STUDIA MATHEMATICA, T. LXXX. (1984)

Multivariate interpolation II of Lagrange and Hermite type

by

HAKOP A. HAKOPIAN (Yerevan)

Abstract. We present the investigation of second pointwise nature multivariate interpolation (MI-II) introduced in [5]. The Lagrange case of this interpolation in algebraic form was found independently by the authors of [2].

Introduction. In this paper we give the remainder formula, the Lagrange and Newton forms and a recurrence relation for the interpolant polynomial. Further we bring an example of application: "Star" numerical integration and a formula for the main determinant (Vandermonde) of this interpolation.

For the similar aspects of ("dual") Multivariate Interpolation I (MI-I) of Lagrange and Hermite type we refer to [5]–[8], see also [1] and [3], [4], [10].

Let $t_0, \ldots, t_r \in R$ and let $m(t_n)$, $n = 0, \ldots, r$, be the multiplicity of t_n, that is $m(t_n)$ is the cardinality of the set $\{m|\ t_m = t_n,\ m = 0, \ldots, r\}$. Then the familiar univariate Lagrange–Hermite interpolant to f at knots $t_0, \ldots, t_r$ is the unique polynomial P_f of degree not exceeding r, with

$$P_f^{(m)}(t_n) = f^{(m)}(t_n), \quad n = 0, \ldots, r, \quad m = 0, \ldots, m(t_n)-1.$$

For distinct knots, i.e., when $m(t_n) = 1$, $n = 0, \ldots, r$, this polynomial can be written in the Lagrange from:

$$P_f(t) = \sum_{n=0}^{r} f(t_n) \prod_{\substack{m=0 \\ m \neq n}}^{r} \frac{t-t_m}{t_n-t_m}. \tag{1}$$

In the general case P_f can be written in the Newton form (which uses divided differences),

$$P_f(t) = \sum_{n=0}^{r} (t-t_0)\ldots(t-t_{n-1})[t_0, \ldots, t_n]f. \tag{2}$$

Hence the remainder of interpolation has the following representation:

$$f(t) - P_f(t) = (t-t_0)\ldots(t-t_r)[t, t_0, \ldots, t_r]f. \tag{3}$$

We will use this formula for the knots $x_0, \ldots, x_r$ which lie on some line l in

R^k. Let $u \in R^k$ be a unit direction-vector of l. Then formula (3) is modified as follows:

(4) $$f(x)-P_f(x) = \varrho(x, x_0)\ldots\varrho(x, x_r)[x, x_0, \ldots, x_r]f,$$

where $x \in l$, $\varrho(x, x_n)$ is the signed distance (with respect to u) of x and x_n. A convenient way of introducing divided difference here is its well-known Hermite–Genocchi representation,

$$[x, x_0, \ldots, x_r]f = \int_{Q^{r+1}} (D_u)^{r+1} f(\nu x+\nu_0 x_0+\ldots+\nu_r x_r)\, d\nu_0 \ldots d\nu_r,$$

where

$$Q^{r+1} = \{(\nu_0, \ldots, \nu_r) \mid \sum_{n=0}^{r} \nu_n \leqslant 1,\ \nu_m \geqslant 0,\ m = 0, \ldots, r\}$$

and

$$\nu = 1-\sum_{n=0}^{r} \nu_n,$$

D_u denoting the directional derivative.

Finally, let us mention the following recurrence relation for interpolant polynomials,

(5) $$P_f(t) = \frac{t-t_0}{t_r-t_0} P_f^0(t)+\frac{t_r-t}{t_r-t_0} P_f^r(t),$$

where $t_0 \neq t_r$ and P_f^n interpolates f at $\{t_0, \ldots, t_r\} \setminus \{t_n\}$, $n = 0, r$.

2. Multivariate interpolation II and the Lagrange form. We start with some notation.

$I_m^n :=$ collection of subsets of $\{0, \ldots, n\}$ of cardinality m. For $q \in I_1^n$ and $i = (i_1, \ldots, i_m) \in I_m^n$, $(q, i) := (q, i_1, \ldots, i_m) \in I_{m+1}^n$ provided $q \notin i$. For $x, y \in R^k$, $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_k)$ and the multiindex $\alpha = (\alpha_1, \ldots, \alpha_k)$ we use the following standard notation:

$$(x, y) = \sum_{n=1}^{k} x_n y_n, \quad |x| = \sqrt{(x, x)}, \quad x^\alpha = x_1^{\alpha_1}\ldots x_k^{\alpha_k},$$

$$|\alpha| = \sum_{n=1}^{k} \alpha_n, \quad \alpha! = \alpha_1!\ldots\alpha_k!, \quad D^\alpha = (\partial/\partial x_1)^{\alpha_1}\ldots(\partial/\partial x_k)^{\alpha_k}.$$

We denote by $\pi_m = \pi_m(R^k)$ the set of k-variate polynomials of total degree not exceeding m.

Let $L_0, \ldots, L_r$ be $(k-1)$-dimensional hyperplanes in R^k and let the equation

$$\lambda_1^n x_1+\ldots+\lambda_k^n x_k+\lambda_{k+1}^n = 0$$

determine L_n, $n = 0, \ldots, r$. We briefly write for $i = (i_1, \ldots, i_m) \in I_m^r$,

$$\{L^i\} := \bigcap_{n=1}^{m} L_{i_n}.$$

Let us call $L_0, \ldots, L_r$ *admissible* if $x_i := \{L^i\}$ $\forall i \in I_k^r$ is a point in R^k. The admissibility of $L_0, \ldots, L_r$ is clearly equivalent to

(6) $$d_\lambda\{i\} := \det \|\lambda_m^{i_n}\|_{m,n=1}^{k} \neq 0 \qquad \forall i = (i_1, \ldots, i_k) \in I_k^r.$$

In what follows it is assumed that the hyperplanes $L_0, \ldots, L_r$ are admissible. Let the knot x_i, $i \in I_k^r$ belong to $m(x_i)$ hyperplanes from $L_0, \ldots, L_r$, i.e., $m(x_i)$ is the multiplicity of x_i and $m(x_i) \geqslant k$. We say that $L_0, \ldots, L_r$ *are in general position* if

$$m(x_i) = k \qquad \forall i \in I_k^r.$$

Denote also by $\varrho(x, L_n)$, $n = 0, \ldots, r$, the signed distance of x from L_n,

$$\varrho(x, L_n) = \frac{\lambda_1^n x_1+\ldots+\lambda_k^n x_k+\lambda_{k+1}^n}{\sqrt{(\lambda_1^n)^2+\ldots+(\lambda_k^n)^2}}.$$

Now we are in a position to present the basic

THEOREM 1. *Let $L_0, \ldots, L_r$ be admissible $(k-1)$-dimensional hyperplanes and $\{x_i \mid i \in J\}$ $(J \subset I_k^r)$ be the set of all distinct points from $\{x_i \mid i \in I_k^r\}$; then we have:*

(i) *For an arbitrary set of real numbers*

$$I = \{\gamma_i^\alpha \mid i \in J,\ |\alpha| \leqslant m(x_i)-k\}$$

there is a unique polynomial $P_I \in \pi_{r-k+1}$, such that

$$D^\alpha P_I(x_i) = \gamma_i^\alpha \qquad \forall i \in J,\ |\alpha| \leqslant m(x_i)-k.$$

(ii) *If $L_0, \ldots, L_r$ are in general position, then we have the analog of Lagrange form* (1) *for P_I, namely*

(7) $$P_I(x) = \sum_{i \in I_k^r} P_I(x_i) \prod_{\substack{n=0\\ n \notin i}}^{r} \frac{\varrho(x, L_n)}{\varrho(x_i, L_n)}.$$

Proof. (ii) can be readily checked from formula (7). To prove (i), we consider the polynomials

$$P_{j,\beta}(x) = \frac{(x-x_j)^\beta}{\beta!} \prod_{\substack{n=0\\ x_j \notin L_n}}^{r} \frac{\varrho(x, L_n)}{\varrho(x_j, L_n)},$$

where $j \in I_k^r$, $|\beta| \leqslant m(x_j)-k$. They have the following properties:

$$P_{j,\beta} \in \pi_{r-k+1}$$

and

$$D^\alpha P_{j,\beta}(x_i) = \begin{cases} 1 & \text{if} \quad i=j,\ \alpha=\beta, \\ 0 & \text{if} \quad i=j,\ |\alpha|\leqslant|\beta|,\ \alpha\neq\beta, \\ 0 & \text{if} \quad i\in I_k^r,\ i\neq j,\ |\alpha|\leqslant m(x_i)-k. \end{cases}$$

This clearly gives us a way of construction of P_I. On the other hand, $\dim \pi_{i-k+1} = \# I$, and that completes the proof. ■

This theorem was presented by the author in [5], [7]. Part (ii) was found independently by W. Dahmen and C. A. Micchelli in [2].

We denote by P_f the above unique polynomial for which

$$D^\alpha P_f(x_i) = D^\alpha f(x_i) \qquad \forall i\in J,\ |\alpha|\leqslant m(x_i)-k.$$

This we shall briefly write

$$P_f = f/(L_0, \ldots, L_r).$$

Let us call $L_0, \ldots, L_r$ *interpolatory hyperplanes.*

If L is an n-dimensional hyperplane in R^k, then $f|_L$ denotes the restriction of f to L and is considered as n-variate function.

Remark 1. Let $i\in I_n^r$, $n<k$,

$$P_f = f/(L_m,\ m=0, \ldots, r).$$

Then we have on the $(k-n)$-dimensional hyperplane $\{L^i\}$,

$$P_f|_{L^i} = f|_{L^i}/(\{L^{m,i}\},\ m\in(0, \ldots, r)\backslash i).$$

Of course, interpolatory hyperplanes here are $(k-n-1)$-dimensional and are contained in $\{L^i\}$.

3. The Newton form, remainder formula and a recurrence relation. Let us first choose the directional vector of the line $l_i := \{L^i\}$, $i = (i_1, \ldots, i_{k-1})\in I_{k-1}^m$, as follows

$$u_i = \begin{vmatrix} e_1 & \ldots & e_k \\ \lambda_1^{i_1} & \ldots & \lambda_k^{i_1} \\ \ldots & \ldots & \ldots \\ \lambda_1^{i_{k-1}} & \ldots & \lambda_k^{i_{k-1}} \end{vmatrix},$$

where $e_1, \ldots, e_k\in R^k$, $(e_n)_m = \delta_m^n$, $n, m = 1, \ldots, k$. Denote for $i\in I_{k-1}^m$, $\lambda^n = (\lambda_1^n, \ldots, \lambda_k^n)\in R^k$,

$$c(n, i) := \frac{|u_i|\cdot|\lambda^n|}{(u_i, \lambda^n)} = \frac{|u_i|\cdot|\lambda^n|}{d_\lambda\{(n, i)\}}.$$

Now we present the Newton form of P_f (cf. (2)).

THEOREM 2. *Let the $(k-1)$-dimensional hyperplanes $L_0, \ldots, L_r$ be in general position and let*

$$P_f = f/(L_0, \ldots, L_r).$$

Then

$$P_f(x) = \sum_{n=k-1}^{r} \sum_{i\in I_{k-1}^{n-1}} \prod_{\substack{m=0\\ m\notin i}}^{n-1} c(m, i)\,\varrho(x, L_m)\cdot[x_{(q,i)},\ q\in(0, \ldots, n)\backslash i]\,f. \tag{8}$$

Proof. Let $P_{f,n}$ be the interpolating polynomial satisfying

$$P_{f,n} = f/(L_0, \ldots, L_n), \qquad n = k-1, \ldots, r, \qquad P_{f,k-2}\equiv 0.$$

We use the Lagrange form, and taking into account the above relation we obtain

$$\begin{aligned} (9)\quad & P_{f,n}(x) - P_{f,n-1}(x) \\ &= \sum_{i\in I_k^n} [P_{f,n}(x_i) - P_{f,n-1}(x_i)] \prod_{\substack{m=0\\ m\notin i}}^{n} \frac{\varrho(x, L_m)}{\varrho(x_i, L_m)} \\ &= \sum_{i\in I_{k-1}^{n-1}} [f(x_{(n,i)}) - P_{f,n-1}(x_{(n,i)})] \prod_{\substack{m=0\\ m\notin i}}^{n-1} \frac{\varrho(x, L_m)}{\varrho(x_{(n,i)}, L_m)}, \qquad n = k-1, \ldots, r. \end{aligned}$$

Applying Remark 1 to the line $l_i = L^i$, $i\in I_{k-1}^{n-1}$, we obtain (interpolatory hyperplanes in this case are zero-dimensional, i.e., they are knots)

$$P_{f,n-1}|_{l_i} = f|_{l_i}/(x_{(m,i)},\ m\in(0, \ldots, n-1)\backslash i).$$

Hence according to (4)

$$f(x_{(n,i)}) - P_{f,n-1}(x_{(n,i)}) = \sum_{\substack{m=0\\ m\notin i}}^{n-1} \varrho(x_{(n,i)}, x_{(m,i)})[x_{(l,i)},\ l\in(0, \ldots, n)\backslash i]\,f. \tag{10}$$

Finally we notice that

$$\frac{\varrho(x_{(n,i)}, x_{(m,i)})}{\varrho(x_{(n,i)}, L_m)} = \frac{1}{\cos(u_i, \lambda^m)} = \frac{|u_i|\,|\lambda^m|}{(u_i, \lambda^m)} = c(m, i). \tag{11}$$

Now it remains to sum up (9) using (10) and (11). ■

Let $\lambda^n = (\lambda_1^n, \ldots, \lambda_k^n)$ be a nonzero vector in R^k and $L_n = L_{x,n}$ the $(k-1)$-dimensional hyperplane with normal λ^n and passing through $x\in R^k$ for $n = r+1, \ldots, r+k$. For the convenient presenting of the remainder formula we denote for $i = (i_1, \ldots, i_n)\in I_n^r$, $n\leqslant k-1$,

$$i^0 := (r-n+k-1, \ldots, r+1, i_1, \ldots, i_n)\in I_{k-1}^{r+k-1}.$$

We mean here that $\emptyset \in I_0^r$ and $\emptyset^0 = (r+k-1, \ldots, r+1)$ and for $k=1$, $\emptyset^0 = \emptyset$. The following theorem gives the remainder formula (cf. (3)).

THEOREM 3. *Let $L_0, \ldots, L_r, L_{x,r+1}, \ldots, L_{x,r+k}$ be in general position. Then*

(12) $$f(x)-P_f(x) = \sum_{n=0}^{k-1} \sum_{i\in I_n^r} \prod_{\substack{m=0\\ m\notin i}}^{r} c(m, i^0)\varrho(x, L_m)[x_{(q,i^0)}, q=r+k-n, q\in(0,\ldots,r)\setminus i]f.$$

Proof. Let

$$\tilde{P}_f = f/(L_0, \ldots, L_r, L_{x,r+1}, \ldots, L_{x,r+k}),$$

where x has been fixed for a moment. We have $\tilde{P}_f(x) = f(x)$ since x is the common point of $L_{x,r+1}, \ldots, L_{x,r+k}$, i.e.,

(13) $$x = x_{(r+1,\ldots,r+k)} = \bigcap_{m=r+1}^{r+k} L_{x,m}.$$

Using the Newton forms of $\tilde{P}_f$ and P_f we readily obtain

$$\tilde{P}_f(y) = P_f(y) + \sum_{n=r+1}^{r+k} \sum_{i\in I_{k-1}^{n-1}} \prod_{\substack{m=0\\ m\notin i}}^{n-1} c(m, i)\varrho(y, L_m)[x_{(q,i)}, q\in(0,\ldots,n)\setminus i]f.$$

Now we put $y = x$ in the above relation. Since $\varrho(x, L_{x,m}) = 0$, $m = r+1, \ldots, r+k$, we have

$$f(x) = \tilde{P}_f(x) = P_f(x) + \sum_{n=r+1}^{r+k} \sum_{i\in I_{k+r-n}^r} \prod_{\substack{m=0\\ m\notin i}}^{n-1} c(m, i^0)\varrho(x, L_m)[x_{(q,i^0)}, q=n, q\in(0,\ldots,r)\setminus i]f. \quad \blacksquare$$

Let us note that the participation of the hyperplane $L_{x,r+k}$ in Theorem 3 is symbolic, in fact it is only used to indicate (13).

Remark 2. The above method of deriving the remainder formula from the Newton form works in every Lagrange–Hermite interpolation setting. In particular, it can be used for MI-I.

COROLLARY 1. *Theorem 2 and Theorem 3 remain valid if we replace the expression "be in general position" by "be admissible" in their hypotheses.*

Proof. We denote by "P_f" the formal Newton form (8) for the admissible hyperplanes. Of course for "P_f" and admissible hyperplanes the remainder formula holds, that is, $f-$"P_f" equals to the right-hand side of (12). This gives

$$D^\alpha[f(x_i) - \text{“}P_f\text{”}(x_i)] = 0 \qquad \forall i\in J, |\alpha| \leqslant m(x_i)-k$$

since for $j\in I_n^r$, $n \leqslant k-1$,

$$D^\alpha[\prod_{\substack{m=0\\ m\notin j}}^{r} \varrho(x_i, L_m)] = 0 \qquad \forall i\in J, |\alpha| \leqslant m(x_i)-k.$$

Thus

$$\text{“}P_f\text{”} \equiv P_f. \quad \blacksquare$$

Now we present a useful recurrence relation which is the analogue of (5).

THEOREM 4. *Let $L_0, \ldots, L_r$ be admissible and*

$$P_f = f/(L_0, \ldots, L_r).$$

Let also $L_{i_0}, \ldots, L_{i_k}$, $i = (i_0, \ldots, i_k) \in I_{k+1}^r$, be in general position. Then

(14) $$P_f(x) = \sum_{n=0}^{k} \frac{(x, L_{i_n})}{\varrho(x_{i\setminus i_n}, L_{i_n})} P_f^n(x),$$

where

$$P_f^n = f/(L_m, m\in(0,\ldots,r)\setminus i_n),$$

and of course

$$x_{i\setminus i_n} = x_{(i_0,\ldots,i_{n-1},i_{n+1},\ldots,i_k)}.$$

Proof. Applying a continuity argument (with the help of Corollary 1) we need to prove (14) for $L_0, \ldots, L_r$ being in general position. In this case it is not hard to obtain it from the relation

$$\sum_{n=0}^{k} \frac{\varrho(x, L_{i_n})}{\varrho(x_{i\setminus i_n}, L_{i_n})} \equiv 1. \quad \blacksquare$$

Remark 3. A similar recurrence relation seems not accessible for MI-I.

4. An application: "Star" numerical integration. In this section we give an interesting application of MI-II to the numerical integration on the disk

$$D = \{(t_1, t_2) \mid t_1^2 + t_2^2 \leqslant 1\}$$

in the plane. Let the points $x_0, \ldots, x_{2q}$ be equidistantly spaced on the circumference

$$S = \{(t_1, t_2) \mid t_1^2 + t_2^2 = 1\}.$$

For convenience we put $x_{2q+1+n} := x_n$, $n = 0, \ldots, q-1$. Let l_n, $n = 0, \ldots, 2q$, be the line passing through x_n and x_{n+q} (with the directional vector $x_{n+q}-x_n$). These lines form a q-star (see Fig. 1 for $q = 3$).

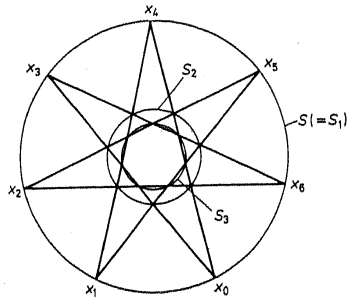

Fig. 1

Let us observe that $x_{n,m} := l_n \cap l_m$ $\forall\, n, m = 0, \ldots, 2q$, $n \neq m$, is a point belonging to D. Moreover, they are equidistantly spaced on the circumferences $S_1 = S, \ldots, S_q$, with common centre,

$$x_{n,n+m} \in S_{q-m+1}, \qquad n = 0, \ldots, 2q, \qquad m = 1, \ldots, q.$$

Now let

$$P_f = f/(l_0, \ldots, l_{2q}).$$

Then by (7)

$$P_f(x) = \sum_{m=1}^{q} \sum_{n=0}^{2q} f(x_{n,n+m}) P_{n,n+m}(x),$$

where

$$P_{n,n+m}(x) = \prod_{\substack{s=0 \\ s \neq n, n+m}}^{2q} \frac{\varrho(x, l_s)}{\varrho(x_{n,n+m}, l_s)}.$$

By the rotational symmetry, $P_{n,n+m}(x)$, $n = 0, \ldots, 2q$, (m fixed) have the same integral over D, that is,

$$\int_D P_{n,n+m}(x)\, dx = \int_D P_{0,m}(x)\, dx := c_m, \qquad n = 0, \ldots, 2q, \qquad m = 1, \ldots, q.$$

Hence we obtain the following simple formula for numerical integration:

$$\int_D f(x)\, dx = \int_D P_f(x)\, dx = \sum_{m=1}^{q} c_m \sum_{n=0}^{2q} f(x_{n,n+m}), \tag{15}$$

which is exact for all two-variate polynomials of total degree not exceeding $(2q-1)$.

Let $r_1, \ldots, r_q$ be the radii of the concentric circumferences $S_1, \ldots, S_q$. Then we easily obtain

$$r_{i+1} = \sin[\pi/(4q+2)]/\sin[(2i+1)\pi/(4q+2)], \qquad i = 1, \ldots, q-1,$$
$$r_1 = 1.$$

If we put in (15) the polynomial

$$f(x) = f(t_1, t_2) = \prod_{\substack{i=1 \\ i \neq n}}^{q} (t_1^2 + t_2^2 - r_i^2)$$

of total degree $(2q-2)$, the following interesting expression for c_n is obtained:

$$c_n = \frac{2\pi}{2q+1} \int_0^1 \prod_{\substack{i=1 \\ i \neq n}}^{q} \frac{(r - r_i^2)}{(r_n^2 - r_i^2)}\, dr.$$

For more detailed consideration and a generalization of this numerical integration see [9].

5. A formula for the main determinant (Vandermonde) of MI-II. First we shall present a quick proof of the following lemma which is interesting in itself (for origins cf. [11], [12]).

LEMMA 1. *Let L be a $(k-1)$-dimensional hyperplane, $P \in \pi_n(R^k)$, and*

$$(D_\lambda)^m P(x) = 0 \qquad \forall x \in L,\ m = 0, \ldots, s-1, \tag{16}$$

where λ has the normal direction of L. Then

$$P(x) = P(x, L)^s P_s(x), \tag{17}$$

with

$$P_s(x) \in \pi_{n-s}(R^k).$$

Proof. Since (17) is independent of the coordinate system, we assume without loss of generality that L is the hyperplane $x_1 = 0$. Next, we can represent $P(x_1, \ldots, x_k)$ in the form

$$P(x_1, \ldots, x_k) = \sum_{m=0}^{s-1} x_1^m P_m(x_2, \ldots, x_k) + x_1^s P_s(x_1, \ldots, x_k),$$

where

$$P_m \in \pi_{n-m}(R^k), \qquad m = 0, \ldots, s.$$

Now (16) implies

$$P_m(x_2, \ldots, x_k) = 0, \qquad m = 0, \ldots, s-1. \ \blacksquare$$

To introduce the analogue of Vandermonde determinant of MI-II we first order the sets I_k^r and $M = \{\alpha = (\alpha_1, \ldots, \alpha_k) |\ |\alpha| \leqslant r-k+1\}$, i.e., we assume that

$$i: \left\{1, \ldots, \binom{r+1}{k}\right\} \to I_k^r$$

and

$$\alpha: \left\{1, \ldots, \binom{r+1}{k}\right\} \to M$$

are one-to-one.

In what follows we assume that the $(k-1)$-dimensional hyperplanes $L_0, \ldots, L_r$ are in general position and that they are given by the following equations

$$\lambda_1^n x_1 + \ldots + \lambda_k^n x_k = 1, \qquad n = 0, \ldots, r, \tag{18}$$

respectively. Let also

$$d_{\lambda,1}\{i\} := \begin{vmatrix} 1 & \ldots & 1 \\ \lambda_1^{i_0} & \ldots & \lambda_1^{i_k} \\ \cdot & \cdots & \cdot \\ \lambda_k^{i_0} & \ldots & \lambda_k^{i_k} \end{vmatrix}$$

for $i = (i_0, \ldots, i_k) \in I_{k+1}^r$, and $d_\lambda\{i\}$, for $i \in I_k^r$, be given as in (6). Then we define

$$V(L_0, \ldots, L_r) := \det \left\| \varphi_{\alpha(m)}(x_{i(n)}) \right\|_{n,m=1}^{\binom{r+1}{k}},$$

where

$$\varphi_\alpha(x) := x^\alpha.$$

THEOREM 5. *We have*

$$V(L_0, \ldots, L_r) = c \frac{\left[\prod_{i \in I_{k+1}^r} d_{\lambda,1}\{i\}\right]^k}{\left[\prod_{i \in I_k^r} d_\lambda\{i\}\right]^{r-k+1}},$$

where c is independent of $L_0, \ldots, L_r$.

Proof. Using Cramer's rule for determining $x_i = L_{i_1} \cap \ldots \cap L_{i_k}$, $i = (i_1, \ldots, i_k) \in I_k^r$, as the unique solution of the linear system of equations of $L_{i_1}, \ldots, L_{i_k}$ it is not hard to show that

$$P_V := V(L_0, \ldots, L_r)\left[\prod_{i \in I_k^r} d_\lambda\{i\}\right]^{r-k+1}$$

is a polynomial of $\lambda^n = (\lambda_1^n, \ldots, \lambda_k^n)$ for each $n = 0, \ldots, r$. Computing the total degree of P_V, then considering it as a polynomial of λ_m^n, $n = 0, \ldots, r$, $m = 1, \ldots, k$ we obtain the sum

$$\sum_{n=0}^{r-k+1} [(r-k+1-n)k + n(k-1)]\binom{n+k-1}{k-1} = k^2\binom{r+1}{k}.$$

Now if for $i = (i_1, \ldots, i_k) \in I_k^r$ and $n \in (0, \ldots, r) \setminus i$,

$$\lambda^n \in L := \Big\{\sum_{m=1}^k \nu_m \lambda^{i_m} \Big|\ \sum_{m=1}^k \nu_m = 1\Big\},$$

then $x_i \in L_n$. Therefore

$$x_{(i_1, \ldots, i_{m-1}, i_{m+1}, \ldots, i_k, n)} = x_i, \qquad m = 1, \ldots, k.$$

It means that in this case $V(L_0, \ldots, L_r)$ will have $(k+1)$ columns equal. Hence

$$(D_{\lambda^n})^m P_V(x) = 0, \qquad \forall x \in L,\ m = 0, \ldots, k-1.$$

Since L is a $(k-1)$-dimensional hyperplane, and

$$\varrho(\lambda^n, L) = c_0 d_{\lambda,1}\{(n, i)\},$$

repeated application of Lemma 1 gives

$$P_V = c \prod_{i \in I_{k+1}^r} [d_{\lambda,1}\{i\}]^k, \tag{19}$$

where c is a polynomial in λ_m^n, $n = 0, \ldots, r$, $m = 1, \ldots, k$,

The total degree of the product on the right-hand side of (19), considered as a polynomial of λ_m^n, $n = 0, \ldots, r$, $m = 1, \ldots, k$, obviously equals $k^2\binom{r+1}{k}$, i.e., it is the same as for P_V. Hence c is a constant. Of course, Theorem 1 (ii) implies $c \neq 0$. ∎

Acknowledgements. The basic Theorem 1 is taken from the author's thesis [5] supervised by Professor Z. Ciesielski. The author is also grateful to him for further very useful discussions.

References

[1] A. S. Cavaretta, Jr., T. N. T. Goodman, C. A. Micchelli and A. Sharma, *Multivariate interpolation and the Radon transform*, Part III: *Lagrange representation*, in: Second Edmonton conference on approximation theory (Edmonton, Alta., 1982), CMS Conf. Proc., 3, Amer. Math. Soc., Providence, R. I. 1983, 37–50.

[2] W. Dahmen and C. A. Micchelli, *On limits of multivariate B-splines*, J. Analyse Math. 39 (1981), 256–278.

[3] W. Dahmen and C. A. Micchelli, *On the linear independence of multivariate B-splines.* II: *Complete configurations*, Math. Comp. 41 (1983), 143–163.
[4] T. N. T. Goodman, *Interpolation in minimum semi-norm and multivariate B-splines*, J. Approx. Theory 37 (1983), 212–223.
[5] H. Hakopian, *Multivariate spline functions, divided differences and polynomial interpolations*, thesis, University of Gdańsk, 1981.
[6] — *Multivariate divided differences and multivariate interpolation of Lagrange and Hermite type*, J. Approx. Theory 34 (1982), 286–305.
[7] — *Multivariate spline-functions, B-spline bases and polynomial interpolations II*, Studia Math. 79 (1984), 91–102.
[8] — *On fundamental polynomials of multivariate interpolation* I *of Lagrange and Hermite type*, Bull. Pol. Acad. Sci. 31 (1983), 137–141.
[9] — *A family of cubature formulas*, manuscript.
[10] P. Kergin, *A natural interpolation of* C^K *functions*, J. Approx. Theory 29 (1980), 279–293.
[11] L. Schwartz, *Analyse Mathématique*, vol. I, Hermann, Paris 1967.
[12] R. A. Nicolaides, *On a class of finite elements generated by Lagrange interpolation*, SIAM J. Numer. Anal. 9 (1972), 435–445.

Received April 27, 1983 (1883)
Revised version September 15, 1983