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Trigonometric approximation in the norms and seminorms

by

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Dedicated to Professor W. Orlicz on the occasion of his 80th birthday

Abstract. The author proves four theorems on approximation by trigonometric polynomials of some periodic functions and their Weyl derivatives. Estimates obtained here can be treated as generalizations of the results of Ganelius [4] and Popov [6].

1. Preliminaries. Let L^p $(1 \leqslant p < \infty)$ be the space of all 2π-periodic complex-valued functions f Lebesgue-integrable to the pth power over the interval $\langle 0, 2\pi\rangle$ with the norm

$$\|f\|_p = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p\,dx\right\}^{1/p}.$$

Write L instead of L^1. Denote by L^∞ the space of all functions $f \in L$ essentially bounded on $\langle 0, 2\pi\rangle$ with the norm

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x\in\langle 0,2\pi\rangle} |f(x)|.$$

The subspace of L^∞ consisting of all real-valued functions bounded on $\langle 0, 2\pi\rangle$ will be denoted by L_0^∞. Moreover, C [resp. AC] will mean the class of all 2π-periodic functions continuous [absolutely continuous] on $\langle 0, 2\pi\rangle$.

Let H_n be the set of all 2π-periodic trigonometric polynomials of order less than n, $n \in N = \{1, 2, \ldots\}$. If $f \in L_0^\infty$, we can also introduce the sets $H_n^-(f)$ and $H_n^+(f)$ of real-valued 2π-periodic trigonometric polynomials t, $T \in H_n$ such that, for all real x,

$$t(x) \leqslant f(x) \quad \text{and} \quad T(x) \geqslant f(x),$$

respectively.

The best trigonometric approximation of an arbitrary function f belonging to the space L^p is defined by

$$E_n(f)_p = \inf_{s\in H_n} \|f-s\|_p.$$

Denote by $\omega(\delta;f)_p$ the modulus of continuity of f with respect to the L^p-norm, i.e.,

$$\omega(\delta;f)_p = \sup_{h\in\langle 0,\delta\rangle} \|\Delta_h f\|_p \quad (0\leqslant\delta<\infty), \tag{1.1}$$

where

$$\Delta_h f(x) = f(x+h)-f(x).$$

For any function $f\in L_0^\infty$, the quantities

$$E_n^-(f)_p = \inf_{t\in H_n^-(f)} \|f-t\|_p, \quad E_n^+(f)_p = \inf_{T\in H_n^+(f)} \|T-f\|_p$$

are called the *best lower one-sided approximation* and the *best upper one-sided approximation* of f in L^p-metrics, respectively. Putting

$$M_\delta f(x) = \sup_{u,u+h\in G_\delta(x)} |\Delta_h f(u)|, \quad G_\delta(x) = \langle x-\delta/2,\ x+\delta/2\rangle,$$

we can define the modified modulus of continuity of f as

$$\tau(\delta;f)_p = \|M_\delta f\|_p \quad (0\leqslant\delta<\infty). \tag{1.2}$$

The moduli (1.1), (1.2) are non-negative, non-decreasing and subadditive functions of δ, and have also some other similar properties. For every $f\in L_0^\infty$ and $p\geqslant 1$, the Jackson type inequality

$$E_n^\pm(f)_p \leqslant C_0\,\tau(n^{-1};f)_p \quad (C_0=\text{const},\ n\in N) \tag{1.3}$$

holds (see [1]).

Suppose that $f\in L^p\ (p\geqslant 1)$ and $g\in L$. Introduce the finite Fourier transform

$$f\hat{}(k) = \frac{1}{2\pi}\int_0^{2\pi} f(u)e^{-iku}\,du \quad (k=0,\ \pm1,\ \pm2,\ldots)$$

and the convolution

$$(f*g)(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x-u)g(u)\,du = \frac{1}{2\pi}\int_0^{2\pi} f(v)g(x-v)\,dv.$$

As is well known, this convolution exists for almost every $x\in(-\infty,\infty)$, is measurable and

$$\|f*g\|_p \leqslant \|f\|_p\|g\|_1.$$

In the case $f\in C$ [resp. $f\in AC$], $f*g\in C$ [$f*g\in AC$].

Write, for arbitrary numbers $\alpha>0$, $n\in N$ and $u\in(-\infty,\infty)$,

$$\Psi_{\alpha,n}(u) = \sum_{\nu=-n}^{n}{}' \frac{e^{i\nu u}}{(i\nu)^\alpha}, \quad (i\nu)^\alpha = |\nu|^\alpha \exp\left(i\frac{\pi\alpha}{2}\operatorname{sign}\nu\right);$$

the prime indicates that the term "corresponding to $\nu=0$" is omitted in the summation. It is easily seen that

$$\Psi_{\alpha,n}(u) = 2\sum_{\nu=1}^{n} \nu^{-\alpha}\cos(\nu u-\tfrac{1}{2}\pi\alpha).$$

Therefore, the function

$$\Psi_\alpha(u) = \lim_{n\to\infty}\Psi_{\alpha,n}(u) = \sum_{\nu=-\infty}^{\infty}{}' \frac{e^{i\nu u}}{(i\nu)^\alpha}$$

is defined for all real u [resp. $u\neq 2k\pi$, $k=0,\pm1,\pm2,\ldots$] if $1\leqslant\alpha<\infty$ [$0<\alpha<1$]. Moreover, $\Psi_\alpha\in L$ for every positive α (see [9], pp. 70, 186).

Consider now a function $f\in L^p$ $(1\leqslant p\leqslant\infty)$ and its Fourier series

$$S[f](x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \quad (c_k = f\hat{}(k)). \tag{1.4}$$

In this case, for any $\alpha>0$, the convolution

$$f_\alpha(x) = (f*\Psi_\alpha)(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x-u)\Psi_\alpha(u)\,du$$

is of class L^p and

$$S[f_\alpha](x) = \sum_{k=-\infty}^{\infty}{}' \frac{c_k}{(ik)^\alpha}e^{ikx} \quad (-\infty<x<\infty).$$

In several cases, the last Fourier series converges for every or almost every x; its sum

$$f^{(-\alpha)}(x) \equiv I_\alpha[f](x),$$

called the *αth integral* of f, coincides with $f_\alpha(x)$ almost everywhere (see [9], pp. 69–70, 36, 93–94, 90 and 77–78).

Write

$$f^{(n)}(x) \quad\text{or}\quad \frac{d^n}{dx^n}f(x)$$

for the ordinary (iterative) derivative of a function f at x of positive integer order n. By convention, let $f^{(0)}(x) \equiv f(x)$. The (Weyl) derivative of $f \in L^p$ ($p \geqslant 1$) of non-integer order $r+\alpha$ ($r+1 \in N$, $\alpha \in (0, 1)$) will be defined by

$$f^{(r+\alpha)}(x) = \frac{d^{r+1}}{dx^{r+1}} I_{1-\alpha}[f](x),$$

provided that the right-hand side exists.

Let f be a complex-valued function defined in the interval $\langle a, b\rangle$. Then

$$V_p(f; a, b) = \sup_{\Pi} \{\sum_{j=0}^{n-1} |f(x_{j+1}) - f(x_j)|^p\}^{1/p} \quad (0 < p < \infty),$$

where Π denotes the partition $\{a = x_0 < x_1 < \ldots < x_n = b\}$, is often called the *p-th variation* of f in $\langle a, b\rangle$. Write

$$V_\infty(f; a, b) = \sup_{u,v \in \langle a,b\rangle} |f(u) - f(v)|.$$

We shall denote by BV_p the set of all 2π-periodic functions f for which

$$V_p(f) \equiv V_p(f; 0, 2\pi) < \infty.$$

A complex-valued function f of period 2π is said to belong to the class $W^\beta L^p$ [resp. $W^\beta BV_p$], with finite $\beta > 0$, if $f \in C$ and

$$f^{(\beta-1)} \in AC, \quad f^{(\beta)} \in L^p \quad [f^{(\beta)} \in BV_p].$$

By convention,

$$W^0 L^p = L^p, \quad W^0 BV_p = BV_p.$$

The aim of this paper is to present some approximation theorems for functions of classes $W^\beta L^p$ and $W^\beta BV_p$ ($\beta \geqslant 0$, $p \geqslant 1$). Their proofs are based on the suitable results announced in [3] and [2]. In our considerations, only the norms $\|\varphi\|_p$ and seminorms $V_p(\varphi)$ will be used.

C_μ, $C_\nu(a, \ldots)$, where $\mu, \nu \in N$, will denote, respectively, some positive absolute constants and positive numbers depending only on the indicated parameters $a, \ldots$

2. Basic estimates for functions of classes $W^\beta L^p$ and $W^\beta BV_p$. Let us start with the following

THEOREM 1. *Suppose that f is a function of class $W^\beta L^p$ ($1 \leqslant \beta < \infty$, $1 \leqslant p \leqslant \infty$). Then*

(2.1) $f^{(\alpha)} \in AC$ *if* $0 \leqslant \alpha \leqslant \beta-1$ *and* $f^{(\alpha)} \in L^p$ *if* $\beta-1 < \alpha \leqslant \beta$.

Moreover, for every $\alpha \in \langle 0, \beta\rangle$,

$$E_n(f^{(\alpha)})_p \leqslant \frac{C_1(\beta-\alpha)}{n^{\beta-\alpha}} E_n(f^{(\beta)})_p \quad (n = 1, 2, \ldots). \tag{2.2}$$

In the case of a real-valued f and $\alpha \in \langle 0, \beta-1\rangle$,

$$E_n^{\pm}(f^{(\alpha)})_p \leqslant \frac{C_2(\beta-\alpha)}{n^{\beta-\alpha}} E_n(f^{(\beta)})_p \quad (n = 1, 2, \ldots). \tag{2.3}$$

Proof. (1) Let (1.4) be the Fourier series of $f \in W^\beta L^p$ and let r be the positive integer such that $r \leqslant \beta < r+1$.

By the assumption,

$$f^{(\beta-1)}(x) = \frac{d^r}{dx^r} I_{r+1-\beta}[f](x) = \sum_{k=-\infty}^{\infty} c_k (ik)^{\beta-1} e^{ikx}$$

for all real x. Moreover,

$$S[f^{(\beta)}](x) = \sum_{k=-\infty}^{\infty}{}' c_k (ik)^\beta e^{ikx}; \tag{2.4}$$

hence

$$c_k = o(|k|^{-\beta}) \quad \text{as} \quad k \to \pm\infty.$$

Consequently,

$$f(x) = \hat{f}(0) + (f^{(\beta)} * \Psi_\beta)(x) \tag{2.5}$$

for all real x. Therefore, under the restriction $\beta \geqslant 1$, $f \in AC$.

In the case $0 < \alpha \leqslant \beta-1$ [resp. $\beta-1 < \alpha < \beta$],

$$f^{(\alpha)}(x) = \sum_{k=-\infty}^{\infty}{}' c_k (ik)^\alpha e^{ikx} = S[f^{(\beta)} * \Psi_{\beta-\alpha}](x),$$

uniformly in x [resp. for almost every x]. Hence

$$f^{(\alpha)}(x) = (f^{(\beta)} * \Psi_{\beta-\alpha})(x) \tag{2.6}$$

for every [almost every] x if $0 < \alpha \leqslant \beta-1$ [resp. $\beta-1 < \alpha < \beta$]. Thus, assertion (2.1) is established.

(2) For $\alpha = 0$, $\beta \geqslant 1$, the estimate (2.2) is known (e.g., see [8], Ths. 1, 3). Therefore, we may suppose that $0 < \alpha < \beta$.

In the class H_n, let $S_{\alpha,n}$ be the trigonometric polynomial of best approximation of $\Psi_{\beta-\alpha}$ with respect to the L-norm, i.e.,

$$\|S_{\alpha,n} - \Psi_{\beta-\alpha}\|_1 = E_n(\Psi_{\beta-\alpha})_1.$$

Consider in H_n the trigonometric polynomial

$$U_{\alpha,n}[f] = f^{(\beta)} * S_{\alpha,n} \quad (n \in N). \tag{2.7}$$

From (2.6) and (2.7) it follows that

$$f^{(\alpha)}(x) - U_{\alpha,n}[f](x) = \frac{1}{2\pi} \int_0^{2\pi} f^{(\beta)}(u) \{\Psi_{\beta-\alpha}(x-u) - S_{\alpha,n}(x-u)\} du$$

(a.e.); hence, by the generalized Minkowski inequality,

$$\|f^{(\alpha)}-U_{\alpha,n}[f]\|_p \leqslant \|\Psi_{\beta-\alpha}-S_{\alpha,n}\|_1\|f^{(\beta)}\|_p.$$

Consequently (see [3], Th. 6),

$$E_n(f^{(\alpha)})_p \leqslant C_3(\beta-\alpha)\,n^{\alpha-\beta}\,\|f^{(\beta)}\|_p \quad (n\in N). \tag{2.8}$$

Denote by $Q_n = Q_n[f^{(\beta)}]$ the trigonometric polynomial of order less than n such that

$$\|f^{(\beta)}-Q_n\|_p = E_n(f^{(\beta)})_p \quad (n\in N).$$

Introduce the auxiliary function

$$\varphi(x) = f(x)-I_\beta[Q_n](x) \quad (-\infty<x<\infty).$$

Clearly,

$$\varphi^{(\beta)}(x) = f^{(\beta)}(x)-Q_n(x)+\hat{Q_n}(0)$$

and

$$|\hat{Q_n}(0)| = \frac{1}{2\pi}\left|\int_0^{2\pi}\{Q_n(u)-f^{(\beta)}(u)\}\,du\right| \leqslant E_n(f^{(\beta)})_p.$$

Hence

$$\|\varphi^{(\beta)}\|_p \leqslant \|f^{(\beta)}-Q_n\|_p+|\hat{Q_n}(0)| \leqslant 2E_n(f^{(\beta)})_p. \tag{2.9}$$

Applying (2.8) to φ, we obtain

$$E_n(\varphi^{(\alpha)})_p \leqslant C_3(\beta-\alpha)\,n^{\alpha-\beta}\,\|\varphi^{(\beta)}\|_p \leqslant 2C_3(\beta-\alpha)\,n^{\alpha-\beta}\,E_n(f^{(\beta)})_p.$$

Now, the obvious identity

$$E_n(\varphi^{(\alpha)})_p = E_n(f^{(\alpha)})_p$$

leads to (2.2).

(3) Given real-valued f, $\beta>1$ and $\alpha\in\langle 0, \beta-1\rangle$, we shall deduce the estimate (2.3) (for $\alpha=0$, $\beta=1$ see [6], Th. 1).

Write

$$f_{|+}^{(\beta)}(u) = \tfrac{1}{2}\{|f^{(\beta)}(u)|+f^{(\beta)}(u)\}, \qquad f_{|-}^{(\beta)}(u) = \tfrac{1}{2}\{|f^{(\beta)}(u)|-f^{(\beta)}(u)\}$$

for all real u. Then

$$f^{(\beta)}(u) = f_{|+}^{(\beta)}(u)-f_{|-}^{(\beta)}(u),$$

and formulae (2.5)–(2.6) imply

$$f(x) = \hat{f}(0)+(f_{|+}^{(\beta)}*\Psi_\beta)(x)-(f_{|-}^{(\beta)}*\Psi_\beta)(x), \tag{2.10}$$

$$f^{(\alpha)}(x) = (f_{|+}^{(\beta)}*\Psi_{\beta-\alpha})(x)-(f_{|-}^{(\beta)}*\Psi_{\beta-\alpha})(x) \quad (0<\alpha\leqslant\beta-1). \tag{2.11}$$

Consider the trigonometric polynomials

$$t_{\alpha,n}\in H_n^-(\Psi_{\beta-\alpha}), \qquad T_{\alpha,n}\in H_n^+(\Psi_{\beta-\alpha})$$

of the best one-sided approximation of $\Psi_{\beta-\alpha}$ in the L-metric. Introduce the trigonometric polynomials

$$U_{0,n}^+[f] = \hat{f}(0)+(f_{|+}^{(\beta)}*T_{0,n})-(f_{|-}^{(\beta)}*t_{0,n}), \tag{2.12}$$

$$U_{0,n}^-[f] = \hat{f}(0)+(f_{|+}^{(\beta)}*t_{0,n})-(f_{|-}^{(\beta)}*T_{0,n}) \tag{2.13}$$

and, for $\alpha\in(0, \beta-1\rangle$,

$$U_{\alpha,n}^+[f] = (f_{|+}^{(\beta)}*T_{\alpha,n})-(f_{|-}^{(\beta)}*t_{\alpha,n}), \tag{2.14}$$

$$U_{\alpha,n}^-[f] = (f_{|+}^{(\beta)}*t_{\alpha,n})-(f_{|-}^{(\beta)}*T_{\alpha,n}). \tag{2.15}$$

By (2.10), (2.11), (2.12) and (2.14),

$$U_{\alpha,n}^+[f](x)-f^{(\alpha)}(x) = \frac{1}{2\pi}\int_0^{2\pi} f_{|+}^{(\beta)}(u)\{T_{\alpha,n}(x-u)-\Psi_{\beta-\alpha}(x-u)\}\,du+$$
$$+\frac{1}{2\pi}\int_0^{2\pi} f_{|-}^{(\beta)}(u)\{\Psi_{\beta-\alpha}(x-u)-t_{\alpha,n}(x-u)\}\,du$$

for all real x. Hence

$$U_{\alpha,n}^+[f](x) \geqslant f^{(\alpha)}(x) \quad (-\infty<x<\infty),$$

i.e.,

$$U_{\alpha,n}^+[f]\in H_n^+(f^{(\alpha)}) \quad \text{when} \quad \alpha\in\langle 0, \beta-1\rangle.$$

Analogously (2.10), (2.11), (2.13), (2.15) imply

$$U_{\alpha,n}^-[f]\in H_n^-(f^{(\beta)}) \quad \text{when} \quad \alpha\in\langle 0, \beta-1\rangle.$$

Arguing further as in [2], pp. 369–370, we obtain

$$\|U_{\alpha,n}^\pm[f]-f^{(\alpha)}\|_p \leqslant C_4(\beta-\alpha)\,n^{\alpha-\beta}\|f^{(\beta)}\|_p \quad (0\leqslant\alpha\leqslant\beta-1). \tag{2.16}$$

Let $Q_n\equiv Q_n[f^{(\beta)}]$ and φ be as in (2). Then, by (2.16) and (2.9),

$$\|U_{\alpha,n}^\pm[\varphi]-\varphi^{(\alpha)}\|_p \leqslant 2C_4(\beta-\alpha)\,n^{\alpha-\beta}\,E_n(f^{(\beta)})_p. \tag{2.17}$$

Since

$$U_{\alpha,n}^-[\varphi](x) \leqslant \varphi^{(\alpha)}(x) \leqslant U_{\alpha,n}^+[\varphi](x)$$

and

$$\varphi^{(\alpha)}(x) = f^{(\alpha)}(x)-Q_n^{(\alpha-\beta)}(x),$$

we have

$$U^-_{\alpha,n}[\varphi](x)+Q_n^{(\alpha-\beta)}(x) \leqslant f^{(\alpha)}(x) \leqslant U^+_{\alpha,n}[\varphi](x)+Q_n^{(\alpha-\beta)}(x)$$

for all real x. Thus

$$E_n^{\pm}(f^{(\alpha)})_p \leqslant \|U^{\pm}_{\alpha,n}[\varphi]+Q_n^{(\alpha-\beta)}-f^{(\alpha)}\|_p = \|U^{\pm}_{\alpha,n}[\varphi]-\varphi^{(\alpha)}\|_p$$

and, in view of (2.17), the estimates (2.3) for all non-negative $\alpha \leqslant \beta-1$ are proved.

By a Jackson type inequality (due to S.B. Stechkin) and the corresponding Marcinkiewicz result ([5], p. 38), we get for the best one-sided approximation the following useful corollary:

COROLLARY 1. *For any real-valued function f of class $W^\beta L^p$ $(1 \leqslant \beta < \infty,$ $1 \leqslant p \leqslant \infty)$ and every number $\alpha \in \langle 0, \beta-1\rangle$,*

$$E_n^{\pm}(f^{(\alpha)})_p \leqslant \frac{3C_2(\beta-\alpha)}{2n^{\beta-\alpha}}\,\omega(\pi/n; f^{(\beta)})_p \qquad (n \in N).$$

In particular, if $f \in W^\beta BV_p$,

$$E_n^{\pm}(f^{(\alpha)})_p \leqslant \frac{3\pi^{1/p}C_2(\beta-\alpha)}{n^{\beta-\alpha+1/p}}\,V_p(f^{(\beta)}) \qquad (n \in N).$$

Remark 1. If $f \in W^\beta L^p$ $(0<\beta<1,\ 1 \leqslant p \leqslant \infty)$, then the representation formulae (2.5), (2.6) also hold for almost every x whenever $\alpha \in (0, \beta)$. Consequently, $f^{(\alpha)} \in L^p$ and the estimate (2.2) remains valid for each $\alpha \in \langle 0, \beta\rangle$ (cf. [8], pp. 21–23).

THEOREM 2. *Consider a function f of class $W^\beta L^p$ $(1 \leqslant \beta < \infty,\ 1 \leqslant p \leqslant \infty)$.*

(i) *Suppose that, for some trigonometric polynomial $s_n \in H_n$, the following inequality holds:*

$$\|f-s_n\|_p \leqslant C_5 E_n(f)_p. \tag{2.18}$$

Then, for every $\alpha \in \langle 0, \beta\rangle$,

$$\|f^{(\alpha)}-s_n^{(\alpha)}\|_p \leqslant \frac{C_6(\alpha, \beta)}{n^{\beta-\alpha}}E_n(f^{(\beta)})_p. \tag{2.19}$$

(ii) *If f is a real-valued function and if, for some trigonometric polynomial $t_n \in H_n^-(f)$ or $T_n \in H_n^+(f)$, we have the corresponding estimates*

$$\|f-t_n\|_p \leqslant C_7 E_n^-(f)_p \quad \text{and} \quad \|T_n-f\|_p \leqslant C_8 E_n^+(f)_p, \tag{2.20}$$

then, for every $\alpha \in \langle 0, \beta\rangle$,

$$\|f^{(\alpha)}-t_n^{(\alpha)}\|_p \leqslant \frac{C_9(\alpha, \beta)}{n^{\beta-\alpha}}E_n(f^{(\beta)})_p \tag{2.21}$$

and

$$\|T_n^{(\alpha)}-f^{(\alpha)}\|_p \leqslant \frac{C_{10}(\alpha, \beta)}{n^{\beta-\alpha}}E_n(f^{(\beta)})_p, \tag{2.22}$$

respectively.

Proof. In the case $\alpha = 0$, (2.19) and (2.21)–(2.22) follow at once from Theorem 1. If $\alpha = \beta$, the estimate (2.19) was obtained in [8], Ths. 2, 4; for positive $\alpha \leqslant \beta$, the proof of (2.19) runs analogously.

Now, we shall deduce the inequality (2.21) when $\alpha \in \langle 0, \beta\rangle$. For these α, $f^{(\alpha)} \in L^p$, by Theorem 1.

Let φ be an arbitrary function in the space L^p $(p \geqslant 1)$. Denote by $S_\nu[\varphi](x)$ the νth partial sum of the Fourier series $S[\varphi](x)$. Introduce the de la Vallée-Poussin mean

$$W_n[\varphi](x) = \frac{1}{n}\sum_{\nu=n-1}^{2n-2} S_\nu[\varphi](x) \tag{2.23}$$

and a polynomial $\overset{*}{s}_n[\varphi] \in H_n$ such that

$$\|\varphi-\overset{*}{s}_n[\varphi]\|_p = E_n(\varphi)_p. \tag{2.24}$$

As is known,

$$W_n[\varphi](x) = \frac{1}{2\pi}\int_0^{2\pi}\varphi(x-u)K_n(u)\,du \qquad (-\infty < x < \infty), \tag{2.25}$$

where

$$K_n(u) = \frac{1}{n}\sum_{\nu=n-1}^{2n-2}\frac{\sin(\nu+\frac{1}{2})u}{\sin\frac{1}{2}u},$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}|K_n(u)|\,du < 4. \tag{2.26}$$

Applying (2.25), (2.26) and the generalized Minkowski inequality, we obtain

$$\|W_n[\varphi]\|_p \leqslant \|\varphi\|_p\|K_n\|_1 \leqslant 4\|\varphi\|_p. \tag{2.27}$$

Clearly,

$$W_n[f^{(\alpha)}](x) = W_n^{(\alpha)}[f](x) \qquad (-\infty < x < \infty);$$

hence, by Minkowski's inequality,

$$\|f^{(\alpha)}-t_n^{(\alpha)}\|_p \leqslant \|f^{(\alpha)}-W_n[f^{(\alpha)}]\|_p + \|\overset{*}{s}{}_n^{(\alpha)}[W_n[f]]-t_n^{(\alpha)}\|_p + \\ +\|W_n^{(\alpha)}[f]-\overset{*}{s}{}_n^{(\alpha)}[W_n[f]]\|_p.$$

In view of (2.24), (2.23) and (2.27),

$$\|f^{(\alpha)}-W_n[f^{(\alpha)}]\|_p \leqslant \|f^{(\alpha)}-\overset{*}{s}_n[f^{(\alpha)}]\|_p+\|\overset{*}{s}_n[f^{(\alpha)}]-W_n[f^{(\alpha)}]\|_p$$
$$= E_n(f^{(\alpha)})_p+\big\|W_n[\overset{*}{s}_n[f^{(\alpha)}]-f^{(\alpha)}]\big\|_p \leqslant 5E_n(f^{(\alpha)})_p.$$

By a Bernstein type inequality (e.g., see [7], p. 392),

$$\big\|\overset{*}{s}{}_n^{(\alpha)}[W_n[f]]-t_n^{(\alpha)}\big\|_p \leqslant 2(n-1)^{\alpha}\big\|\overset{*}{s}_n[W_n[f]]-t_n\big\|_p,$$
$$\big\|W_n^{(\alpha)}[f]-\overset{*}{s}{}_n^{(\alpha)}[W_n[f]]\big\|_p \leqslant 2(2n-2)^{\alpha}\big\|W_n[f]-\overset{*}{s}_n[W_n[f]]\big\|_p$$
$$\leqslant 2^{\alpha+1}n^{\alpha}E_n(W_n[f])_p.$$

Further, (2.24), (2.20) and (2.23) lead to

$$\big\|\overset{*}{s}_n[W_n[f]]-t_n\big\|_p \leqslant \big\|\overset{*}{s}_n[W_n[f]]-W_n[f]\big\|_p+$$
$$+\|W_n[f]-f\|_p+\|f-t_n\|_p \leqslant E_n(W_n[f])_p+5E_n(f)_p+C_7E_n^-(f)_p$$

and

$$E_n(W_n[f])_p \leqslant \|W_n[f]-\overset{*}{s}_n[f]\|_p = \big\|W_n[f-\overset{*}{s}_n[f]]\big\|_p \leqslant 4E_n(f)_p.$$

Consequently,

$$\|f^{(\alpha)}-t_n^{(\alpha)}\|_p \leqslant 5E_n(f^{(\alpha)})_p+2n^{\alpha}\{E_n(W_n[f])_p+5E_n(f)_p+C_7E_n^-(f)_p\}+$$
$$+2^{\alpha+1}n^{\alpha}E_n(W_n[f])_p$$
$$\leqslant 5E_n(f^{(\alpha)})_p+(18+2^{3+\alpha})n^{\alpha}E_n(f)_p+2n^{\alpha}C_7E_n^-(f)_p.$$

Applying Theorem 1, we get (2.21). The estimate (2.22) can be obtained in a parallel manner.

COROLLARY 2. *Given a real-valued function $f \in W^{\beta}BV_p$ $(1 \leqslant \beta < \infty$, $1 \leqslant p \leqslant \infty)$ and a positive integer n, there are trigonometric polynomials $t_n \in H_n^-(f)$, $T_n \in H_n^+(f)$ such that, for every $\alpha \in \langle 0, \beta\rangle$,*

$$\|f^{(\alpha)}-t_n^{(\alpha)}\|_p+\|T_n^{(\alpha)}-f^{(\alpha)}\|_p \leqslant 3\pi\frac{C_9(\alpha,\beta)+C_{10}(\alpha,\beta)}{n^{\beta-\alpha+1/p}}V_p(f^{(\beta)})$$

(cf. Corollary 1, and Theorem III of [4]).

Remark 2. Under the same hypothesis as in Remark 1, inequality (2.18) implies (2.19) for every $\alpha \in \langle 0, \beta\rangle$ (cf. Th. 4 of [8]).

3. Estimates in the norms and seminorms. We begin with two auxiliary results.

LEMMA 1. *If $f \in BV_p$ $(1 \leqslant p < \infty)$, then*

$$\tau(\delta; f)_p \leqslant 4V_p(f)\delta^{1/p} \quad \textit{for all } \delta \in \langle 0, \pi\rangle.$$

Proof. Given any $\delta \in (0, \pi\rangle$, let us take the positive integer $m = m(\delta)$ satisfying the condition

$$\pi(m+1)^{-1} < \delta \leqslant \pi m^{-1}.$$

Write

$$Z_\delta = \int_0^{2\pi}|M_\delta f(x)|^p dx = \int_{-\pi}^{\pi}\{\sup_{u,v\in\langle\xi,\xi+\delta\rangle}|f(u)-f(v)|\}^p d\xi.$$

Clearly,

$$Z_\delta \leqslant \sum_{v=-m}^{m+1}\int_{(v-1)\delta}^{v\delta}\{\sup_{u,v\in\langle\xi,\xi+\delta\rangle}|f(u)-f(v)|\}^p d\xi.$$

Substituting $\lambda = \xi-\xi_v$, $\xi_v = v\pi/m$, we have

$$Z_\delta \leqslant \sum_{v=-m}^{m+1}\int_{(v-1)\delta-\xi_v}^{v\delta-\xi_v}\{\sup_{u,v\in\langle\lambda+\xi_v,\lambda+\xi_v+\delta\rangle}|f(u)-f(v)|\}^p d\lambda$$
$$\leqslant \int_{-3\delta}^{\delta}\sum_{v=-m}^{m+1}\sup_{u,v\in\langle\lambda+\xi_v,\lambda+\xi_v+\delta\rangle}|f(u)-f(v)|^p d\lambda.$$

Further, for every $\varepsilon > 0$ and every $\lambda \in \langle -3\delta, \delta\rangle$, the last sum does not exceed

$$\sum_{v=-m}^{m+1}\left\{|f(u_{\varepsilon,\lambda,v})-f(v_{\varepsilon,\lambda,v})|^p+\frac{\varepsilon}{2^{|v|}}\right\}$$

for some $u_{\varepsilon,\lambda,v}$, $v_{\varepsilon,\lambda,v} \in \langle\lambda+\xi_v, \lambda+\xi_v+\delta\rangle$. Hence

$$Z_\delta \leqslant \int_{-3\delta}^{\delta}\{V_p(f; -\pi-3\delta, 2\pi+2\delta)\}^p d\lambda+\int_{-3\delta}^{\delta}\sum_{v=-\infty}^{\infty}\frac{\varepsilon}{2^{|v|}}d\lambda.$$

Passing to the limit as $\varepsilon \to 0+$, we obtain

$$Z_\delta \leqslant \{V_p(f; -4\pi, 4\pi)\}^p\cdot 4\delta \quad \text{for every } \delta \in (0, \pi\rangle.$$

Consequently,

$$\tau(\delta; f)_p = \left(\frac{1}{2\pi}Z_\delta\right)^{1/p} \leqslant \left(\frac{2\delta}{\pi}\right)^{1/p}V_p(f; -4\pi, 4\pi)$$

and this immediately implies the lemma.

LEMMA 2. *Suppose that the real-valued function f is, alternately, non-increasing [non-decreasing] and non-decreasing [non-increasing] in the neighbouring intervals*

$$\langle u_{v-1}, u_v\rangle, \quad \langle u_v, u_{v+1}\rangle \quad (v = 1, 2, \ldots, r-1).$$

Then, for any number $p \geqslant 1$,

$$V_p(f;\, u_0,\, u_r) = \{\sum_{j=0}^{r-1} |f(u_{j+1})-f(u_j)|^p\}^{1/p}.$$

Proof. Consider the special case $r = 2$. Write $a = u_0$, $b = u_2$, $c = u_1$. Choose a partition $\{a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\}$ in which $x_m \leqslant c$, $x_{m+1} > c$ $(0 \leqslant m \leqslant n-1)$.

By the well-known inequality

$$\sum_{k=l}^{l+\mu} A_k^p \leqslant (\sum_{k=l}^{l+\mu} A_k)^p \quad (A_k \geqslant 0,\ l+1,\ \mu+1 \in N), \tag{3.1}$$

$$\sum_{k=0}^{n-1} |f(x_{k+1})-f(x_k)|^p = \sum_{k=0}^{m-1} |f(x_{k+1})-f(x_k)|^p + \\ +|f(x_{m+1})-f(x_m)|^p + \sum_{k=m+1}^{n-1} |f(x_{k+1})-f(x_k)|^p \\ \leqslant |f(x_m)-f(x_0)|^p + |f(x_{m+1})-f(x_m)|^p + |f(x_n)-f(x_{m+1})|^p.$$

Since $|f(x_{m+1})-f(x_m)|$ does not exceed $|f(c)-f(x_m)|$ or $|f(x_{m+1})-f(c)|$, we have

$$\sum_{k=0}^{n-1} |f(x_{k+1})-f(x_k)|^p \leqslant |f(x_m)-f(a)|^p + |f(c)-f(x_m)|^p + \\ +|f(x_{m+1})-f(c)|^p + |f(b)-f(x_{m+1})|^p \\ \leqslant |f(c)-f(a)|^p + |f(b)-f(c)|^p.$$

Consequently,

$$V_p(f;\, a,\, b) \leqslant \{|f(c)-f(a)|^p + |f(b)-f(c)|^p\}^{1/p},$$

and the desired assertion is established.

For $r > 2$ the proof runs analogously.

THEOREM 3. *Suppose that f is a real-valued function of class BV_p $(1 \leqslant p < \infty)$. Then, for every $n \in N$, there are trigonometric polynomials $t_n \in H_n^-(f)$, $T_n \in H_n^+(f)$ such that*

$$\|f-t_n\|_p + \|T_n-f\|_p \leqslant C_{11}\, n^{-1/p}\, V_p(f), \tag{3.2}$$

$$V_p(f-t_n) + V_p(T_n-f) \leqslant C_{12}\, V_p(f). \tag{3.3}$$

Proof. Starting with the points $x_k = k\pi/n$, $y_k = x_k - \pi/(2n)$ $(k = 0, \pm 1, \pm 2, \ldots)$, we construct, as in [1], the polygonal lines

$$J_n(x) = \begin{cases} \inf\limits_{v \in \langle x_{k-1}, x_k \rangle} f(v) & \text{for} \quad x = y_k, \\ \min(J_n(y_k),\, J_n(y_{k+1})) & \text{for} \quad x = x_k, \\ \text{linear for } x \in \langle y_k,\, x_k \rangle \text{ and } x \in \langle x_k,\, y_{k+1} \rangle, \end{cases}$$

$$S_n(x) = \begin{cases} \sup\limits_{v \in \langle x_{k-1}, x_k \rangle} f(v) & \text{for} \quad x = y_k, \\ \max(S_n(y_k),\, S_n(y_{k+1})) & \text{for} \quad x = x_k, \\ \text{linear for } x \in \langle y_k,\, x_k \rangle \text{ and } x \in \langle x_k,\, y_{k+1} \rangle. \end{cases}$$

Obviously, $J_n, S_n \in AC$ and

$$J_n(x) \leqslant f(x) \leqslant S_n(x) \quad \text{for all real } x.$$

In the open intervals (y_k, x_k) and (x_k, y_{k+1}) the ordinary derivatives $J_n'(x)$, $S_n'(x)$ are equal to some real constants. Assuming that

$$J_n'(x_k) = \tfrac{1}{2}\{J_n'(x_k+0)+J_n'(x_k-0)\}, \qquad J_n'(y_l) = \tfrac{1}{2}\{J_n'(y_l+0)+J_n'(y_l-0)\}$$

$(k, l = 0, \pm 1, \pm 2, \ldots)$ and that $S_n'(x_k)$, $S_n'(y_l)$ are defined similarly, we get a pair of 2π-periodic step functions J_n', S_n'.

By the Lebesgue dominated convergence theorem,

$$J_n(x) = \hat{J_n}(0) + (J_n' * \Psi_1)(x),$$
$$S_n(x) = \hat{S_n}(0) + (S_n' * \Psi_1)(x)$$

for all real x. Putting

$$J_{n|+}'(v) = \tfrac{1}{2}\{|J_n'(v)| + J_n'(v)\}, \qquad J_{n|-}'(v) = \tfrac{1}{2}\{|J_n'(v)| - J_n'(v)\},$$

we obtain

$$J_n(x) = \hat{J_n}(0) + (J_{n|+}' * \Psi_1)(x) - (J_{n|-}' * \Psi_1)(x)$$

for all x. Analogously,

$$S_n(x) = \hat{S_n}(0) + (S_{n|+}' * \Psi_1)(x) - (S_{n|-}' * \Psi_1)(x).$$

Consider the trigonometric polynomials

$$t_{0,n} \in H_n^-(\Psi_1), \qquad T_{0,n} \in H_n^+(\Psi_1)$$

of the best one-sided approximation of Ψ_1 in the L-metric. Introduce the trigonometric polynomials

$$t_n(x) = \hat{J_n}(0) + (J_{n|+}' * t_{0,n})(x) - (J_{n|-}' * T_{0,n})(x),$$
$$T_n(x) = \hat{S_n}(0) + (S_{n|+}' * T_{0,n})(x) - (S_{n|-}' * t_{0,n})(x).$$

It can easily be verified that $t_n \in H_n^-(J_n)$, $T_n \in H_n^+(S_n)$ and

$$\|J_n - t_n\|_p \leqslant \frac{C_{13}}{n}\|J_n'\|_p, \qquad \|T_n - S_n\|_p \leqslant \frac{C_{13}}{n}\|S_n'\|_p$$

(see (2.16)). Hence

$$\|f-t_n\|_p \leqslant \|T_n-t_n\|_p \leqslant \|T_n-S_n\|_p + \|S_n-J_n\|_p + \|J_n-t_n\|_p \\ \leqslant \frac{C_{13}}{n}\{\|S_n'\|_p + \|J_n'\|_p\} + \|S_n-J_n\|_p.$$

But

$$\|S_n'\|_p+\|J_n'\|_p \leqslant \frac{4n}{\pi}\tau\left(\frac{4\pi}{n};f\right)_p, \qquad \|S_n-J_n\|_p \leqslant \tau\left(\frac{2\pi}{n};f\right)_p.$$

Consequently,

$$\|f-t_n\|_p \leqslant \left(\frac{8C_{13}}{\pi}+1\right)\tau\left(\frac{2\pi}{n};f\right)_p,$$

and we have the same inequality for $\|T_n-f\|_p$ (see [1], pp. 801–802, 794). Aplying Lemma 1, we get (3.2).

Clearly,

$$V_p(f-t_n) \leqslant V_p(f)+V_p(J_n)+V_p(J_n-t_n). \tag{3.4}$$

Moreover, in view of Lemma 2,

$$V_p(J_n) = \{\sum_{\nu=0}^{r-1}|J_n(u_{\nu+1})-J_n(u_\nu)|^p\}^{1/p},$$

where $\langle u_\nu, u_{\nu+1}\rangle$ denotes the νth interval of monotonicity of J_n $(0=u_0<u_1<\ldots<u_{r-1}<u_r=2\pi)$. Evidently, we may suppose that every point $u_\nu\in(0, 2\pi)$ coincides with some y_l $(1\leqslant l\leqslant 2n)$.

Easy calculation shows that

$$V_p(J_n) \leqslant 4V_p(f). \tag{3.5}$$

For example, let $r=3$ and let

$$u_0=x_0, \quad u_1=y_8, \quad u_2=y_{n+4}, \quad u_3=x_{2n} \quad (n\geqslant 5).$$

Then

$$V_p(J_n) = \{|J_n(y_8)-J_n(x_0)|^p+|J_n(y_{n+4})-J_n(y_8)|^p+|J_n(x_{2n})-J_n(y_{n+4})|^p\}^{1/p}.$$

Assuming that J_n is non-decreasing in $\langle x_0, y_8\rangle$, we have

$$\begin{aligned}V_p(J_n) &\leqslant \{(f(y_8)-J_n(x_0))^p+(f(y_8)-J_n(y_{n+4}))^p+(f(x_{2n})-J_n(y_{n+4}))^p\}^{1/p}\\
&\leqslant \{(\sup_{u\in\langle x_7,x_8\rangle} f(u)-\inf_{v\in\langle x_{-1},x_0\rangle} f(v))^p+(\sup_{u\in\langle x_7,x_8\rangle} f(u)-\inf_{v\in\langle x_{n+3},x_{n+4}\rangle} f(v))^p+\\
&\qquad +(\sup_{u\in\langle x_{2n-1},x_{2n}\rangle} f(u)-\inf_{v\in\langle x_{n+3},x_{n+4}\rangle} f(v))^p\}^{1/p}\\
&\leqslant \{\sup_{u,v\in\langle x_{-1},x_8\rangle}|f(u)-f(v)|^p+\sup_{u,v\in\langle x_7,x_{n+4}\rangle}|f(u)-f(v)|^p+\\
&\qquad +\sup_{u,v\in\langle x_{n+3},x_{2n}\rangle}|f(u)-f(v)|^p\}^{1/p}.\end{aligned}$$

Applying now the inequalities

$$(A+B)^p \leqslant 2^p(A^p+B^p) \quad (A,\ B\geqslant 0), \tag{3.6}$$

$$(A+B)^{1/p} \leqslant A^{1/p}+B^{1/p} \quad (A,\ B\geqslant 0), \tag{3.7}$$

we obtain

$$\begin{aligned}V_p(J_n) &\leqslant 2\{\sup_{u,v\in\langle x_{-1},x_0\rangle}|f(u)-f(v)|^p+\sup_{u,v\in\langle x_0,x_8\rangle}|f(u)-f(v)|^p+\\
&\qquad +\sup_{u,v\in\langle x_7,x_8\rangle}|f(u)-f(v)|^p+\sup_{u,v\in\langle x_8,x_{n+4}\rangle}|f(u)-f(v)|^p+\\
&\qquad +\sup_{u,v\in\langle x_{n+3},x_{n+4}\rangle}|f(u)-f(v)|^p+\sup_{u,v\in\langle x_{n+4},x_{2n}\rangle}|f(u)-f(v)|^p\}^{1/p}\\
&\leqslant 2\{\sup_{u,v\in\langle x_0,x_8\rangle}|f(u)-f(v)|^p+\sup_{u,v\in\langle x_8,x_{n+4}\rangle}|f(u)-f(v)|^p+\\
&\qquad +\sup_{u,v\in\langle x_{n+4},x_{2n}\rangle}|f(u)-f(v)|^p\}^{1/p}+2\{\sup_{u,v\in\langle x_7,x_8\rangle}|f(u)-f(v)|^p+\\
&\qquad +\sup_{u,v\in\langle x_{n+3},x_{n+4}\rangle}|f(u)-f(v)|^p+\sup_{u,v\in\langle x_{2n-1},x_{2n}\rangle}|f(u)-f(v)|^p\}^{1/p}.\end{aligned}$$

This immediately implies (3.5).

To evaluate the pth variation of the function

$$F(x) = J_n(x)-t_n(x),$$

we observe that, for all real x,

$$\begin{aligned}F(x) &= \frac{1}{2\pi}\int_0^{2\pi} J'_{n|+}(x-u)\{\Psi_1(u)-t_{0,n}(u)\}\,du+\\
&\qquad +\frac{1}{2\pi}\int_0^{2\pi} J'_{n|-}(x-u)\{T_{0,n}(u)-\Psi_1(u)\}\,du.\end{aligned}$$

Therefore, for any partition $\{0=\alpha_0<\alpha_1<\ldots<\alpha_{l-1}<\alpha_l=2\pi\}$,

$$\begin{aligned}2\pi\{\sum_{j=0}^{l-1}|F(\alpha_{j+1})-F(\alpha_j)|^p\}^{1/p} &\leqslant \int_0^{2\pi}|\Psi_1(u)-t_{0,n}(u)|\{\sum_{j=0}^{l-1}|J'_{n|+}(\alpha_{j+1}-u)-J'_{n|+}(\alpha_j-u)|^p\}^{1/p}du+\\
&\qquad +\int_0^{2\pi}|T_{0,n}(u)-\Psi_1(u)|\{\sum_{j=0}^{l-1}|J'_{n|-}(\alpha_{j+1}-u)-J'_{n|-}(\alpha_j-u)|^p\}^{1/p}du;\end{aligned}$$

hence

$$V_p(F) \leqslant V_p(J_n';-2\pi,\ 2\pi)\{\|\Psi_1-t_{0,n}\|_1+\|T_{0,n}-\Psi_1\|_1\}.$$

Consequently (see Lemma 1 in [4]),

(3.8) $$V_p(J_n-t_n) \leqslant \frac{4\pi}{n} V_p(J'_n).$$

In view of Lemma 2,

$$V_p(J'_n) = \{\sum_{\nu=0}^{m-1} |J'_n(\tilde{u}_{\nu+1})-J'_n(\tilde{u}_\nu)|^p\}^{1/p},$$

where $\langle \tilde{u}_\nu, \tilde{u}_{\nu+1}\rangle$ $(\nu = 0, 1, \ldots, m-1)$ denote the subintervals of $\langle 0, 2\pi\rangle$ in which the step function J'_n is alternately non-decreasing [non-increasing] and non-increasing [non-decreasing]. We may suppose that every $\tilde{u}_\nu \in (0, 2\pi)$ is of the form $\frac{1}{2}(x_{k_\nu}+y_{k_\nu+1})$ or $\frac{1}{2}(y_{l_\nu}+x_{l_\nu})$ $(0 \leqslant k_\nu \leqslant 2n-1,\ 1 \leqslant l_\nu \leqslant 2n)$.

By Minkowski's inequality and (3.6),

$$V_p(J'_n) \leqslant \frac{2n}{\pi}\left\{\sum_{\mu=1}^{m-1}\left|J'_n(\tilde{u}_\mu)\frac{\pi}{2n}\right|^p+\left|J'_n(2\pi)\frac{\pi}{2n}\right|^p\right\}^{1/p}+$$
$$+\frac{2n}{\pi}\left\{\left|J'_n(0)\frac{\pi}{2n}\right|^p+\sum_{\nu=1}^{m-1}\left|J'_n(\tilde{u}_\nu)\frac{\pi}{2n}\right|^p\right\}^{1/p}$$
$$\leqslant \frac{2n}{\pi}\{\sum_{k=0}^{2n-1}|J_n(y_{k+1})-J_n(x_k)|^p+\sum_{k=1}^{2n}|J_n(x_k)-J_n(y_k)|^p+$$
$$+|J_n(x_{2n})-J_n(y_{2n})|^p+|J_n(y_{2n+1})-J_n(x_{2n})|^p\}^{1/p}+$$
$$+\frac{2n}{\pi}\{|J_n(x_0)-J_n(y_0)|^p+|J_n(y_1)-J_n(x_0)|^p+$$
$$+\sum_{k=0}^{2n-1}|J_n(y_{k+1})-J_n(x_k)|^p+\sum_{k=1}^{2n}|J_n(x_k)-J_n(y_k)|^p\}^{1/p}.$$

Next, inequality (3.7) leads to

(3.9) $$V_p(J'_n) \leqslant \frac{8n}{\pi} V_p(J_n).$$

Applying (3.4), (3.5), (3.8), (3.9) and the parallel estimates involving T_n and S_n, we obtain the desired assertion (3.3).

Remark 3. For $p = \infty$, Theorem 3 remains valid ($1/p$ is treated as zero). The proof runs on the same lines.

Under the assumption $f \in L_0^\infty \cap BV_p$ $(p \geqslant 1)$, estimate (1.3) implies only (3.2) by Lemma 1.

THEOREM 4. *Let f be a function of class $W^\beta BV_p (1 \leqslant \beta,\ p < \infty)$. Then the derivatives $f^{(\alpha)}$ are in AC $[C \cap BV_p]$ if $0 \leqslant \alpha \leqslant \beta-1$ [resp. $\beta-1 < \alpha < \beta-1+1/p$]. Moreover, in the case of a real-valued f, to any $n \in N$ there exist trigonometric polynomials $t_n \in H_n^-(f)$, $T_n \in H_n^+(f)$ such that, for every $\alpha \in \langle 0, \beta-1+1/p)$,*

(3.10) $$\|f^{(\alpha)}-t_n^{(\alpha)}\|_p+\|T_n^{(\alpha)}-f^{(\alpha)}\|_p \leqslant \frac{C_{14}(\alpha, \beta)}{n^{\beta-\alpha+1/p}} V_p(f^{(\beta)}),$$

(3.11) $$V_p(f^{(\alpha)}-t_n^{(\alpha)})+V_p(T_n^{(\alpha)}-f^{(\alpha)}) \leqslant \frac{C_{15}(\alpha, \beta)}{n^{\beta-\alpha}} V_p(f^{(\beta)}).$$

Proof. (1) From (2.4) it follows that the Fourier coefficients $c_k = f^\wedge(k)$ are of the order $O(|k|^{-\beta-1/p})$ as $k \to \pm\infty$ (see [5], p. 38). Consequently, in the case $0 \leqslant \alpha < \beta-1+1/p$,

$$f^{(\alpha)}(x) = \sum_{k=-\infty}^{\infty} c_k (ik)^\alpha e^{ikx}$$

uniformly in $x \in (-\infty, \infty)$; hence $f^{(\alpha)} \in C$. If $0 \leqslant \alpha \leqslant \beta-1$, identities (2.5) and (2.6) imply $f^{(\alpha)} \in AC$.

Denote by $c_k^{(\beta)}$ the kth Fourier coefficient of $f^{(\beta)}$. Then

$$c_k^{(\beta)} = \frac{1}{2\pi}\int_0^{2\pi} f^{(\beta)}(u) e^{-iku}\,du = c_k (ik)^\beta \quad (k = 0, \pm 1, \pm 2, \ldots).$$

Consequently, for every $\alpha \in (0, \beta-1+1/p)$ and all real x,

$$f^{(\alpha)}(x) = \sum_{k=-\infty}^{\infty}{}' c_k^{(\beta)} (ik)^{\alpha-\beta} e^{ikx}$$
$$= \frac{1}{2\pi}\lim_{\nu\to\infty}\int_0^{2\pi} f^{(\beta)}(u) \sum_{k=-\nu}^{\nu}{}' (ik)^{\alpha-\beta} e^{ik(x-u)}\,du$$
$$= \frac{1}{2\pi}\int_0^{2\pi} f^{(\beta)}(u)\, \Psi_{\beta-\alpha}(x-u)\,du = (f^{(\beta)} * \Psi_{\beta-\alpha})(x),$$

by the Lebesgue dominated convergence theorem (see [9], p. 191). This implies

$$V_p(f^{(\alpha)}) \leqslant 2V_p(f^{(\beta)})\|\Psi_{\beta-\alpha}\|_1 < \infty.$$

Thus,

$$f^{(\alpha)} \in C \cap BV_p \quad \text{if} \quad 0 < \alpha < \beta-1+1/p.$$

(2) Let f be a real-valued function of class $W^\beta BV_p$. In view of Theorem 3, there are real-valued trigonometric polynomials

$$Q_n(u) = \sum_{|k|<n} \gamma_k e^{iku} \quad (n = 1, 2, \ldots)$$

satisfying the inequalities

$$\|f^{(\beta)}-Q_n\|_p \leqslant C_{11} n^{-1/p} V_p(f^{(\beta)}), \quad V_p(f^{(\beta)}-Q_n) \leqslant C_{12} V_p(f^{(\beta)}).$$

Considering the modified polynomials

$$Q_n^*(u) = Q_n(u) - \gamma_0,$$

we have

$$\|f^{(\beta)} - Q_n^*\|_p \leqslant \|f^{(\beta)} - Q_n\|_p + \|\gamma_0\|_p \leqslant C_{11}\, n^{-1/p}\, V_p(f^{(\beta)}) + |\gamma_0|.$$

Further,

$$|\gamma_0| = \frac{1}{2\pi}\left|\int_0^{2\pi} \{Q_n(u) - f^{(\beta)}(u)\}\, du\right| \leqslant \|Q_n - f^{(\beta)}\|_p.$$

Consequently,

$$\|f^{(\beta)} - Q_n^*\|_p \leqslant 2C_{11}\, n^{-1/p}\, V_p(f^{(\beta)}) \qquad (n = 1, 2, \ldots).$$

Moreover,

$$V_p(f^{(\beta)} - Q_n^*) = V_p(f^{(\beta)} - Q_n) \leqslant C_{12}\, V_p(f^{(\beta)}).$$

Write

$$g(x) = f(x) - I_\beta[Q_n](x) \qquad (-\infty < x < \infty).$$

Then, $g \in W^\beta BV_p$ and

$$g^{(\beta)}(x) = f^{(\beta)}(x) - Q_n^*(x) \qquad (-\infty < x < \infty).$$

Retain the symbols $f_{|+}^{(\beta)}$, $f_{|-}^{(\beta)}$ and $t_{\alpha,n}$, $T_{\alpha,n}$ used in the proof of Theorem 1. Introduce trigonometric polynomials $Y_n \in H_n^+(g)$ $(n = 1, 2, \ldots)$ such that

$$(3.12)\qquad Y_n(x) - g(x) = \frac{1}{2\pi}\int_0^{2\pi} g_{|+}^{(\beta)}(u)\{T_{0,n}(x-u) - \Psi_\beta(x-u)\}\, du + \frac{1}{2\pi}\int_0^{2\pi} g_{|-}^{(\beta)}(u)\{\Psi_\beta(x-u) - t_{0,n}(x-u)\}\, du$$

for all real x. Then (see (2.16)),

$$\|Y_n - g\|_p \leqslant C_4(\beta)\, n^{-\beta}\, \|g^{(\beta)}\|_p = C_4(\beta)\, n^{-\beta}\, \|f^{(\beta)} - Q_n^*\|_p \leqslant 2C_4(\beta)\, C_{11}\, n^{-\beta-1/p}\, V_p(f^{(\beta)}).$$

Taking the polynomials $T_n = Y_n + I_\beta[Q_n]$, we observe that $T_n - f = Y_n - g$. Hence $T_n \in H_n^+(f)$ and

$$(3.13)\qquad \|T_n - f\|_p \leqslant 2C_4(\beta)\, C_{11}\, n^{-\beta-1/p}\, V_p(f^{(\beta)}).$$

Moreover (see the proofs of estimates (3.8) and (2.16)),

$$V_p(T_n - f) = V_p(Y_n - g) \leqslant 2V_p(g^{(\beta)})\{\|T_{0,n} - \Psi_\beta\|_1 + \|\Psi_\beta - t_{0,n}\|_1\},$$

i.e.,

$$(3.14)\qquad V_p(T_n - f) \leqslant C_{16}(\beta)\, n^{-\beta}\, V_p(f^{(\beta)}).$$

Analogously, we can construct polynomials $t_n \in H_n^-(f)$ such that inequalities (3.13), (3.14) in which T_n is replaced by t_n remain valid. Thus, for $\alpha = 0$, the desired estimates (3.10), (3.11) are proved.

(3) Considering f as in (2), we observe that the Fourier coefficients $g\hat{}(k) = f\hat{}(k) - I_\beta[Q_n]\hat{}(k)$ are of the order $O(|k|^{-\beta-1/p})$. Hence, under the assumption $0 < \alpha < \beta - 1 + 1/p$,

$$(3.15)\qquad Y_n^{(\alpha)}(x) - g^{(\alpha)}(x) = \frac{1}{2\pi}\int_0^{2\pi} g_{|+}^{(\beta)}(u)\{T_{0,n}^{(\alpha)}(x-u) - \Psi_\beta^{(\alpha)}(x-u)\}\, du + \frac{1}{2\pi}\int_0^{2\pi} g_{|-}^{(\beta)}(u)\{\Psi_\beta^{(\alpha)}(x-u) - t_{0,n}^{(\alpha)}(x-u)\}\, du$$

for all real x (see (3.12), (2.5) and (2.6)).

By Theorem 2 and Remark 2 (see also [2], p. 360, [4], and [3], Th. 6),

$$\|\Psi_\beta^{(\alpha)} - t_{0,n}^{(\alpha)}\|_1 \leqslant C_6(\alpha, \alpha)\, E_n(\Psi_\beta^{(\alpha)})_1 = C_6(\alpha, \alpha)\, E_n(\Psi_{\beta-\alpha})_1,$$

$$\|T_{0,n}^{(\alpha)} - \Psi_\beta^{(\alpha)}\|_1 \leqslant C_6(\alpha, \alpha)\, E_n(\Psi_\beta^{(\alpha)})_1 = C_6(\alpha, \alpha)\, E_n(\Psi_{\beta-\alpha})_1;$$

hence

$$\|T_{0,n}^{(\alpha)} - \Psi_\beta^{(\alpha)}\|_1 + \|\Psi_\beta^{(\alpha)} - t_{0,n}^{(\alpha)}\|_1 \leqslant C_{17}(\alpha, \beta)\, n^{\alpha-\beta}.$$

Consequently,

$$\|Y_n^{(\alpha)} - g^{(\alpha)}\|_p \leqslant \|g_{|+}^{(\beta)}\|_p \|T_{0,n}^{(\alpha)} - \Psi_\beta^{(\alpha)}\|_1 + \|g_{|-}^{(\beta)}\|_p \|\Psi_\beta^{(\alpha)} - t_{0,n}^{(\alpha)}\|_1 \leqslant C_{17}(\alpha, \beta)\, n^{\alpha-\beta}\, \|g^{(\beta)}\|_p = C_{17}(\alpha, \beta)\, n^{\alpha-\beta}\, \|f^{(\beta)} - Q_n^*\|_p.$$

But $T_n^{(\alpha)} - f^{(\alpha)} = Y_n^{(\alpha)} - g^{(\alpha)}$. Therefore,

$$(3.16)\qquad \|T_n^{(\alpha)} - f^{(\alpha)}\|_p \leqslant \frac{2C_{11}\, C_{17}(\alpha, \beta)}{n^{\beta-\alpha+1/p}}\, V_p(f^{(\beta)}).$$

Denote by $G(x)$ the right-hand side of (3.15). Then, for an arbitrary partition $\{0 = v_0 < v_1 < v_2 < \ldots < v_{m-1} < v_m = 2\pi\}$,

$$\Big\{\sum_{j=1}^{m-1} |G(v_{j+1})-G(v_j)|^p\Big\}^{1/p}$$

$$\leqslant \frac{1}{2\pi}\int_0^{2\pi} \Big\{\sum_{j=1}^{m-1} |g_{+}^{(\beta)}(v_{j+1}-v)-g_{+}^{(\beta)}(v_j-v)|^p\,|T_{0,n}^{(\alpha)}(v)-\Psi_\beta^{(\alpha)}(v)|^p\Big\}^{1/p} dv +$$

$$+\frac{1}{2\pi}\int_0^{2\pi} \Big\{\sum_{j=1}^{m-1} |g_{-}^{(\beta)}(v_{j+1}-v)-g_{-}^{(\beta)}(v_j-v)|^p\,|\Psi_\beta^{(\alpha)}(v)-t_{0,n}^{(\alpha)}(v)|^p\Big\}^{1/p} dv$$

$$\leqslant V_p(g_{+}^{(\beta)};-2\pi,\,2\pi)\,\|T_{0,n}^{(\alpha)}-\Psi_\beta^{(\alpha)}\|_1 + V_p(g_{-}^{(\beta)};-2\pi,\,2\pi)\,\|\Psi_\beta^{(\alpha)}-t_{0,n}^{(\alpha)}\|_1 .$$

Consequently,

$$V_p(G) \leqslant 2C_{17}(\alpha,\beta)\,n^{\alpha-\beta}\,V_p(g^{(\beta)}).$$

Thus

$$V_p(T_n^{(\alpha)}-f^{(\alpha)}) \leqslant 2C_{12}C_{17}(\alpha,\beta)\,n^{\alpha-\beta}\,V_p(f^{(\beta)}). \tag{3.17}$$

Observing that in (3.16) and (3.17) the Weyl derivatives $T_n^{(\alpha)}$ can be replaced by $t_n^{(\alpha)}$, we get (3.10) and (3.11) for all numbers $\alpha \in (0, \beta-1+1/p)$.

Remark 4. Theorems 1, 2 ensure that Theorem 4, in which $p=\infty$, is also true.

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