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THE LAGRANGE MULTIPLIERS THEOREM FOR LOCALLY CONVEX METRIC SPACES

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Introduction

An application of the method of contractor directions to nonlinear programming problems in Banach space has been given in a previous investigation. The objective is there to investigate optimization problems under rather weaker hypotheses than it is known in the literature. The objective of this investigation is to extend the theory of optimization to locally convex vector spaces endowed with an increasing sequence of seminorms. For this purpose the diagonal method of contractor directions is very much instrumental. This method provides algorithms for solving nonlinear operator equations in locally convex metric spaces. The key issue of the present investigation is to establish the existence of tangent directions in such spaces. As a consequence of this fact, one can prove a theorem on Lagrange multipliers for the class of spaces mentioned above. Furthermore, an application of the diagonal method of contractor directions along with the general extremum principle (see [4]) provide the basic tools which are needed for the extension of the theory of optimization problems with constraints of mixed type, i.e., with inequalities and equality constraints, to locally convex metric spaces. This extension will be investigated in a separate paper.

1. Tangent directions in locally convex metric spaces

Let X be a vector space endowed with an increasing sequence of seminorms

$$\|x\|_0 \leqslant \|x\|_1 \leqslant \ldots, \qquad \text{for all } x \text{ of } X.$$

Let Y be another vector space of the same type, i.e., with the sequence of seminorms

$$\|y\|_0 \leqslant \|y\|_1 \leqslant \ldots, \qquad \text{for all } y \text{ of } Y.$$

Let $P\colon X \to Y$ be a nonlinear continuous mapping, X and Y being complete in the usual sense. We assume that P is twice continuously differentiable in the Fréchet sense and denote by P', P'' the first and second Fréchet derivative, respectively.

DEFINITION 1.1. Let V be the vector space X with the topology defined by a single siminorm $\|x\|_k$. Suppose that $k = 0$, $Px_0 = 0$ for $x_0 \in X$ and that $P'(x_0)h = 0$, for $h \neq 0$. Then h is called a *tangent direction at* x_0 if there exists a number $t(h) > 0$ with the following property. For each t with $|t| < t(h)$, there exists an element $\eta(t) \in X$ such that

$$P\big(x_0 + th + \eta(t)\big) = 0 \quad \text{and} \quad \|\eta(t)\|_0/t \to 0 \quad \text{as} \quad t \to 0.$$

This definition obviously generalizes the notion of tangent directions in Banach spaces.

Put $U_0 = S(x_0, r) = [x\colon \|x - x_0\|_0 < r]$. We assume that the first and second Fréchet derivatives $P'(x)$ and $P''(x)$ exist and are continuous at all $x \in U_0$ and have the following properties:

(i) For each $x \in U_0$, there exists an element $h(x)$ such that $P'(x)h(x) = -Px$.

(ii) For each $k = 0, 1, \ldots$, there exist a function $B_k \in \boldsymbol{B}$ (see Appendix) and an integer $p(k) \geqslant 0$, $p(0) = p \geqslant 0$ such that

$$\|h(x)\|_k \leqslant B_k(\|Px\|_{k+p(k)}) \quad \text{for } k = 0, 1, \ldots, \tag{1.1}$$

where $B_0(s) = Cs$ for some constant $C > 0$.

THEOREM 1.1. *Suppose, in addition to the hypotheses* (i), (ii), (1.1), *that the following relations are satisfied. There exist positive constants* $C_0, C_1, \ldots$ *such that*

$$\max_{0 \leqslant t \leqslant 1} \|P''(x+th)(h, h)\|_i \leqslant C_i \|Px\|_i \quad \textit{for all } i = 0, 1, \ldots,\ x \in U_0, \tag{1.2}$$

where $P'(x)h = -Px$, *and*

$$\sum_{n=0}^{\infty} 1 / \max_{0 \leqslant i \leqslant n} C_i = \infty. \tag{1.3}$$

If $Px_0 = 0$ *and* $P'(x_0)h = 0$, *then* h *is a tangent direction.*

Proof. We apply Theorem 1 of the Appendix, there x_0 is replaced by $x_0 + th$ with $\|th\|_0 < r/2$. For $0 < q < 1$ and $a = \exp(1-q)C\|P(x_0+th)\|_p$ we get

$$r(t) = (1-q)^{-1} \int_0^a s^{-1} B_0(s)\, ds = (1-q)^{-1} a < r/2$$

if $|t| < \sigma$. Now put $t(h) = \min(\sigma, r/2h)$. By Theorem 1 of the Appendix, where x_0 and r are replaced by x_0+th and $r(t)$, respectively, there exists an element $x^* \in S(x_0+th, r(t))$ such that $Px^* = 0$. Put $x_0+th-x^* = -\eta(t)$; then we obtain that $P(x_0+th+\eta(t)) = 0$ and $\|\eta(t)\|_0 \leqslant r(t)$. Relations $Px_0 = 0$ and $P'(x_0)h = 0$ imply that $\|P(x_0+th)\|_p/t \to 0$ as $t \to 0$. Hence, it follows that $\|\eta(t)\|_0/t \to 0$ as $t \to 0$, and the proof is completed.

Lusternik [5] has proved the existence of tangent directions in Banach spaces under the assumption that the Fréchet derivative $P'(x)$ exists in the neighborhood of x_0 and is continuous at x_0, and $P'(x_0)(X) = Y$. A generalization of Lusternik's theorem in Banach spaces is given in [3].

2. The Lagrange multipliers theorem for locally convex metric spaces

Let $P\colon X \to Y$ be a nonlinear mapping which satisfies the hypotheses of Theorem 1.1, and let F be a nonlinear real-valued functional defined on the neighborhood U_0 of x_0. The problem is to find an element x_0 in U_0 which minimizes F on the set of all x in U_0 such that $Px = 0$, i.e.,

$$F(x_0) = \min\{F(x) \mid x \in U_0 \text{ and } Px = 0\}. \tag{2.1}$$

We assume that $U_0 = [x\colon \|x-x_0\|_0 < r]$. We also assume that F is differentiable in the following sense.

(a) There exists a linear continuous functional $F'(x_0)$ which has the following property. If the linear functional $F'(x_0)$ is continuous with respect to the seminorm $\| \ \|_k$, then

$$[F(x_0+tz)-F(x_0)]/t \to F'(x_0)x \quad \text{as } t\to 0 \text{ and } \|z-x\|_k \to 0. \tag{2.2}$$

Without loss of generality, we may assume that $k = 0$. The following theorem gives necessary conditions for the local minimization problem (2.1).

THEOREM 2.1. *Suppose that the nonlinear mapping $P\colon X\to Y$ satisfies the hypothese of Theorem* 1.1, *and that $P'(x_0)(X) = Y$. If F is differentiable in the sense* (a) *and if x_0 is a solution of the minimization problem* (2.1), *then there exists a linear continuous functional l defined on Y such that*

$$F'(x_0)x = l(Px) \quad \textit{for all } x \in X, \tag{2.3}$$

i.e., if $F'(x_0)$ is also the Fréchet derivative, then

$$\Phi'(x_0) = 0, \quad \Phi(x) = F(x)+l(Px).$$

Proof. We first prove that $F'(x_0)h = 0$ whenever $P'(x_0)h = 0$. In fact, by Theorem 1.1, if $P'(x_0)h = 0$ and t is sufficiently small, then there exist $O(t) \in X$ such that

$$P(x_0+th+O(t)) = 0 \quad \text{and} \quad \|O(t)\|_0/t \to 0 \quad \text{as } t\to 0.$$

Suppose that $F'(x_0)h = a \neq 0$. Then we obtain

$$F\big(x_0+th+O(t)\big)-F(x_0) = tF'(x_0)+F'(x_0)O(t)+\varepsilon(t),$$

where $\varepsilon(t)/t\to 0$ as $t\to 0$, by relation (2.2) with $z = h+O(t)/t$ and $x = h$, since $\|O(t)\|_k/t\to 0$ as $t\to 0$ and $k = 0$ by assumption. Hence, it follows that both, the difference

$$F\big(x_0+th+O(t)\big)-F(x_0) \quad \text{and} \quad tF'(x_0)h = ta$$

have the same sign which changes if h is replaced by $-h$. Therefore, x_0 is not a minimum point and, consequently, we obtain that $F'(x_0)h = 0$. Now put $T_0 = [h\colon\, h\in X$ and $P'(x_0)h = 0]$ and denote by X/T_0 the quotient space of equivalent classes $T\in X/T_0$ with seminorms defined by the corresponding seminorms of X, in the usual way. Then we can define a functional ψ by the formula

$$\psi(T) = F'(x_0)h, \tag{2.4}$$

where h is an arbitrary element of T, since $h_1\in T$ and $h_2\in T$ imply $h_1-h_2\in T_0$ and $F'(x_0)(h_1-h_2) = 0$. Since $F'(x_0)h$ is continuous in the seminorm $\|h\|_0$, by assumption, it follows that there exists a constant M such that for arbitrary $h\in T$ we have

$$|\psi(T)| = |F'(x_0)h| \leqslant M\|h\|_0.$$

Hence,

$$|\psi(T)| \leqslant M\|T\|_0, \tag{2.5}$$

and $\psi(T)$ is a linear continuous functional defined on X/T_0. Let $A\colon X/T_0\to Y$ be the linear continuous operator defined by the formula

$$A(T) = P'(x_0)h = y \qquad \text{for arbitrary } h\in T.$$

Since $P'(x_0)$ is a mapping onto Y, it follows from a theorem of Banach that the linear operator $T = A^{-1}y$ exists and is continuous. Hence, by virtue of (2.4) and (2.5), we obtain

$$F'(x_0)h = \psi(T) = \psi(A^{-1}y) = l(y) = l\big(P'(x_0)h\big),$$

where $y = P'(x_0)h$, $h\in T$. Hence, we conclude that l is a linear continuous functional defined on Y, and the proof is completed.

Lusternik [5] has proved the Lagrange multipliers theorem for Banach spaces under the hypotheses that the Fréchet derivative $P'(x)$ exists in some neighborhood of x_0 and is continuous at x_0, and $P'(x_0)$ is a mapping onto Y. The Fréchet derivative $F'(x_0)$ is also needed in his proof. A generalization of Lusternik's theorem for Banach spaces is given in [3].

Appendix

Let $P\colon X \to Y$ be a nonlinear continuous mapping, where X and Y are the same as in Section 1. Denote by $\boldsymbol{B}$ the class of increasing continuous functions B such that

$$B(0) = 0 \quad \text{and} \quad B(s) > 0 \quad \text{for } s > 0;$$

$$\int_0^a s^{-1} B(s)\,ds < \infty \quad \text{for some positive } a.$$

Suppose that P satisfies the hypotheses of Section 1. Given x_0, $0 < q < 1$, and $U_0 = S(x_0, r) = [x\colon \|x - x_0\| < r]$, we define the following algorithm:

$$x_{n+1} = x_n + \varepsilon_n h_n \quad \text{for } n = 0, 1, \ldots, \quad \text{where } P'(x_n) h_n = -Px_n \tag{1}$$

and relation (1.1) is satisfied with $x = x_n$ and $h_n = h(x_n)$. The positive numbers ε_n in (1) are defined as follows:

$$\varepsilon_n = \min(1, q/\max_{0 \leqslant i \leqslant n} C_i). \tag{2}$$

THEOREM 1 (see Theorem 4.1 [2]). *Suppose that the hypotheses* (1.1)–(1.3) *of Theorem* 1.1 *are satisfied for* U_0 *with*

$$r \geqslant (1-q)^{-1} \int_0^a s^{-1} B_0(s)\,ds \quad \text{with } a = e^{1-q} \|Px_0\|_0, \text{ and } B_0 \in \boldsymbol{B}. \tag{3}$$

Then the sequence $\{x_n\}$ *defined by* (1) *and* (2) *lies in* U_0 *and converges to a solution of equation* $Px = 0$.

The general theory of contractor directions is presented in [1]. The diagonal method of contractor directions is discussed in [2], where more relevant references are given.

References

[1] M. Altman, *Contractors and Contractor Directions, Theory and Applications*, Marcel Dekker, 1977.

[2] —, *An inverse function theorem for Fréchet spaces*, this volume, pp. 683–699.

[3] —, *An application of the method of contractor directions to nonlinear programming*, to appear.

[4] —, *A general extremum principle for optimization problems*, Nonlin. Analys., in the press.

[5] L. A. Lusternik and V. I. Sobolev, *Elements of Functional Analysis*, Moscow 1951.

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