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THE NUMERICAL TREATMENT OF STIFF INITIAL VALUE PROBLEMS

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We consider the initial value problem (IVP) in $\boldsymbol{R}^M$ (with Euclidean norm $\|\cdot\|$)

$$\frac{dx}{dt} = f(t, x), \quad x(0) = x_0$$

with f: $[0, \infty) \times \boldsymbol{R}^M \to \boldsymbol{R}^M$ Lipschitz-continuous.

Assuming the variable t to be a continuously differentiable function φ of a parameter τ we get the autonomous system

$$\frac{dx}{d\tau} = \frac{d\varphi}{d\tau} f(t, x), \quad \frac{dt}{d\tau} = \frac{d\varphi}{d\tau}, \quad x(0) = x_0, \quad t(0) = 0.$$

This transformation may also be used for a stepsize control, for instance the statement

$$\varphi(\tau) = \int_0^\tau \big(1 + \|f(t(s), x(s))\|^2\big)^{-1/2} ds$$

leads to the autonomous system

$$\frac{dx}{d\tau} = \big(1 + \|f(t, x)\|^2\big)^{-1/2} f(t, x), \quad \frac{dt}{d\tau} = \big(1 + \|f(t, x)\|^2\big)^{-1/2},$$

and with a constant τ-stepsize we get segments of equal arclength for the solution curve $x(t)$.

Basing on this observation, we will only treat autonomous systems $dx/dt - f(x) = 0$, $x(0) = x_0$, with f: $\boldsymbol{R}^M \to \boldsymbol{R}^M$ Lipschitz-continuous, with a constant stepsize h.

Discretization of the problem

The initial value problem

$$\frac{dx}{dt}-f(x) = 0, \quad x(0) = x_0 \tag{1}$$

will be solved in an interval $[0, T]$. If we put $t_j = jh$ with $j = 0, 1, \ldots, n$, $(n-1)h < T \leqslant nh$, a *numerical integration method* (NIM) is defined by a formula for the calculation of vectors y_j, which have to approximate the solution vectors $x_j = x(t_j) = x(jh)$. With

$$E_h = E_h^0 = (\boldsymbol{R}^M)^{n+1}, \quad Y_h = \{y_j\}_{j=0}^n, \quad X_h = \{x_j\}_{j=0}^n \in E_h,$$

the defining instruction can be written as

$$\Phi_h(Y_h) = 0 \in E_h^0; \quad \Phi_h\colon E_h \to E_h^0. \tag{2}$$

Introducing suitable norms $\|\cdot\|_h$ and $\|\cdot\|_h^0$ in E_h and E_h^0, we state the fundamental demand that the NIM has to converge discretely with an order $s > 0$, i.e. that there are positive constants $\hat{h}, \gamma$ with the property that $\|X_h - Y_h\|_h \leqslant \gamma h^s$ for $0 < h \leqslant \hat{h}$.

The NIM is called *consistent with the order s* if there are positive constants $\hat{h}_c, \gamma_c$ such that $\|\Phi_h(X_h)\|_h^0 \leqslant \gamma_c h^s$ for $0 < h \leqslant \hat{h}_c$. It is called *realizable* if there is a positive $\hat{h}_r$ such that for all h with $0 < h \leqslant \hat{h}_r$ the equation $\Phi_h(Y_h) = 0$ is uniquely solvable. The NIM is called *stable* if there are constants $\hat{h}_s, \gamma_s > 0$ such that for all $Y_j^1, Y_j^2 \in E_h$, the inequalities $0 < h \leqslant \hat{h}_s$ imply

$$\|Y_h^1 - Y_h^2\|_h \leqslant \gamma_s \|\Phi_h(Y_h^1) - \Phi_h(Y_h^2)\|_h^0.$$

The main theorem of this discretization concept suggested by Stetter [14] states that consistency with the order s, realizability and stability jointly lead to discrete convergence with the order s. This fact follows immediately from

$$\|Y_h - X_h\|_h \leqslant \gamma_s \|\Phi_h(Y_h) - \Phi_h(X_h)\|_h^0 = \gamma_s \|\Phi_h(X_h)\|_h^0 \leqslant \gamma_s \gamma_c h^s.$$

In a general k-step method the equation $[\Phi_h(Y_h)]_j = 0$ for the jth component of Φ_h serves as instruction for the calculation of y_j from the known vectors $y_{j-1}, \ldots, y_{j-k}$. More concretely, we have

$$[\Phi_h(Y_h)]_j = \frac{1}{h}\sum_{r=0}^{k} a_{k-r} y_{j-r} - F_j \quad \text{with} \quad F_j = F(h, y_j, \ldots, y_{j-k}). \tag{3R}$$

Obviously, this instruction may be used only for the running phase $j = k, \ldots, n$. For the starting phase $j = 0, \ldots, k-1$ we need another system of equations:

$$\text{(3S)}\qquad [\Phi_h(Y_h)]_j = \frac{1}{h}\sum_{r=0}^{k-1} a_{jr}Y_r - \hat{F}_j \quad \text{with} \quad \hat{F}_j = \hat{F}_j(h, y_{k-1}, \ldots, y_0).$$

Generally we assume $a_k = 1, \sum_{r=0}^{k-1} a_r = 0$, $\det(a_{jr})_{j,r=0}^{k-1} = 0$ and the functions $F_j, \hat{F}_j$ to be uniformly Lipschitz-continuous with respect to y_i on any finite h-interval.

Any k-step method of this kind is realizable. By virtue of the Dahlquist theorem it is stable (in the maximum norm $\|Y_h\|_h = \|Y_h\|_h^0 = \max_{j=0,\ldots,n} \|y_j\|$) if the rootcondition is fulfilled for the polynomial $\chi(z) = \sum_{r=0}^{k} a_r z^r$, i.e. from $\chi(z_0) = 0$ it follows that either $|z_0| = 1$ and $\chi'(z_0) \neq 0$ or $|z_0| < 1$. Under these conditions discrete convergence with the given order of consistency is guaranteed. If the strict rootcondition holds, i.e. if

$$\text{(4)}\qquad \chi(1) = 0, \quad \chi'(1) \neq 0$$

and if, for $z_0 \neq 1$, $\chi(z_0) = 0$ leads to $|z_0| < 1$, then we get for $s > 1$ additionally the estimation

$$\max_{j=1,\ldots,n} \|x'(jh) - h^{-1}(y_j - y_{j-1})\| \leqslant \hat{\gamma} h^{s-1},$$

which guarantees the convergence of the difference quotients of the approximate solution to the derivative of the exact one.

Stiff differential equation systems

The statements of the preceding section (see also [1], [5]) give criteria easy to use. Therefore the problem of development of practicable, i.e. discretely convergent methods, seems to be solved completely. But unfortunately we have to face the fact that the constants γ, $\hat{h}$ for each concrete method depend strongly on T and the differential equation system. With increasing Lipschitz constants L_f of f also γ and $\hat{h}^{-1}$ increase unboundedly; the same is valid for the T-dependence of γ_s (and therefore of γ).

In many practical problems, especially in the computational simulation of electrical networks with continuous behaviour, dynamics of chemical processes and kinetics of atomic reactors, the so-called stiff differential equation systems occur. They are distinguished by the fact that the Jacobian Df of f has eigenvalues not only of moderate magnitude but also ones with very large negative real parts:

$$\text{(5)}\qquad \Lambda(Df) = \Lambda_1 \cup \Lambda_2 \quad \text{with } |\lambda_2| \ll -\mathrm{Re}(\lambda_1) \quad \text{for } \lambda_i \in \Lambda_i.$$

The eigenvalues from Λ_1 do not disturb the stability behaviour of the exact solution, but they lead to very large L_f because of $|\mathrm{Re}(\lambda_1)| \leqslant |\lambda_1| \leqslant \|Df\| \leqslant L_f$, i.e. their occurence requires, in general, extremely small stepsizes for the numerical integration. Therefore the question arises if there are also NIM's which lead to bounded values of $\hat{h}^{-1}$ and γ for certain classes of stiff problems.

Linear systems with constant coefficients

The question closing the last section was at first treated by Dahlquist [2] for linear IVP's with constant coefficients

$$\frac{dx}{dt} = Ax, \qquad x(0) = x_0. \tag{6}$$

The corresponding iteration equation for the running phase is

$$\sum_{r=0}^{k} p_{k-r}(hA)y_{j-r} = 0, \tag{7}$$

where the $p_r(w)$ are polynomials with $p_k(0) = 1$, $\sum_{r=0}^{k} p_r(0) = 0$; especially for extrapolative NIM's we get $p_k(w) \equiv 1$.

Any difference equation of the kind

$$\sum_{r=0}^{k} p_{k-r}(hA)y_{j-r} = hu_j \tag{8}$$

can be written as

$$y_j + \sum_{r=1}^{k} q_{k-r}(hA)y_{j-r} = hPu_j \tag{9}$$

with $q_r(w) = p_r(w)/p_k(w)$ and $P = [p_k(hA)]^{-1}$, or as iteration formula

$$\hat{y}_j = \hat{Q}\hat{y}_{j-1} + h\hat{P}\hat{u}_j \tag{10}$$

with

$$\hat{y}_j = \begin{bmatrix} y_j \\ y_{j-1} \\ \cdot \\ \cdot \\ \cdot \\ y_{j-k+1} \end{bmatrix}, \quad \hat{u}_j = \begin{bmatrix} u_j \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad \hat{Q} = \begin{bmatrix} -q_{k-1}(hA) & \dots & -q_1(hA) & -q_0(hA) \\ I & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & I & 0 \end{bmatrix}$$

and $\hat{P} = \mathrm{diag}(P, \ldots, P)$. From (10) immediately follows

$$\hat{y}_j = Q^{j-k+1}\hat{y}_{k-1} + h\hat{P}\sum_{l=k}^{j}\hat{Q}^{j-l}\hat{u}_l. \tag{11}$$

If we put specifically $u_j = [\Phi(Y^1) - \Phi(Y^2)]_j$, we get for fixed h and arbitrarily large n the obvious estimation

$$\max_{j=1,\ldots,n} \|y_j^1 - y_j^2\| \leqslant \gamma_8\big(\max_{j=k,\ldots,n}\|u_j\| + \max_{j=0,\ldots,k-1}\|y_j^1 - y_j^2\|\big),$$

provided there are constants β and $\tilde{\beta}$ with the property that the estimates

$$\|\hat{Q}^j\| \leqslant \beta \quad \text{and} \quad \sum_{r=0}^{j}\|\hat{Q}^r\| \leqslant \tilde{\beta}$$

hold for all naturals j. This condition is notoriously fulfilled iff all eigenvalues of $\hat{Q}$ are contained in the open unit circle $\mathscr{E} = \{z \,|\, |z| < 1\}$. The spectrum of $\hat{Q}$ consists of the zeros z_0 of $P(h\lambda, z)$ with $\lambda \in \Lambda(A)$,

$$P(w, z) = \sum_{r=0}^{k} p_r(w)z^r;$$

therefore Dahlquist called the set

$$\mathscr{H} = \{w \mid \mu(w) < 1\} \quad \text{with} \quad \mu(w) = \max\{|z_0| \mid P(w, z_0) = 0\} \tag{12}$$

the *domain of absolute stability*. The relation $\Lambda(hA) \subseteq \mathscr{H}$ ensures that γ_8 is independent of n or T. By the transformation $t = \tau/\sigma$ this result for unboundedly increasing T may be translated into a statement about the system $dx/dt = (1/\sigma)Ax$ for a fixed integration interval, which has an unboundedly increasing Lipschitz constant as σ tends to 0.

Since the exact solution of (6) decreases extremely fast if all eigenvalues λ of A have very large negative real parts, we wish to get for $\mathrm{Re}(\lambda) \to -\infty$ the relation $\|\hat{Q}\| \to 0$. Therefore a NIM is most suitable for the numerical integration of stiff systems if it fulfils the limit condition

$$\mu(w) \to 0 \quad \text{for} \quad \mathrm{Re}(w) \to -\infty. \tag{LC}$$

According to the properties of $\mathscr{H}$ and μ, a NIM is called A-stable for $\mathscr{H} \supseteq C^-$, the negative halfplane (Dahlquist [2]), asymptotically exact for $\mathscr{H} = C^-$ (Griepentrog [5]), $A(\alpha)$-stable for $\mathscr{H} \supseteq \mathscr{W}_\alpha = \{w \mid \pi - \alpha < \arg(w) < \pi + \alpha\}$ (Widlund [16]), $A(0)$-stable if there is an α with $\mathscr{H} \supseteq \mathscr{W}_\alpha$ (Widlund [16]), A_0-stable for $\mathscr{H} \supseteq R^-$, the negative halfaxis (Cryer 1973, s.a. [12]), L-stable if $\mathscr{H} \supseteq C^-$ and (LC) holds (Ehle [4]), stiffly stable if $\mathscr{H} \supseteq \mathscr{W}_\alpha$ and (LC) holds (for some α, Gear [10]). Using the results of complex function theory many authors have analysed and developed methods, which have some of the above properties. The most comprehensive and systematic representation seems to be given by Jeltsch [11]; summarized expositions are also to be found in the textbooks of Grigorieff, Gear, Lapidus–Seinfeld and Stetter ([8], [10], [13], [14]).

All the above mentioned properties include, in particular, the unboundedness of $\mathscr{H}$. Therefore, according to the Vietà theorem and $p_k(w) \equiv 1$, no explicit NIM can fulfil any one of these conditions. Dahlquist proved [2] that no A-stable linear k-step method (i.e. a k-step method in which F is a linear combination of the $f(y_{j-r})$) can have an order of convergence $s > 2$. To get a higher order we must take nonlinear implicit methods, for instance implicit or semi-implicit Runge–Kutta-methods, or we have to weaken the requirement of A-stability, for instance to the $A(\alpha)$-stability of the Gear-methods. Griepentrog proved that only onestep methods with $p_1(w) = -p_0(-w)$, $p_0(w)$ Hurwitz-polynomial, are asymptotically exact. This restriction leads to $|p_0(w)/p_1(w)| \to 1$ as $\operatorname{Re}(w) \to -\infty$ and therefore asymptotical exactness and the limit condition are incompatible.

Statements for more general IVP's

Dahlquist investigated systems $dx/dt = f(x)$ for which a symmetric positive definite matrix S exists with

$$[S(x_1 - x_2)]^T [f(x_1) - f(x_2)] \leqslant 0;$$

for $f(x) = Ax$, $\Lambda(A) \subseteq C^-$, S is of the form

$$S = \int_0^\infty [\exp(tA)]^T \exp(tA)\,dt.$$

All solutions of such systems are stable because of

$$\frac{d}{dt}\left[\left(x_1(t) - x_2(t)\right)^T S\left(x_1(t) - x_2(t)\right)\right] \leqslant 0,$$

and Dahlquist proved that A-stable linear k-step methods always produce stable approximations. But we must note that $s \leqslant 2$ for this class of NIM's and that the statement cannot be extended to nonlinear methods.

It is well known that for any matrix A with $\Lambda(A) \subseteq C^-$ there exists a positive ω such that all solutions of $dx/dt = Ax + f(t, x)$ are asymptotically stable, if the Lipschitz constant L_f of f does not exceed ω. The running phase of a NIM is for this system given by the equation

$$\sum_{r=0}^{k} p_{k-r}(hA) y_{j-r} = hF(h, y_j, \ldots, y_{j-k}) = hF_j \tag{13}$$

(F uniformly Lipschitz-continuous on each interval $0 < h \leqslant h_0$). Considering the angle domain $\mathscr{W}_\alpha$ and the polynomial $P(w, z) = \sum_{r=0}^{k} p_r(w) z^r$ we introduce the terminology: the pair $(\mathscr{W}_\alpha, P)$ fulfils the fundamental conditions (FC) if the following relations hold:

FC1: $\deg(p_k) > \deg(p_r)$ for $r \neq k$ (limit condition),

FC2: $p_k(z_0) = 0$ implies $z_0 \notin \overline{\mathscr{W}_\alpha}$ (unconditional realizability),

FC3: $P(0, z_0) = 0$ and $z_0 \neq 1$ implies $|z_0| < 1$ (strict stability),

FC4: $P(0, 1) = 0$ and $\dfrac{\partial P}{\partial z}(0, 1) = -\dfrac{\partial P}{\partial w}(0, 1) \neq 0$ (order $s \geqslant 2$),

FC5: $P(w, z_0) = 0$ and $w \in \mathscr{W}_\alpha$ implies $|z_0| < 1$ ($A(\alpha)$-stability).

In [7] the following theorem corresponding to the foregoing stability statement is proved; it ensures that the convergence constant γ is independent of T.

THEOREM 1. *If $\Lambda(A) \subseteq \mathscr{W}_\alpha$ and $(\mathscr{W}_\alpha, P)$ satisfies the fundamental conditions* (FC), *then there exists a positive Ω such that for $L_F \leqslant \Omega$ the estimate*

$$\|y_n - x_n\| \leqslant \text{const} \cdot \{\max_{j=0}^{k-1} \|y_j - x_j\| + h^s\}$$

holds independently of n.

The proof of this statement is essentially based on certain norm estimates for the powers of $\hat{Q}$ given in Theorem 2 and also stated in [7].

THEOREM 2. *If $\Lambda(A) \subseteq \mathscr{W}_\alpha$ and $(\mathscr{W}_\alpha, P)$ fulfils the* (FC), *then there are constants $\beta, \tilde{\beta}, \hat{\beta}$ with*

$$\|\hat{Q}^n\| \leqslant \beta, \qquad h \sum_{=k}^{n} \|\hat{Q}^j\| \leqslant \tilde{\beta}, \qquad \|hP\| \sum_{j=0}^{n} \|\hat{Q}^j\| \leqslant \hat{\beta} \tag{14}$$

independent of $n \in N$ and $h \in (0, \infty)$.

In order to investigate the usefulness of a NIM for stiff problems Stetter [15] suggests to consider systems

$$\frac{dx}{dt} = f(x, z), \qquad \frac{dz}{dt} = \frac{1}{\sigma} g(x, z) \ \text{for} \ \sigma \to 0$$

with rapidly decreasing $z(t)$. He called a NIM *R-stable* if the part of the approximate solution which corresponds to z does also rapidly decrease, while the component corresponding to x discretely converges to x uniformly for $\sigma \to 0$. Indeed, this demand is so strong that till now R-stability is only known to hold for the Euler-backward-rule. However, it turns out that interesting results can be derived by realizing Stetters suggestion for a suitably restricted class of problems, the class of systems with constant stiff part.

Systems with constant stiff part

They describe processes in which stiffness is produced by constant material parameters; usually this model assumption is adequate. The mathematical description is given by the equation

$$\frac{d\xi}{dt} = \frac{1}{\sigma} B\xi + \varphi(\xi), \qquad \xi(0) = \xi_0, \tag{15}$$

where $\sigma \to 0$ produces stiffness and $B = S\begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} S^{-1}$ with $\Lambda(A) \subseteq C^-$ holds. Putting $a = \max\{\mathrm{Re}(\lambda) \mid \lambda \in \Lambda(A)\}$ and $\xi = S\begin{bmatrix} x \\ z \end{bmatrix}$, $\varphi = S\begin{bmatrix} f \\ g \end{bmatrix}$, we get, for x and z, the solution components of $\xi(t)$, the differential equation system

$$\frac{dx}{dt} = f(x, z), \qquad \frac{dz}{dt} = \frac{1}{\sigma} Az + g(x, z) \tag{16}$$

which leads to the estimates

$$\|x(t)\|, \left\|\frac{dx}{dt}\right\| \leqslant c, \qquad \|z(t)\| \leqslant c\left\{\exp\left(\frac{at}{2\sigma}\right) + \sigma\right\}, \qquad \left\|\frac{dz}{dt}\right\| \leqslant c\left\{\frac{1}{\sigma}\exp\left(\frac{at}{2\sigma}\right) + 1\right\}$$

valid on the interval $[0, T]$ (c denotes some constant).

Obviously $z(t)$ decreases very rapidly for small values σ and very soon it produces only a negligible contribution to the solution $\xi(t)$ of (15). The NIM leads to the difference equations corresponding to (16):

running phase:

$$\sum_{r=0}^{k} p_{k-r}(0) y_{j-r} = hF_j, \qquad \sum_{r=0}^{k} p_{k-r}\left(\frac{h}{\sigma} A\right) w_{j-r} = hG_j, \tag{17R}$$

starting phase:

$$\sum_{r=0}^{k-1} \hat{p}_{jr}(0) y_r = h\hat{F}_j, \qquad \sum_{r=0}^{k-1} \hat{p}_{jr}\left(\frac{h}{\sigma} A\right) w_r = h\hat{G}_j, \tag{17S}$$

needed for the calculation of the approximations y_j for x_j and w_j for z_j. Here $F_j, G_j, \hat{F}_j, \hat{G}_j$ depend Lipschitz-continuously on the y_i and w_i. Now (17R) and FC1 imply

$$w_j + \sum_{r=1}^{k} q_{k-r}\left(\frac{h}{\sigma}A\right) w_{j-r} = h\left[p_k\left(\frac{h}{\sigma}A\right)\right]^{-1} G_j$$

and

$$\lim_{\sigma\to 0} q_r\left(\frac{h}{\sigma}A\right) = \lim_{\sigma\to 0}\left[p_k\left(\frac{h}{\sigma}A\right)\right]^{-1} = 0,$$

i.e. the w_j also decrease for small σ very rapidly and do not then influence the approximation $S\begin{bmatrix} y_j \\ w_j \end{bmatrix}$ of $\xi(t_j)$. Wanted are NIM's with the property that for any $\hat{\sigma} > 0$ the sequences $Y_h = \{y_j\}_{j=0}^{n}$ discretely converge to the solution component $x(t)$, uniformly in $0 < \sigma \leqslant \hat{\sigma}$.

In [6] we proved the following statement.

THEOREM 3. *For every L-stable onestep method and for arbitrary T, $\hat{\sigma} > 0$ there exist constants $\gamma, \hat{h}$ independent of $\sigma \in (0, \hat{\sigma}]$ with*

$$\max_{j=1,\ldots,n} \|y_j - x_j\| + \frac{1}{n}\sum_{j=1}^{n} \|w_j - z_j\| \leqslant \gamma h$$

for $0 < h \leqslant \hat{h}$, $(n-1)h < T \leqslant nh$, $y_0 = x_0$, $w_0 = z_0$.

This result now is generalized by Gronau for multistep methods, where of course additional conditions for the starting phase are necessary.

THEOREM 4 (Gronau [9]). *For the problem* (16) *with* $\Lambda(A) \subseteq \mathscr{W}_a$ *a NIM realizing the following conditions is used:*

Starting phase:

$$\mathscr{P}(w) = \left(\hat{p}_{jr}(w)\right)_{j,r=1}^{k-1} = \sum_{l=0}^{m} \mathscr{P}_l w^l$$

with

$$m = \max\{\deg(\hat{p}_{jr}) \mid j, r = 1, \ldots, k-1\}$$

fulfils

$$\det \mathscr{P}_m \neq 0, \qquad \det \mathscr{P}(w) \neq 0 \quad \textit{for } w \in \overline{\mathscr{W}}_a,$$

$$\sum_{r=0}^{k-1} \hat{p}_{jr}(0) = 0 \qquad (j = 1, \ldots, k-1).$$

Running phase: $(\mathscr{W}_a, P)$ *fulfils the* (FC).

Then for any T, $\hat{\sigma} > 0$ there are constants γ, $\hat{h} > 0$ not depending on $\sigma \in (0, \hat{\sigma}]$ such that the estimate

$$\max_{j=1,\ldots,n} \|y_j - x_j\| + \frac{1}{n}\sum_{j=1}^{n} \|w_j - z_j\| \leqslant \gamma h$$

holds for $0 < h \leqslant \hat{h}$, $(n-1)h < T \leqslant nh$, $y_0 = x_0$, $w_0 = z_0$.

The proofs of Theorems 3 and 4 basing on (14) are very complicated and cannot be outlined here. It should be noted that $\max\|w_j - z_j\|$ tends to zero as $h \to 0$, not uniformly in $\sigma \in (0, \hat{\sigma}]$. Therefore we have to expect greater errors for (15) in the first integration steps, but because of the above theorems they do not influence further integration.

Some remarks about the conditions should now be done. As regards the running phase, we need nothing more than in the linear case with constant coefficients, and the conditions for the starting phase are fulfilled if, for instance, for the starting steps A-stable onestep methods or Gear-formulas of increasing order are used, provided that their $\mathscr{W}_a$ contain the spectrum of A. In the first case we have $\det\mathscr{P}(w) = \prod_{j=1}^{k-1} \hat{p}_{jj}(w)$ with nowhere on $\bar{C}^-$ vanishing functions $\hat{p}_{jj}(w)$ and $\mathscr{P}_m$ is a lower triangular matrix with not vanishing diagonal elements. In the second case we get $m = 1$, $\det\mathscr{P}(w) = \pm\prod_{j=1}^{k-1}(b_j - c_j w)$ with positive values b_j, c_j and $\mathscr{P}_1 = \pm\prod_{j=1}^{k-1} c_j$.

Finally we have to remark that no higher order of convergence than $s = 1$ is obtainable independently of $\sigma \in (0, \hat{\sigma}]$, i.e. the criteria given by Theorems 3 and 4 are not improvable and therefore they are rather complete for systems with constant stiff parts. This is easy to observe by applying a stiffly stable onestep method on $\frac{dz}{dt} = -\frac{1}{\sigma}z$, $z(0) = 1$. We get, for $u = -h/\sigma$,

$$|w_j - z_j| = |[q(u)]^j - [\exp(u)]^j| \quad \text{and} \quad \sum_{j=1}^{\infty} |w_j - z_j| = g(u)$$

with

$$g(u) = \left|\frac{q(u) - \exp(u)}{\bigl(1 - q(u)\bigr)\bigl(1 - \exp(u)\bigr)}\right|.$$

This expression is bounded on R^- and is of order $O(u^{s-1})$ as $u \to 0$. For a fixed σ we get for $\frac{1}{n}\sum_{j=1}^{n} |w_j - z_j| \leqslant \frac{h}{T} g(u) = O(h^s)$ the convergence order s, but for an estimation independent of σ we can only use the boundedness of $g(u)$, i.e. the best we can get is $\frac{1}{n} g(u) = O(h)$.

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