c\mu(B_i^k)$ because of (1.1) (compare with $B(x_i^k, 2)$ if r_i^k is close to 1), we conclude

$$\left|\int f\pi_l \varphi_i^k \, d\mu\right| \leqslant c\mu(B_i^k)^{1/2} m_\sigma f(y_i^k) \leqslant c\mu(B_i^k)^{1/2} 2^k.$$

From this and (3.4), the lemma follows.

When $2B_i^k$ is of type II, let $P_i^k = 0$. Following [11], p. 393, we now denote by P_{ij}^{k+1} the projection of $(f - P_j^{k+1})\varphi_i^k$ into V_j^{k+1}, if $2B_j^{k+1}$ is of type I. Otherwise $P_{ij}^{k+1} = 0$. As in Lemma 3.3, one can show that

$$|P_{ij}^{k+1}| \leqslant c2^k \tag{3.5}$$

in $2B_j^{k+1}$. Notice that

$$\sum_i P_{ij}^{k+1} = 0. \tag{3.6}$$

Write $f = g_k + b_k$ on F, where

$$g_k = f\chi_{F \setminus \Omega_k} + \sum_i P_i^k \varphi_i^k \tag{3.7}$$

and thus

$$b_k = \sum_i (f - P_i^k)\varphi_i^k,$$

if $\Omega_k \neq F$, and $g_k = 0$, $b_k = f$ if $\Omega_k = F$. This gives a preliminary decomposition of f:

Lemma 3.4. *In the distribution sense,*

$$f = \sum_{k=-\infty}^{+\infty} (g_{k+1} - g_k) = \sum_k (b_k - b_{k+1}).$$

Proof. We need only verify that $g_k \to 0$ as $k \to -\infty$ and $g_k \to f$ as $k \to +\infty$. Now m_σ is of weak type (1,1), which via differentiation of integrals

implies that $|f| \leqslant cm_\sigma f$ μ-a.e. Because of Lemma 3.3, $|g_k| \leqslant c2^k$, so $g_k \to 0$ as $k \to -\infty$, μ-a.e. and in the sense of distributions.

As $k \to +\infty$, $\mu(\Omega_k) \to 0$, so the first term in (3.7) tends to f μ-a.e. and in the distribution sense. To show that the second term tends to 0, we take a $\psi \in C_0^\infty$ and denote by pr the projection operator into V_i^k. Then pr is symmetric and

$$\int P_i^k \psi \varphi_i^k \, d\mu = \int f \operatorname{pr} \psi \varphi_i^k \, d\mu.$$

But $\left|\int \psi \pi_l \varphi_i^k \, d\mu\right| \leqslant c\mu(B_i^k)^{1/2} \sup|\psi|$ because of (3.4), so

$$|\operatorname{pr}\psi| = \left|\sum_l \pi_l \int \psi \pi_l \varphi_i^k \, d\mu\right| \leqslant c \sup|\psi|.$$

Since $\sum_i \varphi_i^k \leqslant c$, this gives

$$\sum_i \left|\int P_i^k \psi \varphi_i^k \, d\mu\right| \leqslant c \int |f| \, d\mu \cdot \sup|\psi|,$$

where the last integral is taken over the union of those $2B_i^k$ of type I which intersect $\operatorname{supp}\psi$. This union is contained in a fixed compact set, and its μ-measure tends to 0 as $k \to +\infty$. It follows that

$$\sum_i \left|\int P_i^k \psi \varphi_i^k \, d\mu\right| \to 0, \quad k \to +\infty,$$

and so $g_k \to f$ in the distribution sense as $k \to +\infty$. Lemma 3.4 is proved.

Assuming to begin with that no Ω_k is F, we have

$$b_k - b_{k+1} = \sum_i \left[(f - P_i^k)\varphi_i^k - \sum_j (f - P_j^{k+1})\varphi_i^k \varphi_j^{k+1}\right],$$

since $\sum_i \varphi_i^k = 1$ in $\Omega_k \supset \Omega_{k+1}$. So because of (3.6) and Lemma 3.4, $f = \sum_{k,i} \beta_i^k$, where

$$\beta_i^k = (f - P_i^k)\varphi_i^k - \sum_j \left((f - P_j^{k+1})\varphi_i^k - P_{ij}^{k+1}\right)\varphi_j^{k+1}.$$

We shall now see that these β_i^k are essentially the atom multiples we are looking for.

The terms in β_i^k containing f cancel in Ω_{k+1}, since $\sum_j \varphi_j^{k+1} = 1$ there. Off Ω_{k+1} we have $|f| \leqslant c2^k$ μ-a.e. Because of Lemma 3.3 and (3.5) it follows that $|\beta_i^k| \leqslant c2^k$. Notice that P_{ij}^{k+1} vanishes when $2B_i^k$ and $2B_j^{k+1}$ are disjoint. Therefore, the support of β_i^k is seen to be contained in the union of $2B_i^k$ and those balls $2B_j^{k+1}$ intersecting it. Since such $2B_j^{k+1}$ must have radii at most cr_i^k/M because of (3.2) and the fact that $\Omega_{k+1} \subset \Omega_k$, this union is contained in a ball $\hat{B}_i^k = B(x_i^k, cr_i^k)$. We may assume that the balls $(2\hat{B}_i^k)_i$ do not intersect $F \setminus \Omega_k$ and have bounded overlap, by choosing a suitable M in (3.2).

Because of (3.2), there exists a $\varrho = \varrho(M) > 0$ such that if $r_i^k \leqslant \varrho$, then $2B_i^k$ and all balls $2B_j^{k+1}$ intersecting it are of type I. For these r_i^k, the β_i^k will have vanishing moments up to order s. Then we may write $\beta_i^k = \lambda_i^k a_i^k$, where

$$\lambda_i^k = c2^k \mu(2\hat{B}_i^k)^{1/p}$$

and a_i^k is a local (p, ∞, s)-atom with supporting ball $\hat{B}_i^k$. When $r_i^k > 1$, no moment condition is necessary, so we define λ_i^k and a_i^k in the same way. For $\varrho < r_i^k \leqslant 1$, we avoid moment conditions by covering supp β_i^k by at most c unit balls $B(x_\nu, 1)$ with $x_\nu \in F \cap \hat{B}_i^k$. Write $\beta_i^k = \sum_\nu \beta_{i,\nu}^k$, where $\operatorname{supp} \beta_{i,\nu}^k \subset B(x_\nu, 1)$ and $|\beta_{i,\nu}^k| \leqslant c2^k$. With $\lambda_{i,\nu}^k = c2^k \mu\big(B(x_\nu, 2)\big)^{1/p}$, we may write $\beta_{i,\nu}^k = \lambda_{i,\nu}^k a_{i,\nu}^k$. This $a_{i,\nu}^k$ will be a local (p, ∞, s)-atom if we let its supporting ball be $B(x_\nu, 1+\varepsilon)$, where ε is so small that

$$\mu\big(B(x_\nu, 2(1+\varepsilon))\big) \leqslant 2\mu\big(B(x_\nu, 2)\big).$$

Altogether, we have

$$f = \sum_{k,i} \lambda_i^k a_i^k + \sum_{k,i,\nu} \lambda_{i,\nu}^k a_{i,\nu}^k,$$

where the first sum is over those k, i for which $r_i^k \leqslant \varrho$ or $r_i^k > 1$, and the second sum over the remaining k, i and at most c values of ν. For λ_i^k we have

$$\sum_{k,i} |\lambda_i^k|^p \leqslant c \sum_{k,i} 2^{kp} \mu(2\hat{B}_i^k) \leqslant c \sum_k 2^{kp} \mu(\Omega_k) \leqslant c \int (m_\sigma f)^p \, d\mu,$$

since the balls $(2\hat{B}_i^k)_i$ have bounded overlap. For $\lambda_{i,\nu}^k$, notice first that

$$B(x_\nu, 1) \subset B\big(x_i^k, \max(2, 2\hat{r}_i^k)\big)$$

where $\hat{r}_i^k$ is the radius of $\hat{B}_i^k$. And $\varrho < r_i^k \leqslant 1$. Because of the doubling condition, we may therefore estimate $\mu\big(B(x_\nu, 1)\big)$ and thus $\mu\big(B(x_\nu, 2)\big)$ by $c\mu(2\hat{B}_i^k)$. This implies that the $\lambda_{i,\nu}^k$ can be estimated like the λ_i^k. It follows that $f \in h^{p,\infty,s}$, and we get the right estimate for the quasi-norm of f.

The case where some $\Omega_k = F$ remains. Denoting by k' the first k with $\Omega_k \neq F$, it is enough to decompose $g_{k'} = g_{k'} - g_{k'-1}$ into atoms since the differences $g_{k+1} - g_k$, $k \geqslant k'$ can be treated as before. Cover F with unit balls B_ν centered on F and having bounded overlap. Since $|g_{k'}| \leqslant c2^{k'}$ as before, we may write $g_{k'} = \sum \alpha_\nu$, where α_ν has support contained in B_ν and $|\alpha_\nu| \leqslant c2^{k'}$. Choose balls B'_ν concentric to B_ν with slightly larger radii so that

(3.8) $$\mu(2B'_\nu) \leqslant 2\mu(2B_\nu).$$

Then $\alpha_\nu = c2^{k'} \mu(2B'_\nu)^{1/p} a_\nu$ for atoms a_ν with supporting balls B'_ν, so that no moment conditions apply. For the coefficients here, we use (3.8) and the bounded overlap of the B_ν to get

$$\sum_\nu 2^{k'p} \mu(2B'_\nu) \leqslant c \int (m_\sigma f)^p \, d\mu$$

since $m_\sigma f > 2^{k'-1}$ in $\Omega_{k'-1} = F$.

This ends the proof of Theorem 3.2.

4. Duality

4.1. The Lipschitz spaces $\Lambda_\alpha(F)$. These spaces have been studied for arbitrary closed sets F in $\boldsymbol{R}^n$, but we define them here only when F has the Markov property. The characterization of $\Lambda_\alpha(F)$ which we take as definition here was given in [9]; there we used cubes instead of balls, but it is easily seen that balls give equivalent spaces.

DEFINITION 4.1. Let $\alpha > 0$, and let F be a set having the Markov property. A function f defined on F belongs to $\Lambda_\alpha(F)$ if and only if there is a constant M such that

(a) for every ball $B = B(x_0, r)$, $x_0 \in F$, $r \leqslant 1$, there exists a polynomial P_B of degree at most $[\alpha]$ such that

(4.1) $$|f(x) - P_B(x)| \leqslant Mr^\alpha, \quad x \in B \cap F,$$

(b)

(4.2) $$|f(x)| \leqslant M, \quad x \in F.$$

The *norm* of f in $\Lambda_\alpha(F)$ is the infimum of the constants M.

For $F = \boldsymbol{R}^n$, the spaces $\Lambda_\alpha(F)$ coincide with the classical Lipschitz spaces $\Lambda_\alpha(\boldsymbol{R}^n)$, defined in § 0.1, see [9]. A basic property of the spaces $\Lambda_\alpha(F)$ is the following trace property: The pointwise restriction to F of a function in $\Lambda_\alpha(\boldsymbol{R}^n)$ belongs to $\Lambda_\alpha(F)$ (this is obvious), and, conversely, every function in $\Lambda_\alpha(F)$ may be extended, by means of a bounded extension operator, to a function in $\Lambda_\alpha(\boldsymbol{R}^n)$. This result, which for non-integer α is a version of the Whitney extension theorem, is given (for an arbitrary F) in [9].

In the proof of the duality theorem for h^p spaces, it is convenient to have characterizations of $\Lambda_\alpha(F)$ for d-sets F where one approximates with polynomials in the $L^p(\mu)$-norm instead of the maximum norm. We give a result of that kind, and to describe it, we introduce certain spaces $\Lambda_\alpha^q(F)$.

DEFINITION 4.2. Let F be a d-set, μ a d-measure on F, $\alpha > 0$ and $1 \leqslant q \leqslant \infty$, or $\alpha = 0$ and $1 \leqslant q < \infty$. A function f defined μ-a.e. on F belongs to $\Lambda_\alpha^q(F)$ if and only if there is a constant M such that for every ball B

$= B(x_0, r)$, $x_0 \in F$, $r \leqslant 1$, there exists a polynomial P_B of degree at most $[\alpha]$ such that if $r \leqslant 1$, then

$$\left(\frac{1}{\mu(B)} \int_B |f-P_B|^q \, d\mu\right)^{1/q} \leqslant Mr^\alpha, \tag{4.3}$$

and if $r = 1$, then

$$\left(\frac{1}{\mu(B)} \int_B |f|^q \, d\mu\right)^{1/q} \leqslant M. \tag{4.4}$$

The *norm* of $f \in \Lambda_\alpha^q(F)$ is defined as the infimum of all possible constants M. Different d-measures μ on F give equivalent norms, and as a canonical choice for the norm in $\Lambda_\alpha^q(F)$, one may take the restriction to F of the d-dimensional Hausdorff measure, cf. § 1.1.

Remark. Using a covering argument, one obtains from (4.4) that

$$\Big(\int_B |f|^q \, d\mu\Big)^{1/q} \leqslant cM\big(\mu(2B)\big)^{1/q} \quad \text{for balls } B = B(x_0, r),\ x_0 \in F,\ r > 1.$$

For $F = \mathbf{R}^n$ the space $\Lambda_0^1(F)$ is the bmo space introduced in [5], and in analogy with this we will below also denote the spaces $\Lambda_0^q(F)$ by $\mathrm{bmo}_q(F)$ and $\Lambda_0^1(F)$ by $\mathrm{bmo}(F)$.

The following theorem is well known in many special cases, cf. [1].

THEOREM 4.1. *Let F be a d-set with the Markov property, $\alpha > 0$, $1 \leqslant q_1, q_2 \leqslant \infty$. Then*

$$\Lambda_\alpha^{q_1}(F) = \Lambda_\alpha^{q_2}(F)$$

with equivalent norms.

Proof. Since it is readily seen that $\Lambda_\alpha^{q_2} \subset \Lambda_\alpha^{q_1}$ continuously if $q_1 < q_2$, it is enough to verify that $\Lambda_\alpha^1 \subset \Lambda_\alpha^\infty$ (then $\Lambda_\alpha^{q_1} \subset \Lambda_\alpha^1 \subset \Lambda_\alpha^\infty \subset \Lambda_\alpha^{q_2}$, $q_1 < q_2$). So take f in Λ_α^1 with norm 1, and let B be a ball of radius $r \leqslant 1$ centered on F. Let P be a polynomial of degree $\leqslant [\alpha]$ such that

$$\frac{1}{\mu(2B)} \int_{2B} |f-P| \, d\mu \leqslant cr^\alpha$$

(if $2r > 1$ one takes $P = 0$).

Take x in $B \cap F$, and put $B_k = B(x, 2^{-k}r)$, $k \geqslant 0$. Choose a polynomial P_k of degree $\leqslant [\alpha]$ so that

$$\frac{1}{\mu(B_k)} \int_{B_k} |f-P_k| \, d\mu \leqslant c(2^{-k}r)^\alpha, \tag{4.5}$$

then for $k \geqslant 1$

$$\frac{1}{\mu(B_k)} \int_{B_k} |P_k - P_{k-1}| \, d\mu \leqslant \frac{1}{\mu(B_k)} \int_{B_k} |f-P_k| \, d\mu + \frac{\mu(B_{k-1})}{\mu(B_k)} \cdot \frac{1}{\mu(B_{k-1})} \int_{B_{k-1}} |f-P_{k-1}| \, d\mu \leqslant c(2^{-k}r)^\alpha.$$

Using III in Theorem 1.1 we get

$$\max_{F \cap B_k} |P_k - P_{k-1}| \leqslant c2^{-k\alpha} r^\alpha. \tag{4.6}$$

Hence, using (4.5) and (4.6),

$$\left|\frac{1}{\mu(B_k)} \int_{B_k} (f-P_0) \, d\mu\right| \leqslant \frac{1}{\mu(B_k)} \int_{B_k} |f-P_0| \, d\mu \leqslant \frac{1}{\mu(B_k)} \int_{B_k} |f-P_k| \, d\mu + \sum_{\nu=1}^{k} \frac{1}{\mu(B_k)} \int_{B_k} |P_\nu - P_{\nu-1}| \, d\mu \leqslant c2^{-k\alpha} r^\alpha + \sum_{\nu=1}^{k} c2^{-\nu\alpha} r^\alpha \leqslant cr^\alpha,$$

and letting k tend to infinity, we obtain for μ-almost all x in $B \cap F$ $|f(x) - P_0(x)| \leqslant cr^\alpha$. Then we also have $|f(x) - P(x)| \leqslant |f(x) - P_0(x)| + |P_0(x) - P(x)| \leqslant cr^\alpha$ μ-a.e. in $B \cap F$, since $|P_0(x) - P(x)| \leqslant cr^\alpha$, $x \in B \cap F$, may be obtained simply in a similar way as we obtained (4.6) (consider $1/\mu(B_0) \int_{B_0} |P_0 - P| \, d\mu$). Thus (4.3) holds for $q = \infty$, and to prove (4.4), consider a ball $B = B(x_0, 1)$, $x_0 \in F$, and let P be as above. Then, as we have seen, $|f(x)| = |f(x) - P(x)| \leqslant c$ μ-a.e. in B, and the proof is complete.

Obviously, a function in $\Lambda_\alpha(F)$ belongs to $\Lambda_\alpha^\infty(F)$. The converse holds in the following sense.

PROPOSITION 4.1. *Let F be a d-set having the Markov property and $\alpha > 0$. Then every function in $\Lambda_\alpha^\infty(F)$ may be changed on a set of μ-measure zero so that it becomes a function in $\Lambda_\alpha(F)$, with norm in $\Lambda_\alpha(F)$ equal to the norm in $\Lambda_\alpha^\infty(F)$.*

Proof. Let $\psi \not\equiv 0$ be a nonnegative function in $C^\infty(\mathbf{R}^n)$ supported in $B(0, 1)$, put

$$\varphi(x, y, t) = \left\{\int \psi\left(\frac{x-u}{t}\right) d\mu(u)\right\}^{-1} \cdot \psi\left(\frac{x-y}{t}\right), \qquad x, y \in F,\ t > 0,$$

and define for $f \in \Lambda_\alpha^\infty(F)$ f_t by

$$f_t(x) = \int f(y)\, \varphi(x, y, t) \, d\mu(y), \qquad x \in F.$$

Then the functions f_t are continuous, and we shall see that $\{f_t\}$ converges uniformly as $t \to 0$, and that $f_t(x) \to f(x)$ μ-a.e., $t \to 0$, which proves the theorem. Take $0 < t_1 < t_2 < 1$. Using $\int \varphi(x, y, t)\, d\mu(y) = 1$ we get

$$|f_{t_1}(x) - f_{t_2}(x)| = \iint \{f(y) - f(z)\}\, \varphi(x, y, t_1)\, \varphi(x, z, t_2)\, d\mu(y)\, d\mu(z).$$

Taking $B = B(x, t_2)$, $\beta < 1$, $\beta < \alpha$, and P_B as in the definition of Λ_α^∞ we obtain

$$\begin{aligned}|f(y) - f(z)| &= |f(y) - P_B(y)| + |P_B(y) - P_B(z)| + |P_B(z) - f(z)| \\ &\leqslant ct_2^\alpha + ct_2^\beta + ct_2^\alpha \leqslant ct_2^\beta\end{aligned}$$

for μ-almost all y and z in $B(x, t_2) \cap F$ (for the estimate of $|P_B(y) - P_B(z)|$, see below). Thus, since the integrand is non-zero only for z, $y \in B(x, t_2)$, we get $|f_{t_1}(x) - f_{t_2}(x)| \leqslant ct_2^\beta$, so we have uniform convergence.

Let next $x_0 \in F$. Then, e.g. for $r \leqslant t < 1/2$,

$$\begin{aligned}&\frac{1}{(B(x_0, r))} \int\limits_{B(x_0, r)} |f_t(x) - f(x)|\, d\mu(x) \\ &= \frac{1}{\mu(B(x_0, r))} \int\limits_{B(x_0, r)} \left|\int \{f(y) - f(x)\}\, \varphi(x, y, t)\, d\mu(y)\right| d\mu(x) \leqslant ct^\beta,\end{aligned}$$

and letting r tend to zero, we get $|f_t(x) - f(x)| \leqslant c|t|^\beta$ μ-almost all x. As a consequence, $f_{t_n} \to f$ μ-a.e. for some sequence $t_n \to 0$ and hence $f_t \to f$ μ-a.e.

The estimate $|P_B(z) - P_B(y)| \leqslant c|z - y|^\beta$, $z, y \in B = B(x, r)$, $\beta < 1$, $r < 1$ follows from the mean value theorem and the estimate $|\mathrm{grad}\, P_B| \leqslant c$ in B, $\alpha > 1$ and $|\mathrm{grad}\, P_B| \leqslant c|\ln r|$, $\alpha = 1$.

These estimates are easily obtained with the aid of Markov's inequality, upon writing $P_B = \sum_{i=1}^{n} (P_{B_{i-1}} - P_{B_i}) + P_{B_n}$, where $B_i = B(x, 2^i r)$ and n is the integer such that $\frac{1}{2} < 2^n r \leqslant 1$.

4.2. Local Hardy spaces as subsets of $(\Lambda_\alpha(F))'$. Let F be a d-set with the Markov property and μ a d-measure on F. We put $\Lambda_0(F) = \mathrm{bmo}(F) = \Lambda_0^1(F)$ and $(\Lambda_\alpha(F))'$ denotes the dual space of $\Lambda_\alpha(F)$. We need

LEMMA 4.1. *If $0 < p \leqslant 1$ and $\alpha = d(1/p - 1)$, then*

$$\left|\int a\varphi\, d\mu\right| \leqslant c\, \|\varphi\|_{\Lambda_\alpha(F)},$$

for all $\varphi \in \Lambda_\alpha(F)$ and all local p-atoms a on F. Here c is independent of φ and a.

Proof. Let $B = B(x_0, r)$, $x_0 \in F$, be a supporting ball of a and P a

polynomial of degree at most $[\alpha]$. By the definitions of local p-atoms, $\Lambda_\alpha(F)$, and d-measure, we get, if P is suitably chosen and $r \leqslant 1$,

$$\begin{aligned}\left|\int a\varphi\, d\mu\right| &= \left|\int\limits_B a(\varphi - P)\, d\mu\right| \leqslant \|a\|_{L^\infty(\mu)} \|\varphi - P\|_{L^1(\mu, B)} \leqslant c\mu(2B)^{-1/p} \|\varphi\|_{\Lambda_\alpha(F)} r^\alpha r^d \\ &\leqslant cr^{-d/p + d + \alpha} \|\varphi\|_{\Lambda_\alpha(F)} = c\, \|\varphi\|_{\Lambda_\alpha(F)}.\end{aligned}$$

For $r > 1$ we get analogously without using P,

$$\left|\int a\varphi\, d\mu\right| \leqslant \mu(2B)^{-1/p} \int\limits_B |\varphi|\, d\mu \leqslant c\, \|\varphi\|_{\Lambda_\alpha(F)},$$

since $\mu(B) \geqslant c' > 0$. This proves the lemma.

Now, assume that a_j, $j = 1, 2, \ldots$, are local p-atoms on F, $0 < p \leqslant 1$, and that λ_j are complex numbers such that $\sum |\lambda_j|^p < \infty$. The element $f = \sum \lambda_j a_j \in h^p(F) \subset \mathscr{D}'$ is determined by the formula $\langle f, \varphi \rangle = \sum \lambda_j \int a_j \varphi\, d\mu$, for $\varphi \in C_0^\infty(\boldsymbol{R}^n)$. For the duality theorem below (Theorem 4.2) it is essential that this formula in a unique way may be extended to hold for all $\varphi \in \Lambda_\alpha(F)$. However, from Lemma 4.1 it follows that $\sum \lambda_j a_j$ also defines an element $f_1 \in (\Lambda_\alpha(F))'$ given by $\langle f_1, \varphi \rangle = \sum \lambda_j \int a_j \varphi\, d\mu$ for all $\varphi \in \Lambda_\alpha(F)$, $\alpha = d(1/p - 1)$. Since $\varphi \in C_0^\infty(\boldsymbol{R}^n)$ implies $\varphi | F \in \Lambda_\alpha(F)$, f is uniquely determined by f_1. Conversely we shall prove

LEMMA 4.2. *f_1 is uniquely determined by f.*

From this lemma it follows that we may identify $f = \sum \lambda_j a_j \in h^p(F) \subset \mathscr{D}'$ with $f_1 = \sum \lambda_j a_j \in (\Lambda_\alpha(F))'$, where $\alpha = d(1/p - 1)$. If $f = \sum \lambda_j a_j \in h^p(F) \subset \mathscr{D}'$, then f is uniquely determined on $\Lambda_\alpha(F)$ by $\langle f, \varphi \rangle = \sum \lambda_j \int a_j \varphi\, d\mu$, $\varphi \in \Lambda_\alpha(F)$, and Lemma 4.1 gives, since $\sum |\lambda_j| \leqslant (\sum |\lambda_j|^p)^{1/p}$,

$$(4.7) \qquad |\langle f, \varphi \rangle| \leqslant c\, \|f\|_{h^p(F)} \|\varphi\|_{\Lambda_\alpha(F)}, \qquad \alpha = d(1/p - 1),$$

i.e. $h^p(F) \subset (\Lambda_\alpha(F))'$ with continuous imbedding.

Proof of Lemma 4.2. (1) *The case $p < 1$.* We take a $\varphi \in \Lambda_\alpha(F)$. Since $\Lambda_\alpha(F)$ is the trace to F of $\Lambda_\alpha(\boldsymbol{R}^n)$, we may assume that $\varphi \in \Lambda_\alpha(\boldsymbol{R}^n)$. This allows us to follow the method in [13], § 5. We first assume that $\mathrm{supp}\, \varphi$ is compact. Take a nonnegative function $\eta \in C_0^\infty(\boldsymbol{R}^n)$ with support in $B(0, 1)$ satisfying $\int \eta = 1$, and put $\eta_t(x) = t^{-n} \eta(x/t)$, for $t > 0$. Then $\eta_t * \varphi \in C_0^\infty(\boldsymbol{R}^n)$ and the norm, in the space $\Lambda_\alpha(\boldsymbol{R}^n)$, of $\eta_t * \varphi$ is less than or equal to the norm of φ, for $t > 0$. By dominated convergence and Lemma 4.1 this gives, as $t \to 0$,

$$\langle f, \eta_t * \varphi \rangle = \sum \lambda_j \int a_j \eta_t * \varphi\, d\mu \to \sum \lambda_j \int a_j \varphi\, d\mu = \langle f_1, \varphi \rangle,$$

i.e. the value of f_1 on $\Lambda_\alpha(\boldsymbol{R}^n) \cap \{\varphi\colon \mathrm{supp}\, \varphi\ \mathrm{compact}\}$ is uniquely determined by f. If $\mathrm{supp}\, \varphi$ is not compact we introduce a function $m \in C_0^\infty(\boldsymbol{R}^n)$ which is 1 in $B(0, 1/2)$ and 0 outside $B(0, 1)$, and define $m^R(x) = m(x/R)$, $R > 1$. If we can prove that, for some constant c independent of R and φ,

$$\|m^R \varphi\|_{\Lambda_\alpha(\mathbf{R}^n)} \leqslant c\|\varphi\|_{\Lambda_\alpha(\mathbf{R}^n)}, \tag{4.8}$$

then we get by dominated convergence, as $R \to \infty$,

$$\langle f_1, m^R \varphi\rangle = \sum \lambda_j \int a_j m^R \varphi\, d\mu \to \sum \lambda_j \int a_j \varphi\, d\mu = \langle f_1, \varphi\rangle,$$

i.e. f_1 is uniquely determined by its values on $\Lambda_\alpha(\mathbf{R}^n) \cap \{\varphi\colon \operatorname{supp}\varphi \text{ compact}\}$ and hence by f. Consequently, Lemma 4.2 follows for $p < 1$ if we prove (4.8). We do that when α is an integer which is the more difficult case. We then assume that $|\Delta_h^2(D^j\varphi(x))| \leqslant M|h|$, for $|j| = \alpha - 1$, $x, h \in \mathbf{R}^n$, and that $|D^j\varphi(x)| \leqslant M$, for $|j| \leqslant \alpha - 1$, $x \in \mathbf{R}^n$; the norm of φ in $\Lambda_\alpha(\mathbf{R}^n)$ is the infimum of the constants M. We have to prove that

$$|\Delta_h^2(D^\beta m^R D^\gamma \varphi)(x)| \leqslant cM|h| \quad \text{for} \quad |\beta| + |\gamma| = \alpha - 1,\ x,\ h \in \mathbf{R}^n,$$

$$|(D^\beta m^R D^\gamma \varphi)(x)| \leqslant cM \quad \text{for} \quad |\beta| + |\gamma| \leqslant \alpha - 1,\ x \in \mathbf{R}^n.$$

The second inequality follows since $|D^\beta m^R| \leqslant cR^{-|\beta|}$, and the first follows immediately if we estimate each term in the right-hand member of the formula

$$\Delta_h^2(GH)(x) = \Delta_h^2 G(x) H(x) + 2\Delta_h G(x+h)\Delta_h H(x) + G(x+2h)\Delta_h^2 H(x)$$

used with $G = D^\beta m^R$ and $H = D^\gamma \varphi$. This proves (4.8) when α is an integer; the case where α is not an integer is simpler since we then have to use first differences only in the definition of $\Lambda_\alpha(\mathbf{R}^n)$.

(2) *The case* $p = 1$. We have $f = \sum \lambda_j a_j \in h^1(F) \subset L^1(\mu)$ and $\langle f, \varphi\rangle = \sum \lambda_j \int a_j \varphi\, d\mu$, for all $\varphi \in L^\infty(\mu)$. Hence, if $\varphi \in \Lambda_0(F) = \operatorname{bmo}(F)$ is bounded, then $\langle f_1, \varphi\rangle = \langle f, \varphi\rangle$. If $\varphi \in \Lambda_0(F)$ is not bounded we introduce, for constants $M \geqslant 1$, φ_M by $\operatorname{Re}\varphi_M = \sup(\inf(\operatorname{Re}\varphi, M), -M)$ and similarly for $\operatorname{Im}\varphi_M$. Then it is easy to show, compare [13], Lemma 5.7, that the norm in $\Lambda_0(F)$ of φ_M is less than a constant, independent of M, times the norm of φ. This gives by dominated convergence as $M \to \infty$,

$$\langle f, \varphi_M\rangle = \sum \lambda_j \int a_j \varphi_M\, d\mu \to \sum \lambda_j \int a_j \varphi\, d\mu = \langle f_1, \varphi\rangle,$$

i.e. f_1 is uniquely determined by f. Lemma 4.2 is proved.

4.3. The duality theorem. The discussion in Section 4.2 (see formula (4.7)) gives one half of the following theorem where $(h^p(F))'$ denotes the dual of $h^p(F)$.

THEOREM 4.2. *Let F be a d-set having the Markov property. If $0 < p \leqslant 1$ and $\alpha = d(1/p - 1)$, then*

$$(h^p(F))' = \Lambda_\alpha(F),$$

with equivalent norms; here $\Lambda_0(F) = \operatorname{bmo}(F)$. The duality holds in the sense that every $\varphi \in \Lambda_\alpha(F)$ defines an element in $(h^p(F))'$ by the formula

$$\langle f, \varphi\rangle = \sum \lambda_j \int a_j \varphi\, d\mu, \quad f = \sum \lambda_j a_j \in h^p(F),$$

and that conversely every element in $(h^p(F))'$ is given by this formula by a unique $\varphi \in \Lambda_\alpha(F)$.

Proof. Let L be a bounded linear functional on h^p with $\|L\| = 1$. Take $B = B(x_0, r)$, $x_0 \in F$, $r > 1$, and $1 < q < \infty$, and let $l \in L^q(\mu)$ be a non-zero function supported in B. Then $a = \mu(2B)^{-1/p}\mu(B)^{1/q}\|l\|_q^{-1} l$ is a local (p, q)-atom, and thus $|La| \leqslant \|a\|_{h^p} \leqslant c\|a\|_{h^{p,q}} \leqslant c$ since h^p and $h^{p,q}$ are equivalent (Corollary 3.1). Consequently, $|Ll| \leqslant c\mu(2B)^{1/p}\mu(B)^{-1/q}\|l\|_q$, $l \in L^q(\mu, B)$, and thus there is a function φ in $L^{q'}(\mu, B)$, $1/q + 1/q' = 1$, with norm $\leqslant c\mu(2B)^{1/p}\mu(B)^{-1/q} \leqslant c$ if $r < 2$, so that $Ll = \int l\varphi\, d\mu$, $l \in L^q(\mu, B)$. Constructing functions φ_1 and φ_2 corresponding to different balls B_1 and B_2 in this way, the functions φ_1 and φ_2 must coincide μ-a.e. on $B_1 \cap B_2 \cap F$, and thus we can obtain a function φ such that $Ll = \int l\varphi\, d\mu$ if $l \in L^q(\mu)$ and l has compact support. In particular, $La = \int a\varphi\, d\mu$ if a is a local p-atom, and it is easy to see that φ is the only locally integrable function having this property.

Consider next a ball $B = B(x_0, F)$, $x_0 \in F$, $r \leqslant 1$, and put $\delta = \inf(\int_B |\varphi - P|^{q'}\, d\mu)^{1/q'}$, where the infimum is taken over all polynomials P of degree $\leqslant [\alpha]$. Then there is a bounded linear functional K on $L^{q'}(\mu, B)$ such that $K(\varphi) = 1$, $K(P) = 0$ if P is a polynomial of degree $\leqslant [\alpha]$, and $\|K\| = 1/\delta$. Let g be the function in $L^q(\mu, B)$ with norm $1/\delta$ such that $Kh = \int hg\, d\mu$, $h \in L^{q'}(\mu, B)$. Then $b = \delta\mu(2B)^{-1/p}\mu(B)^{1/q} g$ is a local (p, q)-atom, so $|Lb| = |\int b\varphi\, d\mu| \leqslant c$. Thus $1 = K(\varphi) = \int \varphi g\, d\mu = \delta^{-1}\mu(2B)^{1/p}\mu(B)^{-1/q}\int \varphi b\, d\mu \leqslant \delta^{-1}\mu(2B)^{1/p}\mu(B)^{-1/q} c$, so $\delta \leqslant c\mu(2B)^{1/p}\mu(B)^{1/q'-1} \leqslant cr^\alpha\mu(B)^{1/q'}$. Together with the estimate for the $L^{q'}(\mu, B)$-norm of φ, $1 < r < 2$, given above, this shows that $\varphi \in \Lambda_\alpha^{q'}(F)$, and consequently, if $p < 1$, by Theorem 4.1 and Proposition 4.1, we have found a unique function φ in $\Lambda_\alpha(F)$ such that $La = \int a\varphi\, d\mu$ if a is a local p-atom, and for $f = \sum \lambda_j a_j \in h^p$ we get $Lf = \sum \lambda_j \int a_j \varphi\, d\mu$.

For $p = 1$, we have shown that L is given by a function φ which belongs to $\Lambda_0^{q'}$, $1 < q' < \infty$, and thus to $\Lambda_0^1(F) = \operatorname{bmo}(F)$.

Remark. The proof of Theorem 4.2 shows that $(h^p(F))' = \operatorname{bmo}_q(F)$ for $1 \leqslant q < \infty$. Thus the spaces $\operatorname{bmo}_q(F)$ are equivalent for $1 \leqslant q < \infty$ if F has the Markov property.

4.4. Sets without the Markov property. It is possible to get a duality theorem for d-sets F which do not have the Markov property, if the elements of the Hardy spaces are defined as elements in the dual of suitable Λ_α^q spaces. This is done in the following way. Let μ be a fixed d-measure on F,

$\alpha = d(1/p-1)$, $1/q+1/q' = 1$, $1 < q < \infty$, and let $\tilde{h}^{p,q}(F)$ consist of all elements in $(\Lambda_\alpha^{q'}(F))'$ of the form $h = \sum \lambda_j a_j$, where a_j are local (p, q)-atoms and $\sum |\lambda_j|^p < \infty$. ($\sum \lambda_j a_j$ converges, in the norm of $(\Lambda_\alpha^{q'}(F))'$, if $\sum |\lambda_j|^p < \infty$.) Then it is easy to verify that every φ in $\Lambda_\alpha^{q'}(F)$ defines a bounded linear functional on $\tilde{h}^{p,q}(F)$ by means of

$$\langle f, \varphi \rangle = \sum \lambda_j \int a_j \varphi \, d\mu, \qquad f = \sum \lambda_j a_j \in \tilde{h}^{p,q}(F),$$

and conversely, the proof of the second part of the duality theorem in § 4.3 shows that every bounded linear functional on $\tilde{h}^{p,q}(F)$ is given by an element φ in $\Lambda_\alpha^{q'}(F)$ by this formula. Thus we have

$$(\tilde{h}^{p,q}(F))' = \Lambda_\alpha^{q'}(F), \qquad \alpha = d(1/p-1),$$

for $1 < q < \infty$ and $p \leqslant 1$, even if F does not have the Markov property. However, properties of the spaces $h^{p,q}$ and Λ_α^q which are fundamental in the classical theory, are in general no longer valid, as we shall see in some examples. We construct a closed set $M \subset \boldsymbol{R}^2$, which in some sense is very regular (a C^∞-manifold), and show that defining $\Lambda_\alpha(M)$ as in Definition 4.1 one obtains a space which does not have the trace property (Example 1), that $\Lambda_\alpha^{q_1}(M) \neq \Lambda_\alpha^{q_2}(M)$, $q_1 \neq q_2$ (Example 2), and that $h^{p,q_1}(M, \mu) \neq h^{p,q_2}(M, \mu)$, $q_1 \neq q_2$ (Example 3).

We start by constructing M. Let φ be a C^∞ function supported in $[-1, 1]$ such that $0 \leqslant \varphi \leqslant 1$ and $\varphi(0) = 1$, put $\varphi_n(x) = 2^{-n^3} \varphi((x-2^{-n})/2^{-n^2})$ and $g(x) = \sum_{n=3}^{\infty} \varphi_n(x)$. Then g is infinitely differentiable on $[-1, 1]$, and we define the set $M \subset \boldsymbol{R}^2$ by

$$M = \{(x, y);\ y = g(x),\ 0 \leqslant x \leqslant 1\}.$$

This set M is a d-set with $d = 1$.

Example 1. In this example, we furnish a function f defined on M which is bounded and satisfies (4.1), but which cannot be extended to a function in $\Lambda_\alpha(\boldsymbol{R}^n)$; in other words, if we attempted to define $\Lambda_\alpha(M)$ as in Definition 4.1, then $\Lambda_\alpha(M)$ would fail to have the restriction property. We introduce some more notation: Put $M_n = \{(x, y) \in M;\ 2^{-n}-2^{-n^2} \leqslant x \leqslant 2^{-n}+2^{-n^2}\}$, $p_n = (2^{-n}, 2^{-n^3})$, $a_n = (2^{-n}-2^{-n^2}, 0)$, and $b_n = (2^{-n}+2^{-n^2}, 0)$. Now we construct f on M as follows. Take $1 < \alpha < 2$, and let for $n \geqslant 3$ $f(x, y)$ coincide on M_n with the first-degree polynomial which is zero at a_n and b_n and $2^{-n\alpha}$ at p_n, and put $f = 0$ elsewhere. Then

$$\frac{|f(p_n)-f(b_n)|}{|p_n-b_n|} = \frac{2^{-n\alpha}}{|p_n-b_n|} \geqslant 2^{-n\alpha}/(2^{-n^2}+2^{-n^3}) \to \infty, \qquad n \to \infty,$$

so f does not satisfy a Lipschitz condition, and hence f cannot be extended to a function in $\Lambda_\alpha(\boldsymbol{R}^n)$.

On the other hand, it is clear that f is bounded, and we shall show that condition (a) in Definition 4.1 is fulfilled. Let B be a ball centered at $(x_0, y_0) \in M$ with radius $r \leqslant 1$, and let the integer n_0 satisfy $2^{-n_0-1} < r \leqslant 2^{-n_0}$. If $x_0 \leqslant 4 \cdot 2^{-n_0}$, put $P_B = 0$; then $|f-P_B| \leqslant (2^{-(n_0-2)})^\alpha \leqslant cr^\alpha$ in B, since if B intersects some M_n with $n \geqslant 3$, then $n \geqslant n_0-2$. If $x_0 > 4 \cdot 2^{-n_0}$, let P_B coincide with f on B (this is possible since then B intersects at most one of the sets M_n, $n \geqslant 3$), so $|f-P_B| = 0$ in B. Thus condition (a) is satisfied.

Example 2. *If* $1 \leqslant q_1 < q_2 \leqslant \infty$, $\alpha \geqslant 1$, *then* $\Lambda_\alpha^{q_1}(M) \neq \Lambda_\alpha^{q_2}(M)$. Construct f as in the preceding example, but define now f at p_n by $f(p_n) = 2^{n^2/q_1 - n(\alpha+1/q_1)}$. Taking P_B as in Example 1, it is easy to verify that $f \in \Lambda_\alpha^{q_1}(M)$, and that f is not even in $L^{q_2}(\mu, M)$ if $q_2 > q_1$; we omit the details.

Example 3. *If* $1 < q_1 < q_2 < \infty$, $p \leqslant 1/2$, *then* $h^{p,q_1}(M, \mu) \neq h^{p,q_2}(M, \mu)$. To see this, let $1/q_i + 1/q_i' = 1$, $i = 1, 2$, (then $q_2' < q_1'$) and take as above $f \in \Lambda_\alpha^{q_2'}(M)$, $f \notin \Lambda_\alpha^{q_1'}(M)$, $\alpha = 1/p-1$, and let f_k be as f if $(x, y) \in M_n$, $n \leqslant k$, and zero elsewhere. Then it is clear that f_k may be extended to a function in $C_0^\infty(\boldsymbol{R}^n)$, and easy to show that $L_k h = \lim_{N\to\infty} \sum_{i=1}^{N} \lambda_i \int a_i f_k \, d\mu$, $h \in h^{p,q_2}$, $h = \sum \lambda_i a_i$, $\sum |\lambda_i|^p < \infty$, a_i (p, q_2)-atoms, defines a bounded linear functional on h^{p,q_2} with norm less than $c\|f_k\|_{\Lambda_\alpha^{q_2'}}$. Now, if h^{p,q_2} is equivalent to h^{p,q_1}, then L_k is also in $(h^{p,q_1})'$, and by the proof of the second part of the duality theorem L_k has a representation $L_k h = \lim_{N\to\infty} \sum_{i=1}^{N} \lambda_i' \int a_i' \varphi_k \, d\mu$, $h = \sum \lambda_i' a_i' \in h^{p,q_1}$, where $\|\varphi_k\|_{\Lambda_\alpha^{q_1'}} \leqslant c\|L_k\|$. It is easy to realize that $\varphi_k = f_k$ a.e. (take as h atoms $\mu(2B)^{-1/p}\{\operatorname{sgn}(\varphi_k - f_k)\}\chi_B$ where B are balls centered in F of radius 2), and thus $\|f_k\|_{\Lambda_\alpha^{q_1'}} \leqslant c\|f_k\|_{\Lambda_\alpha^{q_2'}}$, which leads to a contradiction as k tends to infinity.

5. Global Hardy spaces

5.1. Global atoms and Hardy spaces. We shall say that μ satisfies the *global doubling condition* if (1.1) holds for balls of arbitrary radii. Similarly, F is called a *global d-set* if there exists a measure $\mu \geqslant 0$, called a *global d-measure*, whose support is F and which satisfies (1.2) for arbitrary $r > 0$ and $x \in F$. As to the Markov property, Theorems 1.1 and 1.2 are seen to hold also if balls of arbitrary radii are considered. The equivalent properties thus obtained are called the *global Markov property*. The analogues of Theorems 1.3 and 1.4 also remain valid. Notice that global d-sets and sets with the global Markov property are always unbounded.

Now assume μ satisfies the global doubling condition and $\operatorname{supp}\mu = F$. Let p and q be admissible as in § 2, and $s \geqslant 0$ an integer. Then (*global*)

(p, q, s)-*atoms* are defined like local (p, q, s)-atoms, except that the moment condition is always present, independently of the radius of the supporting ball. The (*global*) *Hardy space* $H^{p,q,s} = H^{p,q,s}(F, \mu)$ is defined by means of sums of such atoms with l^p coefficients, like $h^{p,q,s}$. When F is a global d-set, we choose $s = [d(1/p-1)]$, as in § 3, and write $H^{p,q}$. Since Lemma 2.2 remains valid for global atoms, $H^{p,q,s}(F, \mu)$ is a space of distributions in $\boldsymbol{R}^n$, and is contained in $L^1(\mu)$ for $p = 1$.

The maximal function needed for $H^{p,q,s}$ is

$$M_\sigma f(x) = \sup_{\varphi \in S_\sigma} \sup_{t>0} \mu(B(x, t))^{-1} \left\langle f, \varphi\left(\frac{x-\cdot}{t}\right)\right\rangle.$$

The proof of the following theorem is analogous to those in § 3 although simpler, since we do not have to treat large balls differently from small balls.

THEOREM 5.1. *Let μ satisfy the global doubling condition, and assume that F has the global Markov property and that p, q are admissible.*

(a) *If s is large enough and $\sigma > s$, then $f \in H^{p,q,s}$ implies $M_\sigma f \in L^p(\mu)$.*

(b) *If $f \in L^1_{\mathrm{loc}}(\mu)$ and $M_\sigma f \in L^p(\mu)$ for some $\sigma > 0$, then $f \in H^{p,\infty,s}$ for any $s > 0$.*

(c) *For large s, $H^{p,q,s}$ is independent of q and s, and of μ in the d-set case. Norm inequalities hold as in § 3.*

We can thus write H^p for $H^{p,q,s}$. When F is a global d-set, $s = [d(1/p-1)]$ is large enough, as in § 3, and $H^p = H^{p,q}$.

5.2. Duality. In this section, F is a global d-set with the global Markov property, and we consider $0 < p < 1$ (the case $p = 1$ is covered in [2]). P^N denotes the space of polynomials in $\boldsymbol{R}^n$ of degree $\leqslant N$.

DEFINITION 5.1. Let $\alpha > 0$. A function f defined on F belongs to $\dot{\Lambda}_\alpha(F)$ if and only if there is a constant M such that for every ball $B(x_0, r)$, $x_0 \in F$, $r > 0$, there exists a polynomial $P_B \in P^{[\alpha]}$ such that $|f(x) - P_B(x)| \leqslant Mr^\alpha$, $x \in B \cap F$.

The infimum of the constants M is a seminorm on $\dot{\Lambda}_\alpha(F)$, and the null space of $\dot{\Lambda}_\alpha(F)$ is $P^{[\alpha]}$.

The spaces $\dot{\Lambda}_\alpha(F)$ coincide for $F = \boldsymbol{R}^n$ with the spaces $\dot{\Lambda}_\alpha(\boldsymbol{R}^n)$ defined in § 0.1, and $\dot{\Lambda}_\alpha(F)$ is the trace to F of $\dot{\Lambda}_\alpha(\boldsymbol{R}^n)$; this is seen by inspection of the proofs of the corresponding statements for the Λ_α-case given in [7] and [9]. The natural analogues of Theorem 4.1 and Proposition 4.1 are also valid. Each element in $H^p(F)$ may be identified with an element in the dual of $\dot{\Lambda}_\alpha(F)$, $\alpha = d(1/p-1)$. To see this one proceeds as in § 4.2; the function m^R used there must, however, for integer α be replaced by the function $\tilde{m}^R$ used in [14].

THEOREM 5.2. *Let $0 < p < 1$ and $\alpha = d(1/p-1)$. Then*

$$(H^p(F))' = \dot{\Lambda}_\alpha(F)/P^{[\alpha]}$$

with equivalent norms. The duality holds in the sense that any $\psi \in \dot{\Lambda}_\alpha(F)/P^{[\alpha]}$ defines an element in $(H^p(F))'$ by the formula

$$\langle f, \psi\rangle = \sum \lambda_j \int a_j \varphi \, d\mu, \quad f = \sum \lambda_j a_j \in H^p(F),$$

where φ is any representative in $\dot{\Lambda}_\alpha(F)$ of ψ, and that conversely every element in $(H^p(F))'$ is given through this formula by a unique $\psi \in \dot{\Lambda}_\alpha(F)/P^{[\alpha]}$.

In the proof of this theorem, in order to find a function φ representing a given functional $L \in (H^p(F))'$, one must argue a little differently than in § 4.2, cf. the proof of the duality theorem for $H^p(\boldsymbol{R}^n)$ e.g. in [13].

5.3. An application to Hardy spaces in $\boldsymbol{R}^n$.

THEOREM 5.3. *Let F be a closed convex bounded set with nonempty interior. If f belongs to the standard Hardy space $H^p(\boldsymbol{R}^n)$, $0 < p \leqslant 1$, and* $\operatorname{supp} f \subset F$, *then f has an atomic decomposition whose atoms are supported in F.*

However, the supporting balls of the atoms need not be contained in F.

Proof. Notice that F is a d-set with $d = n$ having the Markov property. As μ we take the restriction of Lebesgue measure to F. We apply our space $h^p(F)$. Clearly, $m_\sigma f \in L^p(\mu)$, but since f need not be in $L^1_{\mathrm{loc}}(\mu)$, we start by approximating f: Assume 0 is an interior point of F. Using the mapping $x \to (1-\varepsilon)x$, we may contract f to a distribution f_ε supported in $(1-\varepsilon)F$. Clearly, $M_\sigma f_\varepsilon \in L^p(\mu)$, uniformly for small $\varepsilon > 0$. Then take a C^∞ approximate identity $\eta_\delta(x) = \delta^{-n}\eta(x/\delta)$, with $\int \eta \, dx = 1$ and $\operatorname{supp} \eta \subset B(0, 1)$. If $\delta(\varepsilon)$ is small, $\tilde{f}_\varepsilon = f_\varepsilon * \eta_{\delta(\varepsilon)}$ is a C^∞ function with support in F. Clearly, $\tilde{f}_\varepsilon \to f$ as $\varepsilon \to 0$, in the distribution sense. Considering convolutions of S_σ functions with η, one verifies that $m_\sigma \tilde{f}_\varepsilon \leqslant cM_\sigma f_\varepsilon$, so by Theorem 3.2 $\tilde{f}_\varepsilon \in h^p(F)$, uniformly in ε.

Thus $\tilde{f}_\varepsilon$ has an atomic decomposition with local (p, ∞, s)-atoms, $s = [n(1/p-1)]$. We want these atoms to be global, i.e., all of them should have vanishing moments up to order s. If we assume that $\operatorname{diam} F$ is small enough, all balls $2B_i^k$ appearing in the proof of Theorem 3.2 will be of type I, and the corresponding atoms global. Since Theorem 5.3 is invariant under scaling, this assumption is no restriction. The assumption also implies that there will be only one ball B_ν and one atom a_ν in the decomposition of $g_{k'}$ in the last part of the same proof. But then the moments of a_ν must also vanish since those of f and $\tilde{f}_\varepsilon$ vanish, and we have decomposed $\tilde{f}_\varepsilon$ into global atoms. Extending these atoms by 0 in $\boldsymbol{R}^n \setminus F$, we obtain standard p-atoms for $H^p(\boldsymbol{R}^n)$, supported in F.

It remains to let $\varepsilon \to 0$. We follow the method of [2], Lemma 4.2 p. 638, or [13], Lemma 5.13, which we only briefly sketch. Indexing the atoms by means of dyadic cubes containing their supporting balls, we may find a sequence $\varepsilon_j \to 0$ for which each atom in the decomposition of $\tilde{f}_{\varepsilon_j}$ converges weakly* in L^∞. The limits must be p-atoms supported in F. The coefficients in the decomposition of $\tilde{f}_{\varepsilon_j}$ are uniformly in l^p, so we can make them converge also, to limits forming an l^p sequence. To verify that the atomic sum converges in the distribution sense, we notice that the terms corresponding to cubes larger than some $\delta > 0$ are finite in number, so their sum converges. We need thus only verify that the integrals of the remaining terms have a small sum when integrated against a test function φ. Now $\varphi \in C^\infty$, so φ is small in the dual space $\dot{\Lambda}_\alpha$ in a small dyadic cube, and this gives the necessary estimate. Hence, the atomic decomposition of $\tilde{f}_{\varepsilon_j}$ converges in the distribution sense to an atomic sum representing an H^p distribution which must be f. This gives the required decomposition of f, completing the proof.

References

[1] S. Campanato, *Proprietà di una famiglia di spazi funzionali*, Ann. Scuola Norm. Sup. Pisa 18 (1964), 137–160.

[2] R. R. Coifman, G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. 83 (1977), 569–645.

[3] C. Fefferman, E. M. Stein, *H^p spaces of several variables*, Acta Math. 129 (1972), 137–193.

[4] O. Frostman, H. Wallin, *Riesz potentials on manifolds*, in: Proceedings of a Conference on Potential Theory (Copenhagen 1979), pp. 106–120. Lecture Notes in Math. 787, Springer, Berlin-Heidelberg-New York 1980.

[5] D. Goldberg, *A local version of real Hardy spaces*, Duke Math. J. 46 (1979), 27–42.

[6] A. Jonsson, H. Wallin, *A Whitney extension theorem in L^p and Besov spaces*, Ann. Inst. Fourier (Grenoble) 28 (1978), 139–192.

[7] —, —, *The trace to closed sets of functions in R^n with second difference of order $O(h)$*, J. Approx. Theory 26 (1979), 159–184.

[8] —, —, *Local polynomial approximation and Lipschitz type conditions on general closed sets*, Department of Math., Univ. of Umeå, No 1, 1980.

[9] —, —, *Local polynomial approximation and Lipschitz functions on closed sets*, to appear in: Proceedings of a Conference on Constructive Function Theory, Varna, Bulgaria, 1981.

[10] R. H. Latter, *A decomposition of $H^p(R^n)$ in terms of atoms*, Studia Math. 62 (1977), 92–101.

[11] R. H. Latter, A. Uchiyama, *The atomic decomposition for parabolic H^p spaces*, Trans. Amer. Math. Soc. 253 (1979), 391–398.

[12] R. A. Macias, C. Segovia, *A decomposition into atoms of distributions on spaces of homogeneous type*, Adv. in Math. 33 (1979), 271–309.

[13] P. Sjögren, *Lectures on atomic H^p space theory in R^n*, Department of Math., Univ. of Umeå, No 5, 1981.

[14] —, *A weak spectral synthesis property for Hardy and Lipschitz spaces*, in: Harmonic analysis, Proceedings 1981, Lecture Notes in Math. 908 Springer, 1982, 285–296.

[15] J. O. Strömberg, A. Torchinsky, *Weights, sharp maximal functions and Hardy spaces*, Bull. Amer. Math. Soc. 3 (1980), 1053-1056.

Received June 24, 1983 (1905)

STUDIA MATHEMATICA, T. LXXX. (1984)

Generalized convolutions III

by

K. URBANIK (Wrocław)

Abstract. This is a study of characteristic functions of generalized convolutions. In particular, we obtain some uniqueness and characterization theorems. Moreover, the concepts of representability and order of generalized convolutions are discussed. The paper is a continuation of [11] and [13].

1. Preliminaries and notation. We denote by $\mathscr{C}_b$ the space of bounded continuous real-valued functions on the positive half-line R^+ with the topology of uniform convergence on every compact subset of R^+. Further, by $\mathfrak{P}$ we shall denote the set of all probability measures defined on Borel subsets of R^+. The set $\mathfrak{P}$ is endowed with the topology of weak convergence. For $a \in R^+$ we define the mapping T_a: $R^+ \to R^+$ by $T_a x = ax$. For a function $f \in \mathscr{C}_b$, $T_a f$ denotes the function $(T_a f)(x) = f(ax)$ and for a measure $\mu \in \mathfrak{P}$, $T_a \mu$ denotes the measure defined by $(T_a \mu)(E) = \mu(a^{-1}E)$ if $a > 0$ and $T_0 \mu = \delta_0$, where $a^{-1}E = \{a^{-1}x \colon x \in E\}$ and δ_c is the probability measure concentrated at the point c. We say that two functions f and g from $\mathscr{C}_b$ are *similar*, in symbols $f \sim g$, if $f = T_a g$ for a certain positive number a. Further, two measures μ and ν from $\mathfrak{P}$ are said to be *similar*, in symbols $\mu \sim \nu$, if $\mu = T_a \nu$ for a certain positive number a.

A continuous commutative and associative $\mathfrak{P}$-valued binary operation $\circ$ defined on $\mathfrak{P}$ is called a *generalized convolution* if the following conditions are fulfilled:

(i) the measure δ_0 is a unit element, i.e. $\mu \circ \delta_0 = \mu$ $(\mu \in \mathfrak{P})$,

(ii) $(c\mu + (1-c)\nu) \circ \lambda = c(\mu \circ \lambda) + (1-c)(\nu \circ \lambda)$ $(0 \leqslant c \leqslant 1;\ \mu, \nu, \lambda \in \mathfrak{P})$,

(iii) $(T_a \mu) \circ (T_a \nu) = T_a(\mu \circ \nu)$ $(a \in R^+,\ \mu, \nu \in \mathfrak{P})$,

(iv) there exists a sequence $c_1, c_2, \ldots$ of positive numbers such that the sequence $T_{c_n} \delta_1^{\circ n}$ converges to a measure different from δ_0. The power $\delta_1^{\circ n}$ is taken here in the sense of the operation $\circ$.

The set $\mathfrak{P}$ with the operation $\circ$ and the operations of convex linear combinations is called a *generalized convolution algebra* and denoted by $(\mathfrak{P}, \circ)$. For basic properties of generalized convolution algebras we refer to [2]–[7] and [10]–[14]. In particular, generalized convolution algebras admitting a non-trivial homomorphism into the algebra of real numbers with the